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GLOBAL EXISTENCE OF LARGE AMPLITUDE SOLUTIONS FOR NONLINEAR MASSLESS DIRAC EQUATION

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Abstract: In this paper we solve the global Cauchy problem for nonlinear relativistic Dirac equation in Minkowski space.

We can take arbitrarily large initial data with the only smallness assumption on the Chiral invariant related to Lochak-Majorana condition. We study the asymptotic behaviour of the solution, particularly the equipartition of energy and the remarkable decay of Lorentz-invariant products.

Introduction

We consider the nonlinear massless Dirac equation in Minkowski space $\mathbf{R}^4_1$ with the Lorentz metric $g_{\mu,\nu} = \mathrm{diag}(1,-1,-1,-1)$:

$$-i\,\gamma^\mu\,\partial_\mu\psi = f(\psi)\ , \tag{1}$$

where

$$\partial_\mu = \frac{\partial}{\partial x^\mu}\,, \qquad 0 \le \mu \le 3, \quad x^0 = t, \quad x = (x^1, x^2, x^3)\ ,$$

and γ^μ are the Dirac matrices defined with the Pauli matrices: σ^μ:

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ ,$$

$$\gamma^0 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}, \qquad 1 \le j \le 3\ .$$

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The Dirac matrices satisfy the relations

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\, g^{\mu,\nu}\, Id \ , \tag{2}$$

$$\tilde{\gamma}^{\mu} = g^{\mu,\mu}\gamma^{\mu} \ , \tag{3}$$

$$^{t}\gamma^{0} = \gamma^{0}, \quad ^{t}\gamma^{2} = \gamma^{2}, \quad ^{t}\gamma^{1} = -\gamma^{1}, \quad ^{t}\gamma^{3} = -\gamma^{3} \ , \tag{4}$$

where ^{t}A and $\tilde{A}$ note respectively the transposate and the conjugate--transposate of matrix A. We introduce the matrice γ^5:

$$\gamma^{5} = -i\,\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3} = \begin{pmatrix} 0 & \sigma^{0} \\ \sigma^{0} & 0 \end{pmatrix} , \tag{5}$$

which satisfies

$$^{t}\gamma^{5} = \tilde{\gamma}^{5} = \gamma^{5}, \quad \gamma^{5}\gamma^{5} = Id, \quad \gamma^{5}\gamma^{\mu} = -\gamma^{\mu}\gamma^{5} , \qquad 0 \leq \mu \leq 3 \ . \tag{6}$$

The quadratic Lorentz-invariant quantities (by respect to the spinorial representation of the Lorentz group) are:

$$\overline{\psi}\,\psi \quad \text{and} \quad \overline{\psi}\,\gamma^{5}\,\psi \ , \tag{7}$$

where $\overline{\psi}$ is the Dirac-conjugate of a spinor ψ:

$$\overline{\psi} = \tilde{\psi}\,\gamma^{0} \ . \tag{8}$$

We are concerned by the cubic Lorentz-invariant nonlinearities $f(\psi)$. Let $\mathcal{M}$ be the vector space of 4×4 matrices:

$$\mathcal{M} = \left\{ \alpha\, Id + i\,\beta\,\gamma^{5}, \ (\alpha,\beta) \in \mathbf{R}^{2} \right\} \ . \tag{9}$$

We assume that $f(\psi)$ satisfies:

$$f(\psi) = F(\overline{\psi}\,\psi, i\,\overline{\psi}\,\gamma^{5}\,\psi)\,\psi \ , \tag{10}$$

$$F \in C^{\infty}(\mathbf{R}^{2}, \mathcal{M}) \ , \tag{11}$$

$$|F(u,v)| = 0(|u| + |v|) , \qquad |u| + |v| \rightarrow 0 \ . \tag{12}$$

Many physical models verify conditions (10), (11), (12): Heisenberg equation, $F(u,v) = u \cdot Id$, pseudoscalar interaction, $F(u,v) = i\,v \cdot \gamma^{5}$, wave equation for a magnetic monopole of G. Lochak [14], $F(u,v) = u\, Id + i\, v\, \gamma^{5}$.

The global Cauchy problem associated to (1) for small initial data was solved by J.P. Dias and M. Figueira [6], [7]. In this paper we prove the existence of global solutions with arbitrarily large energy. Nevertheless we do an assumption of smallness but only on the Chiral invariant of the initial data $\rho^2 = |\psi\,\psi|^2 + |\psi\,\gamma^5\,\psi|^2$. This hypothesis means roughly that the Lochak-Majorana condition [15] is "near satisfied", i.e., $\psi - z\,\gamma^2\,\psi^+$ is small for some $z \in \mathbf{C}$, $|z| = 1$, and ψ^+ is the complex conjugate of ψ. That implies the relativistic quantities $\psi\,\psi$ and $\psi\,\gamma^5\,\psi$ are small and, in fact, we solve the global Cauchy problem for large data for which the nonlinearity $f(\psi)$ is nevertheless small. The very particular case $\rho \equiv 0$ was studied by J. Chadam and R. Glassey for the Dirac-Klein-Gordon system [4]; then, the system is decoupled and the Klein-Gordon equation is free. In [3] we have solved the global Cauchy problem for Yukawa mass models with small ρ. The fundamental point in the delicate study of these nonlinear hyperbolic systems is the algebraic properties of the nonlinearities which allow to obtain global small solutions [2]; to get large solutions we add algebraic condition on the polarization of initial data; Lochak-Majorana condition or smallness of the Chiral invariant that we study in part I. In part II we solve the global Cauchy problem for such a data. We study the asymptotic behaviour of the solution in part III: the solution is asymptotically free and a very remarkable consequence of the Lorentz invariance is the decay of relativistic quantities $\overline{\psi}\,\psi$ and $\overline{\psi}\,\gamma^5\,\psi$ which is better than for ordinary products: particularly $\overline{\psi}\,\psi$ and $\overline{\psi}\,\gamma^5\,\psi$ are uniformly $0(t^{-3})$ instead of $0(t^{-2})$ as usual and their integral on $\mathbf{R}^3_x$ is $0(t^{-1})$: it is the equipartition of energy [1]. These facts are well-known in the linear cases and are characteristic of the algebraic condition of compatibility between a sesquilinear form and a differential system [1], [8], [9], [10].

I – Chiral invariant and Lochak-Majorana Condition

To estimate a Lorentz-invariant non linearity $F(\psi\,\psi, i\,\psi\,\gamma^5\,\psi)$, we introduce, following G. Lochak [15], the Chiral invariant of ψ, $\rho(\psi)$:

$$\rho^2 = |\overline{\psi}\,\psi|^2 + |\overline{\psi}\,\gamma^5\,\psi|^2\ . \tag{13}$$

We are concerned by the spinors ψ for which the Chiral invariant is null. In the case of a free solution of the linear Dirac equation a necessary and sufficient condition to $\rho = 0$ is Lochak-Majorana condition:

$$\exists\, z \in \mathbf{C}, \quad |z| = 1, \qquad \psi = z\,\gamma^2\,\psi^+ , \tag{14}$$

where ψ^+ is the complex conjugate of ψ. We prove the same result for the Dirac system with a time dependent potential A

$$-i\,\gamma^\mu\,\partial_\mu\psi = A\,\psi , \tag{15}$$

where A satisfies

$$A, \partial_\mu A \in L^\infty_{\text{loc}}(\mathbf{R}_t, L^\infty(\mathbf{R}^3_x; \mathcal{M})) . \tag{16}$$

Proposition I.1. *Let ψ be a solution of (15) and*

$$\psi \in C^0(\mathbf{R}_t, (L^2(\mathbf{R}^3_x))^4) , \qquad \psi|_{t=0} = \psi_0 \in (L^2(\mathbf{R}^3_x))^4 .$$

Then the following assertions are equivalent:

(i) $\exists\, z \in \mathbf{C},\ |z| = 1,\ \psi_0 = z\,\gamma^2\,\psi_0^+$;

(ii) $\forall\, x \in \mathbf{R}^3,\ \rho(\psi_0(x)) = 0$;

(iii) $\forall\, (t,x) \in \mathbf{R}^{1+3},\ \rho(\psi(t,x)) = 0$.

Proof: We use the bispinorial representation by putting

$$\psi = \frac{1}{\sqrt{2}}(\gamma^0 + \gamma^5)\begin{pmatrix}\xi\\ \eta\end{pmatrix}, \qquad \xi, \eta \in \mathbf{C}^2 . \tag{17}$$

We verify easily that

$$\overline{\psi}\,\psi = \xi^+\cdot\eta + \eta^+\cdot\xi , \qquad \overline{\psi}\,\gamma^5\,\psi = \xi^+\cdot\eta - \eta^+\cdot\xi .$$

Therefore $\rho = 0$ if and only if

$$\xi^+\cdot\eta = 0 .$$

This condition is equivalent to

$$\exists\, z \in \mathbf{C}, \quad |z| = 1, \qquad \xi = z\,\sigma^2\,\eta^+ .$$

By using (17) we see that this means

$$\psi = z\,\gamma^2\,\psi^+$$

and we conclude that

$$\psi = z\,\gamma^2\,\psi^+ \quad \Longleftrightarrow \quad \rho(\psi) = 0\,. \tag{18}$$

Now it is obvious that it is sufficient to prove (ii)⇔(iii) for $\psi_0 \in (H^1(\mathbf{R}^3_x))^4$ with compact support. Equation (15) can be written

$$\partial_0\psi + \sum_{j=1}^{3}\gamma^0\,\gamma^j\,\partial_j\psi = i\,\gamma^0\,A\,\psi\,. \tag{19}$$

By multiplying (19) by $\tilde{\psi}$ we find

$$\partial_0|\psi|^2 + \sum_{j=1}^{3}\partial_j(\tilde{\psi}\,\gamma^0\,\gamma^j\,\psi) = 0\,. \tag{20}$$

We integrate (20) over $\mathbf{R}^3_x$ and we obtain the charge conservation:

$$\int_{\mathbf{R}^3}|\psi(t,x)|^2\,dx = \mathrm{Cst}\,. \tag{21}$$

Now we multiply (19) by ${}^{\mathrm{t}}\psi\,\gamma^2$ and it follows:

$$\partial_0({}^{\mathrm{t}}\psi\,\gamma^2\,\psi) + \sum_{j=1}^{3}\partial_j({}^{\mathrm{t}}\psi\,\gamma^2\,\gamma^0\,\gamma^j\,\psi) = 0\,.$$

We integrate over $\mathbf{R}^3_x$ again and we obtain the conservation law:

$$\int_{\mathbf{R}^3}{}^{\mathrm{t}}\psi(t,x)\,\gamma^2\,\psi(t,x)\,dx = \mathrm{Cst}\,. \tag{22}$$

Let z be a complex number of modulus one. We have

$$|\psi - z\,\gamma^2\,\psi^+|^2 = 2\,|\psi|^2 + 2\,\mathrm{Re}(\bar{z}\,{}^{\mathrm{t}}\psi\,\gamma^2\,\psi)\,.$$

Then we have thanks to (21) and (22):

$$\int_{\mathbf{R}^3}|\psi(t,x) - z\,\gamma^2\,\psi^+(t,x)|^2\,dx = \mathrm{Cst}\,. \tag{23}$$

By (18) and (23) we conclude that $\rho(\psi_0) = 0$ is equivalent to $\rho(\psi) \equiv 0$. ∎

II – Large global solutions

We consider the Cauchy problem:

$$\begin{cases} -i\,\gamma^\mu\,\partial_\mu\psi = F(\overline{\psi}\,\psi, i\,\overline{\psi}\,\gamma^5\,\psi)\,\psi\,, \\ \psi|_{t=0} = \Psi_0 + \varepsilon\,\chi_0\,, \end{cases} \tag{24}$$

where F satisfies (11) and (12) and the initial data $\psi|_{t=0}$ is in a neighbourhood of the cone defined by the Lochak-Majorana condition (14). For simplicity we choose test functions Ψ_0, χ_0, and Ψ_0 verifies (14):

$$\Psi_0, \chi_0 \in \mathcal{D}(\mathbf{R}^3_x, \mathbf{C}^4)\,, \qquad 0 \leq \varepsilon\,, \tag{25}$$

$$\exists\, z \in \mathbf{C}, \quad |z| = 1\,, \qquad \Psi_0 = z\,\gamma^2\,\Psi_0^+\,. \tag{26}$$

Note that Ψ_0 can be as large as we want.

Theorem II.1. *There exists $\varepsilon_0 > 0$ depending only on the derivatives of Ψ_0 and χ_0 of order ≤ 6, such that for $0 \leq \varepsilon \leq \varepsilon_0$, the Cauchy problem (24) has a unique solution $\psi \in C^\infty(\mathbf{R}^4)$.*

Remark 1. For simplicity and to use easily the powerful $L^2 - L^\infty$ estimates method, we consider only initial data that are test functions; an interesting open problem would be to relax this assumption and to solve the Cauchy problem (24) (26) under the only hypothesis of regularity used by J.P. Dias and M. Figueira [6, 7] for small data.

Remark 2. We could use the conformal method and the Penrose transform following Y. Choquet-Bruhat and D. Christodoulou [5]; but in fields theory it is very important to use invariance properties as weak as possible (e.g. in the case of mass and massless fields interacting [2]); here we need only the Lorentz invariance; particularly we don't use the scaling invariance to prove the global existence.

Remark 3. By using the scaling invariance we can relax the assumptions of regularity of initial data; in fact, ε_0 depends only of derivatives of Ψ_0, χ_0 of order ≤ 4.

Proof: We note $(\Gamma_\sigma)_{1\leq\sigma\leq 10}$ the generators of the Poincaré group:

$$(\Gamma_\sigma)_{1\leq\sigma\leq 10} = (\partial_\mu, x_\mu\,\partial_\nu - x_\nu\,\partial_\mu)_{0\leq\mu,\nu\leq 3}\,. \tag{27}$$

We define the Sobolev norms associated with the Lorentz metric: for any test function $u \in \mathcal{D}(\mathbf{R}_t \times \mathbf{R}^3_x)$ and for any integer N we put

$$\|u(t)\|_N^2 = \sum_{|\lambda| \leq N} \|\Gamma^\lambda u(t)\|^2_{L^2(\mathbf{R}^3_x)} , \tag{28}$$

$$|u(t)|_N = \sup_{|\lambda| \leq N} |\Gamma^\lambda u(t)|_{L^\infty(\mathbf{R}^3_x)} , \tag{29}$$

where

$$\lambda \in \mathbf{N}^{10} , \quad |\lambda| = \lambda_1 + \ldots + \lambda_{10}, \quad \Gamma^\lambda = \Gamma_1^{\lambda_1} \cdots \Gamma_{10}^{\lambda_{10}} .$$

To estimate a spinorial field we replace in (27), (28), (29) the operators $(\Gamma_\sigma)_{1 \leq \sigma \leq 10}$ by the Fermi operators

$$(\tilde{\Gamma}_\sigma)_{1 \leq \sigma \leq 10} = \left(\partial_\mu, x_\mu \, \partial_\nu - x_\nu \, \partial_\mu + \frac{1}{2} \, \gamma_\mu \, \gamma_\nu \right)_{0 \leq \mu, \, \nu \leq 3} .$$

The crucial property of these operators is the exact commutation with the Dirac system:

$$[\tilde{\Gamma}_\sigma, -i \, \gamma^\mu \, \partial_\mu] = 0 . \tag{30}$$

Obviously the norms defined by $(\tilde{\Gamma}_\sigma)_{1 \leq \sigma \leq 10}$ and (28) and (29) are equivalent.

Now let Ψ be the solution of

$$\begin{cases} -i \, \gamma^\mu \, \partial_\mu \, \Psi = 0 , \\ \Psi|_{t=0} = \Psi_0 . \end{cases} \tag{31}$$

Lochak-Majorana condition (26) and proposition I.1 imply

$$\overline{\Psi} \, \Psi = \overline{\Psi} \, \gamma^5 \, \Psi \equiv 0 . \tag{32}$$

We write

$$\psi = \Psi + \chi \tag{33}$$

and the Cauchy problem (24) becomes

$$\begin{cases} -i \, \gamma^\mu \, \partial_\mu \, \chi = f_1(\chi; \Psi) , \\ \chi|_{t=0} = \varepsilon \, \chi_0 , \end{cases} \tag{34}$$

where thanks to (32), (11), (12), f_1 satisfies

$$|f_1(\chi;\Psi)| = O(|\chi|\,(|\chi| + |\Psi|)^2)\,, \quad |\chi| \to 0\,. \tag{35}$$

We define the sequence χ^ν, $\nu \in \mathbf{N}$, by putting:

$$\chi^0 \equiv 0\,, \tag{36}$$

$$\begin{cases} -i\,\gamma^\mu\,\partial_\mu\,\chi^\nu = f_1(\chi^{\nu-1};\Psi), & \nu \geq 1\,, \\ \chi^\nu|_{t=0} = \varepsilon\,\chi_0, & \nu \geq 1\,. \end{cases} \tag{37}$$

The charge conservation and (35) imply

$$\|\chi^\nu(t)\|_N \leq C\left(\varepsilon + \left|\int_0^t \|\chi^{\nu-1}(s)\|_N \left(|\chi^{\nu-1}(s)|_{[\frac{N}{2}]} + |\Psi(s)|_N\right)^2 ds\right|\right)\,, \tag{38}$$

where $[N/2]$ is the integer part of $N/2$. We recall the $L^2 - L^\infty$ estimate of [11]

$$|u(t)|_k \leq C\,(1+|t|)^{-1} \sup_{s\in\mathbf{R}} \|u(s)\|_{k+3}\,. \tag{39}$$

Ψ being a free solution we have

$$(1+|t|)\,|\Psi(t)|_N \leq C \sup_{s\in\mathbf{R}} \|\Psi(s)\|_{N+3} < +\infty\,. \tag{40}$$

We introduce

$$a_n(t) = \sup_{\substack{|s|\leq|t| \\ 0\leq\nu\leq n}} \|\chi^\nu(s)\|_N\,, \qquad A_n = \sup_{t\in\mathbf{R}} a_n(t)\,. \tag{41}$$

We deduce from (38), (39), (40) that for $N \geq 6$

$$a_n(t) \leq C\left(\varepsilon + \left|\int_0^t a_{n-1}(s)\,(1+A_{n-1})^2\,(1+|s|)^{-2}\,ds\right|\right)\,. \tag{42}$$

Suppose $A_{n-1} < +\infty$; then (42) implies $a_n \in L^1_{\mathrm{loc}}(\mathbf{R}_t)$ and the sequence a_n being increasing we have

$$a_n(t) \leq C\left(\varepsilon + \left|\int_0^t a_n(s)\,(1+A_{n-1})^2\,(1+|s|)^{-2}\,ds\right|\right)\,.$$

Gronwall lemma gives

$$A_n \leq C\,\varepsilon \exp\{C\,A_{n-1}\,(1+A_{n-1})\}\,, \tag{43}$$

where $C > 0$ does not depend on n. Let $\varepsilon_0 > 0$ be such that

$$2\,C\,\varepsilon_0 < 1\ ,$$

$$4\,C^2\,\varepsilon_0 < \log 2\ ;$$

If A_{n-1} verifies

$$A_{n-1} \leq 2\,C\,\varepsilon_0\ , \tag{44}$$

(43) implies

$$A_n \leq C\,\varepsilon_0 \exp\{C\,(2\,C\,\varepsilon_0)\,2\} \leq 2\,C\,\varepsilon_0\ . \tag{45}$$

As (44) is trivial for A_0 we obtain

$$\sup_{t,\nu} \|\chi^\nu(t)\|_N = \sup_n A_n \leq 2\,C\,\varepsilon_0 < +\infty\ . \tag{46}$$

Now the Sobolev inequality (39), and (40) and (46) assure

$$\sup\{|\chi^\nu(t,x)|, |\Psi(t,x)|;\ t,x,\nu\} = r < +\infty\ . \tag{47}$$

We have at each (t,x)

$$\begin{aligned}\Big|f_1(\chi^{\nu-1}(t,x);\Psi(t,x)) - f_1(\chi^{\nu-2}(t,x);\Psi(t,x))\Big| &\leq \\ \leq |\chi^{\nu-1}(t,x) - \chi^{\nu-2}(t,x)| &\sup_{\substack{\xi,\eta\in\mathbf{C}^4\\ |\xi|,|\eta|\leq r}} |f_1'(\xi,\eta)|\ .\end{aligned}$$

The Dirac propagator being unitary on $(L^2(\mathbf{R}^3_x))^4$ we calculate that

$$\|\chi^\nu(t) - \chi^{\nu-1}(t)\|_{L^2(\mathbf{R}^3_x)} \leq C \int_0^t \|\chi^{\nu-1}(s) - \chi^{\nu-2}(s)\|_{L^2(\mathbf{R}^3_x)}\, ds\ .$$

This inequality implies

$$\|\chi^\nu(t) - \chi^{\nu-1}(t)\|_{L^2(\mathbf{R}^3_x)} \leq \varepsilon\, \|\chi_0\|_{L^2(\mathbf{R}^3_x)}\, \frac{(C\,t)^\nu}{\nu!}\ ,$$

therefore the sequence χ^ν is converging to some χ in $C^0(\mathbf{R},(L^2(\mathbf{R}^3_x))^4)$ and thanks to (47), $f_1(\chi^\nu;\Psi)$ is converging to $f_1(\chi;\Psi)$ in $C^0(\mathbf{R}_t,(L^2(\mathbf{R}^3_x))^4)$; finally, χ is solution of (34) and by (33) and (46), ψ is solution of (24) with

$$\sup_{t\in\mathbf{R}} \|\psi(t)\|_N < +\infty\ , \tag{48}$$

for some N, choosen ≥ 6, and particularly we have

$$\sup_{t,x} |\psi(t,x)| < +\infty \ .$$

Then thanks to Gagliardo-Nirenberg inequality, for any integer k, we have

$$\|F(\bar\psi\,\psi, i\,\bar\psi\gamma^5\,\psi)\,\psi(t)\|_{H^k(\mathbf{R}^3_x)} \leq C_k\,\|\psi(t)\|_{H^k(\mathbf{R}^3_x)} \ . \tag{49}$$

The local Cauchy problem for (24) is well posed $C^0(0,T_k;(H^k(\mathbf{R}^3_x))^4)$ for some $0 < T_k$ and the Dirac propagator is unitary on Sobolev space; then we obtain with (49):

$$0 \leq t < T_k\,, \quad \|\psi(t)\|_{H^k(\mathbf{R}^3_x)} \leq C'_k + \int_0^t C_k\,\|\psi(s)\|_{H^k(\mathbf{R}^3_x)}\,ds \ .$$

We conclude by Gronwall lemma that $T_k = +\infty$ and $\psi(t) \in C^\infty(\mathbf{R}^3_x)$; by using the equation we have also $\psi \in C^\infty(\mathbf{R}^4)$. ∎

Now we justify remark 3. We add the radiation operator $\Gamma_0 = x^\mu\,\partial_\mu$ to family (27):

$$(\Gamma_\sigma)_{0\leq\sigma\leq 10} = (\partial_\mu, x_\mu\,\partial_\nu - x_\nu\,\partial_\mu, x^\alpha\,\partial_\alpha)_{0\leq\mu,\,\nu\leq 3} \ . \tag{27 bis}$$

We introduce the norms $\|\ \|_N^*$, $|\ |_N^*$ instead of $\|\ \|_N$, $|\ |_N$:

$$\|u(t)\|_N^{*\,2} = \sum_{|\lambda|\leq N} \|\Gamma^\lambda\,u(t)\|^2_{L^2(\mathbf{R}^3_x)} \ , \tag{28 bis}$$

$$|u(t)|_N^* = \sum_{|\lambda|\leq N} |\Gamma^\lambda\,u(t)|_{L^\infty(\mathbf{R}^3_x)} \ , \tag{29 bis}$$

where

$$\Gamma^\lambda = \Gamma_0^{\lambda_0}\cdots\Gamma_{10}^{\lambda_{10}}\,, \quad \lambda \in \mathbf{N}^{11},\ \ \Gamma_0 = x^\mu\,\partial_\mu \ .$$

We add so $\widetilde\Gamma_0 = \Gamma_0$ to family $\widetilde\Gamma_\sigma$ and the scaling invariance means:

$$[\widetilde\Gamma_0, -i\,\gamma^\mu\,\partial_\mu] = i\,\gamma^\mu\,\partial_\mu \ . \tag{30 bis}$$

Following [13] we replace Sobolev inequality (39) by

$$|u(t)|_k^* \leq C\,(1+|t|)^{-1}\,\|u(t)\|_{k+2}^* \ . \tag{39 bis}$$

Thanks to (30bis) we can prove as above

$$\sup_{t\in\mathbf{R}} \|\chi(t)\|_N^* < 2\,C\,\varepsilon_0 \,, \tag{46 bis}$$

$$\sup_{t\in\mathbf{R}} \|\psi(t)\|_N^* < +\infty \,, \tag{48 bis}$$

for some N choosen ≥ 4 and $\varepsilon_0 > 0$ small enough.

III – Asymptotic behaviour – Equipartition of energy

The rate of the uniform decay of ψ and the order of the nonlinearity are large enough to assure the time integrability of the energy of the second member of (1) and the solution is asymptotically free:

Theorem III.1. *Let ψ be the solution of (24) given by theorem II.1. Then there exists $\psi^{\pm}$ satisfying*

$$\psi^{\pm} \in \bigcap_k C^k(\mathbf{R}_t, (H^{6-k}(\mathbf{R}_x^3))^4)\,, \qquad -i\,\gamma^{\mu}\,\partial_{\mu}\,\psi^{\pm} = 0\,,$$

$$\lim_{t\to\pm\infty} \|\partial_t^k\psi(t) - \partial_t^k\psi^{\pm}(t)\|_{H^{6-k}(\mathbf{R}_x^3)} = 0\,, \quad \forall\, k \in \mathbf{N}\,.$$

So we are interested by comparing the ponctual decay of ψ with the ponctual decay of free solutions. We know by (39) and (48) that

$$|\psi(t)|_{L^{\infty}(\mathbf{R}_x^3)} = 0(|t|^{-1})\,, \tag{50}$$

like free solution, but inside the light cone, the decay is better:

Theorem III.2. *Let $0 \leq C < 1$ be; then ψ satisfies for $\varepsilon_0 > 0$ small enough:*

$$|\psi(t)|_{L^{\infty}(\{|x|\leq C\,|t|\})} = 0(|t|^{-2})\,. \tag{51}$$

Now we want point out the remarkable properties of asymptotic behaviour of relativistic quantities $\overline{\psi}\,\psi$ and $\overline{\psi}\,\gamma^5\,\psi$.

Let ψ^0 be a regular wave packet free solution of:

$$-i\,\gamma^{\mu}\,\partial_{\mu}\,\psi^0 = 0\,.$$

Obviously we have

$$\|\psi^0(t)\|_{L^2(\mathbf{R}^3_x)} = \mathrm{Cst}\ ,$$

$$|\psi^0(t)|_{L^\infty(\mathbf{R}^3_x)} = 0(|t|^{-1})\ .$$

Therefore, for any sesquilinear form Q on $\mathbf{C}^4$ we have

$$\sup_{t\in\mathbf{R}}\left|\int_{\mathbf{R}^3} Q(\psi^0(t,x),\psi^0(t,x))\,dx\right| < +\infty\ ,$$

$$|Q(\psi^0(t),\psi^0(t))|_{L^\infty(\mathbf{R}^3_x)} = 0(|t|^{-2})\ .$$

But we know that

$$\left|\int_{\mathbf{R}^3} \overline{\psi}^0(t,x)\,\psi^0(t,x)\,dx\right| + \left|\int_{\mathbf{R}^3} \overline{\psi}^0(t,x)\,\gamma^5\,\psi^0(t,x)\,dx\right| = \\ = 0(|t|^{-1})\ ,\qquad |t|\to\infty\ , \tag{52}$$

and

$$|\overline{\psi}^0(t)\,\psi^0(t)|_{L^\infty(\mathbf{R}^3_x)} + |\overline{\psi}^0(t)\,\gamma^5\,\psi^0(t)|_{L^\infty(\mathbf{R}^3_x)} = 0(|t|^{-3})\ . \tag{53}$$

In fact these properties of equipartition of energy are characteristic of the Lorentz invariance of sesquilinear forms, which is equivalent to the "compatibility" of sesquilinear forms with the Dirac system [1], [2], [9]. Here we prove that the solution of the relativistic nonlinear Dirac equation satisfies also (53) and a stronger form of (52):

Theorem III.3. *The solution ψ of (24) given by theorem II.1 verifies as $|t|\to\infty$:*

$$|\overline{\psi}(t)\,\psi(t)|_{L^1(\mathbf{R}^3_x)} + |\overline{\psi}(t)\,\gamma^5\,\psi(t)|_{L^1(\mathbf{R}^3_x)} = 0(|t|^{-1})\ , \tag{54}$$

$$|\overline{\psi}(t)\,\psi(t)|_{L^\infty(\mathbf{R}^3_x)} + |\overline{\psi}(t)\,\gamma^5\,\psi(t)|_{L^\infty(\mathbf{R}^3_x)} = 0(|t|^{-3})\ . \tag{55}$$

Note that (54) is much stronger form of equipartition of energy than the usual result for general hyperbolic systems [1] of type (52), i.e., the modulus outside the integral. An analogous result is known, [12], for the partition of energy of free solution of wave equation, $\Box\, u = 0$, $u|_{t=0}, \partial_t u|_{t=0} \in \mathcal{D}(\mathbf{R}^3_x)$:

$$\|\partial_\mu u(t)\,\partial^\mu u(t)\|_{L^1(\mathbf{R}^3_x)} = 0(|t|^{-1})\ .$$

There are two arguments:

i) a factorization

$$\partial_\mu u\, \partial^\mu u(t) = t^{-1}\left\{(x^\mu\, \partial_\mu u)\, \partial_0 u - (x^\mu\, \partial_0 u - x^0\, \partial^\mu u)\, \partial_\mu u\right\};$$

ii) the fact that the L^2-norms of factors $x^\mu\, \partial_\mu u$, $\partial_0 u$, $x^\mu\, \partial_0 u - x^0\, \partial^\mu u$, $\partial_\mu u$ are bounded because these terms are free solutions of wave equation with test functions initial data. An analogous factorization of $\bar\psi\, \psi$ and $\bar\psi\, \gamma^5\, \psi$ is given by proposition II.2 following with a certain factor ϕ estimated in proposition III.1: We associate to spinor ψ a new spinor ϕ defined by

$$\phi = x_\mu\, \gamma^\mu\, \psi\ . \tag{56}$$

Proposition III.1. *Let ψ be the solution given by theorem II.1 for $\varepsilon_0 > 0$ small enough. Then ϕ satisfies*

$$\sup_{t\in\mathbf{R}} \|\phi(t)\|_4^* < +\infty\ , \tag{57}$$

where the norm $\|\ \|_4^*$ is defined by (28bis).

The next proposition is a special case of the theorem of characterization of compatibility of B. Hanouzet and J.L. Joly [8] by factorization of relativistic quantities by the symbol of Dirac operator.

Proposition III.2. *Let P be the 4×4 complex matrix homogeneous of 1-order defined by:*

$$P = (t^2 + |x|^2)^{-1}\, x_\mu\, \gamma^\mu\, \gamma^0\ . \tag{58}$$

Then we have for any spinors ψ_j

$$\bar\psi_1\, \psi_2 = \tilde\phi_1\, P\, \psi_2 + \tilde\psi_1\, \tilde P\, \phi_2\ , \tag{59}$$

$$\bar\psi_1\, \gamma^5\, \psi_2 = \tilde\phi_1\, \gamma^5\, P\, \psi_2 + \tilde\psi_1\, \tilde P\, \gamma^5\, \phi_2\ , \tag{60}$$

where ϕ_j is given by (56) with ψ_j.

Proof of theorem III.1: Let $D(t)$ the Dirac propagator:

$$D(t) = \exp i\, t\, A\ , \qquad A = \sum_{j=1}^{3} i\, \gamma^0\, \gamma^j\, \partial_j\ .$$

To obtain $\psi^{\pm}$ is is sufficient to prove the convergence of $D(-t)\,\psi(t)$ in $(H^6(\mathbf{R}^3_x))^4$ as $t \to \pm\infty$. Now we have

$$D(-t)\,\psi(t) = \psi(0) + \int_0^t D(-s)\, f(\psi(s))\, ds \ .$$

Following (48) $f(\psi(s))$ satisfies

$$\|f(\psi(s))\|_{H^6(\mathbf{R}^3_x)} \le C\,(1+|s|)^{-2} \in L^1(\mathbf{R}_s) \ ,$$

therefore we conclude $D(-t)\,\psi(t)$ is converging in $(H^6(\mathbf{R}^3_x))^4$. ∎

Proof of proposition III.1: We note that $(i\,\gamma^\mu\,\partial_\mu)\,(-i\,\gamma^\nu\,\partial_\nu) = -\,\Box$, so we have:

$$\begin{aligned}\Box\,\psi &= i\,\gamma^\mu\,\partial_\mu(F(\overline{\psi}\,\psi, i\,\overline{\psi}\gamma^5\,\psi))\,\psi \\ &\quad + i\,\gamma^\mu\,F(\overline{\psi}\,\psi, i\,\overline{\psi}\gamma^5\,\psi)\,\partial_\mu\psi \ ,\end{aligned} \tag{61}$$

$$\begin{aligned}\Box\,\phi &= i\,x_\nu\,\gamma^\nu\,\gamma^\mu\,\partial_\mu(F(\overline{\psi}\,\psi, i\,\overline{\psi}\gamma^5\,\psi))\,\psi \\ &\quad + i\,x_\nu\,\gamma^\nu\,\gamma^\mu\,F(\overline{\psi}\,\psi, i\,\overline{\psi}\gamma^5\,\psi)\,\partial_\mu\psi \\ &\quad + 2\,i\,F(\overline{\psi}\,\psi, i\,\overline{\psi}\gamma^5\,\psi)\,\psi \ .\end{aligned} \tag{62}$$

Following decomposition (33) we write

$$\phi = \underline{\Psi} + \varphi \ , \tag{63}$$

where

$$\underline{\Psi} = x_\mu\,\gamma^\mu\,\Psi \ , \qquad \varphi = x_\mu\,\gamma^\mu\,\chi \ , \tag{64}$$

satisfy

$$\Box\,\underline{\Psi} = 0 \ , \qquad \Box\,\varphi = \Box\,\phi \ . \tag{65}$$

We note that thanks to (6), (9) and (11)

$$\begin{aligned}&x_\nu\,\gamma^\nu\,\gamma^\mu\,\partial_\mu\,(F(\overline{\psi}\,\psi, i\,\overline{\psi}\gamma^5\,\psi))\,\psi = \\ &= (x_\nu\,\partial_\mu - x_\mu\,\partial_\nu)\,(F(\overline{\psi}\,\psi, i\,\overline{\psi}\gamma^5\psi))\,\gamma^\nu\,\gamma^\mu\,\psi + \partial_\nu\,(F(\overline{\psi}\,\psi, i\,\overline{\psi}\gamma^5\psi))\,\gamma^\nu\phi\end{aligned} \tag{66}$$

and

$$\begin{aligned}&i\,x_\nu\,\gamma^\nu\,\gamma^\mu\,F(\overline{\psi}\,\psi, i\,\overline{\psi}\gamma^5\,\psi)\,\partial_\mu\,\psi = \\ &= -x_\nu\,F(\overline{\psi}\,\psi, i\,\overline{\psi}\gamma^5\,\psi)\,\gamma^\nu\,F(\overline{\psi}\,\psi, i\,\overline{\psi}\gamma^5\,\psi)\,\psi \ .\end{aligned} \tag{67}$$

Now we recall the energy estimate of [12] for $0 \leq t$

$$\|\varphi(t)\|_4^* \leq C \left\{ \varepsilon + \int_0^t (1+s) \, \|(\Box \varphi)(s)\|_3^* \, ds \right\} . \tag{68}$$

We obtain with (62), (65), (66), (67),

$$\|(\Box \varphi)(s)\|_3^* \leq \tag{69}$$

$$\leq C \Big\{ \|F(s)\|_4^* \, |\psi(s)|_2^* + |F(s)|_2^* \, \|\psi(s)\|_4^* + \|F(s)\|_4^* \, |\phi(s)|_2^* + |F(s)|_2^* \, \|\phi(s)\|_4^*$$
$$+ (1+s) \, |F(s)|_2^* \left(\|F(s)\|_4^* \, |\psi(s)|_2^* + |F(s)|_2^* \, \|\psi(s)\|_4^* \right) \Big\} ,$$

where we have put for simplicity

$$F(s) = F(\overline{\psi} \, \psi, i \, \overline{\psi} \gamma^5 \, \psi)(s) . \tag{70}$$

But (32) gives

$$F(s) = F\Big(\overline{\Psi} \chi + \overline{\chi} \Psi + \overline{\chi} \chi, i \, (\overline{\Psi} \gamma^5 \chi + \overline{\chi} \gamma^5 \Psi + \overline{\chi} \gamma^5 \chi) \Big)(s) . \tag{71}$$

Now we use proposition III.2 and (71) to obtain for $0 \leq s$

$$\|F(s)\|_4^* \leq C \, (1+s)^{-1} \Big\{ \|\varphi(s)\|_4^* \left(|\chi(s)|_2^* + |\Psi(s)|_2^* \right) \tag{72}$$
$$+ |\varphi(s)|_2^* \, \|\chi(s)\|_4^* + |\chi(s)|_2^* \left(\|\Psi(s)\|_4^* + \|\underline{\Psi}(s)\|_4^* + \|\chi(s)\|_4^* \right) \Big\}$$

$$\|F(s)\|_2^* \leq C \, (1+s)^{-1} \Big\{ |\varphi(s)|_2^* \left(|\chi(s)|_2^* + |\Psi(s)|_2^* \right) \tag{73}$$
$$+ |\chi(s)|_2^* \left(|\Psi(s)|_2^* + |\underline{\Psi}(s)|_2^* + |\chi(s)|_2^* \right) \Big\}$$

We recall that Ψ and $\underline{\Psi}$ are free solutions of wave equation with test functions initial data and then

$$\sup_{s \in \mathbf{R}} \left(\|\Psi(s)\|_4^* + \|\underline{\Psi}(s)\|_4^* \right) < +\infty . \tag{74}$$

We have so by (46bis) and (48bis)

$$\sup_{s \in \mathbf{R}} \|\chi(s)\|_4^* \leq 2 \, C \, \varepsilon_0 , \tag{75}$$

$$\sup_{s \in \mathbf{R}} \|\psi(s)\|_4^* < +\infty , \tag{76}$$

and by (39)

$$\sup_{s\in\mathbf{R}}(1+|s|)\,|\chi(s)|_2^* \leq 2\,C'\,\varepsilon_0\ , \tag{77}$$

$$\sup_{s\in\mathbf{R}}\Big\{(1+|s|)\,\Big(|\psi(s)|_2^* + |\Psi(s)|_2^* + |\underline{\Psi}(s)|_2^*\Big)\Big\} < +\infty\ . \tag{78}$$

On the other hand Sobolev type inequality (39bis) gives

$$|\varphi(s)|_2^* \leq C\,(1+s)^{-1}\,\|\varphi(s)\|_4^*\ . \tag{79}$$

We conclude from inequalities (69) to (79) that

$$\|(\Box\,\varphi)(s)\|_3^* \leq C\,(1+s)^{-3}\,[B(s)\,(1+B(s))+\varepsilon_0]\ , \tag{80}$$

where

$$B(s) = \sup_{0\leq\sigma\leq s}\ \|\varphi(\sigma)\|_4^*\ . \tag{81}$$

Now (68) and (81) imply for $0 \leq t$

$$B(t) \leq C\,\Big\{\varepsilon_0 + \int_0^t (1+s)^{-2}\,B(s)\,(1+B(s))\,ds\Big\} \tag{82}$$

and by using Gronwall lemma:

$$B(t) \leq C_0\,\varepsilon_0 \exp C_0\,B(t)\ , \tag{83}$$

where $C_0 > 0$ does not depend on t. Let $0 < T_M$ be such that

$$T_M = \sup\Big\{T;\ 0 \leq t \leq T,\ \ B(t) \leq 2\,C_0\,\varepsilon_0\Big\}\ .$$

We choose $\varepsilon_0 > 0$ such that

$$2\,C_0^2\,\varepsilon_0 < \log 2\ . \tag{84}$$

Suppose $T_M < +\infty$. $B(t)$ being a continuous increasing function of t, $B(T_M)$ satisfies

$$B(T_M) \leq C_0\,\varepsilon_0 \exp C_0\,B(T_M)\ , \qquad B(T_M) \leq 2\,C_0\,\varepsilon_0\ . \tag{85}$$

Thus (84) and (85) imply

$$B(T_M) \leq C_0\,\varepsilon_0 \exp 2\,C_0^2\,\varepsilon_0 < 2\,C_0\,\varepsilon_0\ . \tag{86}$$

Again by continuity of $B(t)$ we conclude from (86) that

$$B(t) \leq 2\,C_0\,\varepsilon_0 \quad \text{for} \;\; 0 \leq t \leq T_M + \eta \; ,$$

for some $0 < \eta$ small enough; it is a contradiction with the definition of T_M and we conclude that $T_M = +\infty$ and

$$\sup_{t \in \mathbf{R}} \|\varphi(s)\|_4^* < +\infty \tag{87}$$

and with (74)

$$\sup_{t \in \mathbf{R}} \|\phi(s)\|_4^* < +\infty \; . \; \blacksquare \tag{88}$$

Proof of theorem III.2. Relations (2) give

$$(t^2 - |x|^2)\,\psi(t,x) = (x_\mu\,\gamma^\mu\,\phi)(t,x) \; , \tag{89}$$

where as usual

$$t = x^0, \quad x = x^1, x^2, x^3, \qquad |x|^2 = |x^1|^2 + |x^2|^2 + |x^3|^2 \; .$$

We remark that

$$\operatorname{supp} \phi \subset \Big\{ |x| \leq |t| + R \Big\}$$

and thus

$$\sup_{t,x} |x_\mu\,\gamma^\mu\,\phi(t,x)| \leq C \sup_t (1 + |t|)\,|\phi(t)|_{L^\infty(\mathbf{R}^3_x)} \; \cdot \tag{90}$$

Now (39), (57), (89) and (90) imply

$$\sup_{t,x} |(t^2 - |x|^2)\,\psi(t,x)| < +\infty \tag{91}$$

that assures (51). $\blacksquare$

Proof of theorem III.3. Proposition III.2 implies

$$|\bar\psi\,\psi(t,x)| + |\bar\psi\,\gamma^5\,\psi(t,x)| \leq C\,|t|^{-1}\,|\phi(t,x)|\,|\psi(t,x)| \; . \tag{92}$$

Now (39) and (48) give

$$\sup_t \Big\{ \|\psi(t)\|_{L^2(\mathbf{R}^3_x)} + (1 + |t|)\,|\psi(t)|_{L^\infty(\mathbf{R}^3_x)} \Big\} < +\infty \; , \tag{93}$$

and (39) and (57) give

$$\sup_t \left\{ \|\phi(t)\|_{L^2(\mathbf{R}^3_x)} + (1+|t|)\,|\phi(t)|_{L^\infty(\mathbf{R}^3_x)} \right\} < +\infty \,. \tag{94}$$

At present (54) and (55) are immediate consequences of (92), (93), (94). ∎

Proof of proposition III.2. Thanks to (2) and (3) we have

$$\widetilde{\psi}_1(x_\mu \widetilde{\gamma}^\mu x_\nu \gamma^\nu \gamma^0 + x_\mu \widetilde{\gamma}^0 \widetilde{\gamma}^\mu x_\nu \gamma^\nu)\,\psi_2 = (t^2+|x|^2)\,\overline{\psi}_1\,\psi_2 \ ,$$

and with (6)

$$\widetilde{\psi}_1(x_\mu \widetilde{\gamma}^\mu \gamma^5 x_\nu \gamma^\nu \gamma^0 + x_\mu \widetilde{\gamma}^0 \widetilde{\gamma}^\mu \gamma^5 x_\nu \gamma^\nu)\,\psi_2 = (t^2+|x|^2)\,\overline{\psi}_1\,\gamma^5\,\psi_2 \ ,$$

that prove (59) and (60). ∎

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