# Coupling of a Fast Multipole Method and a Microlocal Discretization for Integral Equations of Electromagnetism

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## 1 Introduction

A numerical solution of the integral equations for the 3-D exterior problem of electromagnetism, leads to the solution of dense linear systems. Those systems have generally a bad conditionement and a size that strongly increases with the frequency. We then consider the Després's Integral Equations to have a well conditioned system ([5], [8]). If $N_{\text{iter}}$ is the number of iterations, the classical complexity of the iterative solution is $\mathcal{O}(N_{\text{iter}}\,\kappa^4)$, where $\kappa$ is the wave number. In order to speed up the solution of the system, we have considered the coupling of two methods, the microlocal discretization method and the fast multipole method (FMM). The microlocal discretization method according to Abboud *et al.* [1], enables one to consider new systems whose size is $\mathcal{O}(\kappa^{2/3} \times \kappa^{2/3})$ instead of $\mathcal{O}(\kappa^2 \times \kappa^2)$ thanks to a coarse discretization of the unknown for convex geometries. However, due to the geometrical approximation of the surface, the fine mesh of a classical solution is still considered. An other method, the fast multipole method ([6], [7], [4]), is one of the most efficient and robust methods used to speed up iterative solutions. Using a multilevel algorithm, it leads to the cost $\mathcal{O}(N_{\text{iter}}\,\kappa^2 \ln \kappa^2)$ instead of $\mathcal{O}(N_{\text{iter}}\,\kappa^4)$. In this paper, the coupling of both methods, using a multilevel algorithm, enables one to reduce the CPU time efficiently for large wave numbers, with the complexity $\mathcal{O}(\kappa^{8/3} \ln \kappa^2 + N_{\text{iter}}\,\kappa^{4/3})$ (see also [2]). The numerical results show that the new method is in certain cases more efficient than a classical use of the multilevel FMM. Such a coupling has been proposed for Helmholtz's equation using a one-level FMM ([3]).

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## 2 Després's integral equations

B. Després suggested new integral formulations for 3-D Maxwell's equations and Helmholtz's equation. The EID (Després's Integral Equations) lead to new systems with such properties that one can use an efficient iterative solution based on the conjugate gradient. The Després's Integral Equations are derived from the separation of the real part and the imaginary part of the Green function and from the introduction of a new unknown. For Maxwell's equations, using a Raviart-Thomas finite element discretization, after discretization of the integrals on the surface, the systems introduce the following discrete matrices, $i, j \in \{1, .., N\}$

$$\mathcal{M}_{ij}^{\Re} = \overline{\alpha_i}\, \alpha_j \mathcal{G}_r(\boldsymbol{x}_i, \boldsymbol{x}_j)\ ,\quad \mathcal{M}_{ij}^{\Im} = \overline{\beta_i}\, \beta_j \mathcal{G}_i(\boldsymbol{x}_i, \boldsymbol{x}_j)\ ,\quad \alpha_i,\, \alpha_j,\, \beta_i,\, \beta_j \in \mathbb{C}\ , \quad (1)$$

where $\mathcal{G}$ depends on the Green function and its derivates. The different matrices may be treated on the same way. Then, we henceforth consider $\mathcal{G}_r(\boldsymbol{x}, \boldsymbol{y}) = G_r(\boldsymbol{x}, \boldsymbol{y}) = (\cos(\kappa|\boldsymbol{x} - \boldsymbol{y}|))/(4\pi|\boldsymbol{x} - \boldsymbol{y}|)$ and $\mathcal{G}_i(\boldsymbol{x}, \boldsymbol{y}) = G_i(\boldsymbol{x}, \boldsymbol{y}) = (\sin(\kappa|\boldsymbol{x} - \boldsymbol{y}|))/(4\pi|\boldsymbol{x} - \boldsymbol{y}|)$. The system to solve which can be obtained with a minimization problem with linear constraints ([5]), is a Stokes-like system.

## 3 Coupling of the microlocal discretization and the fast multipole method

The first step consists in the consideration of the approximation of the phase function leading to the microlocal discretization. We assume that the obstacle $\Omega$ is a bounded open *convex* domain in $\mathbb{R}^3$ of boundary $\Gamma$ and $u^{\text{inc}}$ is an incident *plane* wave. The previous matrices are given by a classical discretization, for a classical mesh $\Gamma_f$ with $N = N_f$ elements, $N_f \sim \kappa^2$. Then, denoting by $\phi_0$ the first degree approximation of the phase function of the unknown $q$, we consider a new unknown $\widetilde{q}$ such as $q = \widetilde{q}\mathrm{e}^{\mathrm{i}\kappa\phi_0}$. Due to the sharp error estimates on $\widetilde{q}$, the change of unknown enables one to define it on a coarse mesh $\Gamma_c$, with a number of elements $N_c$ equal to $\mathcal{O}(\kappa^{2/3})$ instead of $\mathcal{O}(\kappa^2)$. However, because of the geometrical approximation of the boundary $\Gamma$, we have also to consider the fine mesh $\Gamma_f$ with a number of elements $N_f = \mathcal{O}(\kappa^2)$, to evaluate the integrals. Those considerations give a new system to solve. Considering $\Gamma_f$ as a refinement of $\Gamma_c$, we can consider the elements of $\Gamma_c$ as groups of elements of $\Gamma_f$. Let $\mathcal{N}_i$ be the number of elements of $\Gamma_f$ included in the $i^{\text{th}}$ one of $\Gamma_c$. So, the $(i_0)^{\text{th}}$ element of $\Gamma_f$ included in the $i^{\text{th}}$ one of $\Gamma_c$ will be denoted $T_{i\,i_0}$. Let $\pi$ be the orthogonal projection from the plane triangles of $\Gamma_c$ to the ones of $\Gamma_f$. Let $\varphi_i$ be the vectorial basis function associated to the $i^{\text{th}}$ edge of the coarse mesh $\Gamma_c$. It is defined on the coarse mesh and the discretization of the integrals on the surface needs to be defined on the fine mesh $\Gamma_f$. Moreover, the basis functions need to be defined by vectors that are tangential to $\Gamma_f$. Then, let $\varphi_i^{\pi}$ be the projection of $\varphi_i$ on the space

of vectors that are tangential to $\Gamma_f$. $\boldsymbol{x}_{i\,i_0}$ denoting the quadrature point on $T_{i\,i_0}$, using the test-functions $\mathrm{e}^{\mathrm{i}\kappa\phi_0}\varphi_i^\pi$, the new discrete matrix $\mathcal{M}^{\Re}$ has the following form

$$\mathcal{M}_{ij}^{\Re} = \sum_{i_0=1}^{\mathcal{N}_i}\sum_{j_0=1}^{\mathcal{N}_j} \alpha_{j\,j_0}\,\mathrm{e}^{\mathrm{i}\kappa\phi_0(\boldsymbol{x}_{j\,j_0})}\,\overline{\alpha_{i\,i_0}}\,\overline{\mathrm{e}^{\mathrm{i}\kappa\phi_0(\boldsymbol{x}_{i\,i_0})}} \cdot\varphi_j^\pi(\pi^{-1}(\boldsymbol{x}_{j\,j_0}))\cdot\overline{\varphi_i^\pi(\pi^{-1}(\boldsymbol{x}_{i\,i_0}))}\,G_r(\boldsymbol{x}_{i\,i_0},\boldsymbol{x}_{j\,j_0})\,. \tag{2}$$

Hence, we obtain a problem whose size is $\mathcal{O}(\kappa^{4/3})$ instead of $\mathcal{O}(\kappa^4)$. Thus, the gain about the memory cost is very interesting. However, due to the construction of the matrices, with the consideration of the fine mesh $\Gamma_f$, the calculation cost is still $\mathcal{O}(\kappa^4)$. To reduce that cost, the authors T. Abboud, J.-C. Nédélec and B. Zhou, suggested the use of the theory of the stationary phase but the numerical approach of that theory implies difficulties not yet solved in 3-D. We suggest then the use of the multilevel FMM (MLFMM) to speed up the calculation of the matrices without the theory of the stationary phase. The FMM is a robust method that speeds up the calculation of the matrix-vector products of iterative solutions. The method is based on two developments. Through clusters of elements, an uncoupling between two points of the boundary $\Gamma_f$ is obtained using the Gegenbauer series and an integral around the unit sphere $S^2$. Let $y_m$, $j_m$ be the spherical Neumann and Bessel functions and $P_m$ the Legendre polynomial. Let $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ in $\Gamma_f$; $\boldsymbol{O}_1$, $\boldsymbol{O}_2$ the centers of the two clusters $C_1$ and $C_2$ containing respectively $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Then, $\boldsymbol{x}_1-\boldsymbol{x}_2=\boldsymbol{r}_0+\boldsymbol{r}$ where $\boldsymbol{r}_0=\boldsymbol{O}_1-\boldsymbol{O}_2$, $\boldsymbol{r}=\boldsymbol{r}_1-\boldsymbol{r}_2$ and $\boldsymbol{r}_i=\boldsymbol{x}_i-\boldsymbol{O}_i$. With $|\boldsymbol{r}_0|>|\boldsymbol{r}|$, the multipole approximation for the real part of the integral kernel of the EID is given by

$$\frac{\cos(\kappa|\boldsymbol{x}_1-\boldsymbol{x}_2|)}{4\pi|\boldsymbol{x}_1-\boldsymbol{x}_2|} \approx \frac{-\kappa}{(4\pi)^2}\sum_{p=1}^{P}\omega_p\,\mathrm{e}^{\mathrm{i}\kappa<\boldsymbol{s}_p,\boldsymbol{r}_1>}\,T_{l,\boldsymbol{r}_0}^{\Re}(\boldsymbol{s}_p)\,\mathrm{e}^{-\mathrm{i}\kappa<\boldsymbol{s}_p,\boldsymbol{r}_2>}$$
$$\text{with } T_{l,\boldsymbol{r}_0}^{\Re}(\boldsymbol{s}) = \sum_{m=0}^{l}(2m+1)\mathrm{i}^m\,y_m(\kappa|\boldsymbol{r}_0|)\,P_m(\cos(\boldsymbol{s},\boldsymbol{r}_o))\,. \tag{3}$$

where $l\approx\kappa d+c(\kappa d)^{1/3}$, $d$ is the diameter of the multipole boxes, $c$ depends only on the desired accuracy and $P=(2l+1)(l+1)$ is the number of quadrature points on the unit sphere. The MLFMM needs also the consideration of an octree. It implies several levels of clusters $C$. A classical MLFMM uses all the levels of the octree, from level three to the fine level. For our use, we will see that we should not consider levels that are coarser than the level $lev_c$ whose the number of clusters is $\mathcal{O}(N_c)$. Let $\overline{V}(C)$ be defined for each cluster $C$ of each level $lev$ by

$$\overline{V}(C) = \left\{\widetilde{C}/\widetilde{C} \text{ far from } C \text{ and } \widetilde{C}_a \text{ close to } C_a\right\} \quad \text{if } lev > lev_c$$
$$\text{and } \overline{V}(C) = \left\{\widetilde{C}/\widetilde{C} \text{ far from } C\right\} \quad \text{if } lev = lev_c$$

where $C_a$ is the antecedent of $C$ in the octree. Let $T_i^\pi$ be the set $\{\boldsymbol{x}_{i\,i_0}, i_0 = 1, ..., \mathcal{N}_i\}$. If we consider the matrix $\mathcal{M}^{\Re}$ given by (2), the multipole approximation of the far part is given by the following algorithm

• Step 1: Transfert functions: For each considered level,
$\forall(\boldsymbol{O}_C - \boldsymbol{O}_{\widetilde{C}})$, $\widetilde{C} \in \overline{V}(C)$, $\forall p \in \{1, ..., P\}$

$$T^{\Re}_{l,\boldsymbol{O}_C-\boldsymbol{O}_{\widetilde{C}}}(\boldsymbol{s}_p) = \sum_{m=0}^{l} (2m+1)\mathrm{i}^m \; y_m(\kappa|\boldsymbol{O}_C - \boldsymbol{O}_{\widetilde{C}}|) \; P_m(\cos(\boldsymbol{s}_p, \boldsymbol{O}_C - \boldsymbol{O}_{\widetilde{C}})) \; .$$

• Step 2: Local translation (radiation functions): For the finest level,
$\forall i \in \{1, ..., N_c\}$, $\forall C$ FMM cluster such that $T_i^\pi \cap C \neq \emptyset$, $\forall p \in \{1, ..., P\}$

$$F_{i\,C}(\boldsymbol{s}_p) = \sum_{i_0/\boldsymbol{x}_{i\,i_0} \in C} \alpha_{i\,i_0} \; \varphi_i^\pi(\pi^{-1}(\boldsymbol{x}_{i\,i_0})) \; \mathrm{e}^{\mathrm{i}\kappa\phi_0(\boldsymbol{x}_{i\,i_0})} \; \mathrm{e}^{\mathrm{i}\kappa<\boldsymbol{s}_p, \boldsymbol{O}_C - \boldsymbol{x}_{i\,i_0}>}$$

The radiation functions at the other levels are obtained by interpolation ([4]).

• Step 3: Matrix approximation: For each considered level, $\forall i, j \in \{1, ..., N_c\}$

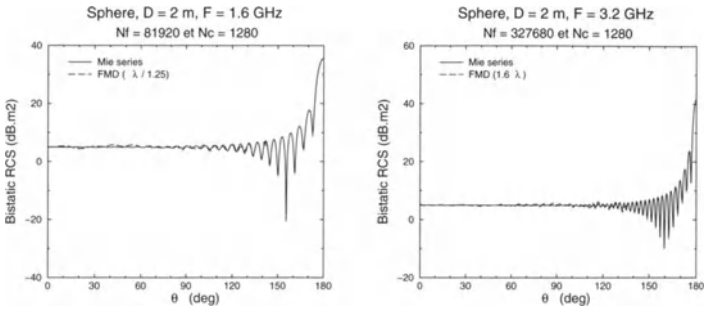
$$(\mathcal{M}^{\Re,\mathrm{far}}_{\mathrm{approx}})_{ij} \longleftarrow (\mathcal{M}^{\Re,\mathrm{far}}_{\mathrm{approx}})_{ij} + \frac{-\kappa}{(4\pi)^2} \sum_{p=1}^{P} \omega_p \sum_{C/T_i^\pi \cap C \neq \emptyset} \overline{F_{i\,C}(\boldsymbol{s}_p)} \sum_{\substack{\widetilde{C}/T_j^\pi \cap \widetilde{C} \neq \emptyset \\ \widetilde{C} \in \overline{V}(C)}} T^{\Re}_{l,\boldsymbol{O}_C-\boldsymbol{O}_{\widetilde{C}}}(\boldsymbol{s}_p) \; F_{j\,\widetilde{C}}(\boldsymbol{s}_p) \; .$$

The same process enables one to approximate the matrix $\mathcal{M}^{\Im}$.

Indeed, a matricial element $\mathcal{M}^{\Re}_{ij}$ corresponds to an interaction between two objects $T_i^\pi$ and $T_j^\pi$. From an intuitive point of vue, it appears no use considering FMM clusters larger than the triangles $T_i$ and $T_j$. Actually, regarding the complexity of the different steps in detail, we clearly have to consider only the clusters smaller than the coarse triangles of $\Gamma_c$. Thus, we work out a suitable multilevel fast multipole algorithm considering a reduced number of levels. Theoreticaly, an iterative solution using that new method has the memory cost $\mathcal{O}(N_f)$ and needs the CPU time $\mathcal{O}(N_f^{4/3} \ln N_f + N_{\mathrm{iter}} N_f^{2/3})$. Numerical results show that it can be less than the cost of a solution using the MLFMM for quite large cases due to $N_{\mathrm{iter}}$ and other constant factors.

## 4 Numerical results

We give now numerical results that were obtained for the perfect conductor unit sphere. For the second result, the sphere has a size of $26\,\lambda$ and the frequency is 3.2 GHz. The solution was based on a mesh with an average edge length about $1.6\,\lambda$ instead of the classical $\lambda/10$ where $\lambda$ is the wavelength. Thus, that case using a fine mesh with $N_f = 327680$ triangles, is solved




discretizing the unknown on a coarse mesh with $N_c = 1280$ triangles. The figures show the bistatic radar cross section (RCS) obtained with our new method denoted by FMD in comparison with the exact solution obtained with the Mie series. For the first case (F = 1.6 GHz), the CPU time required by the new method is 3 hours and 20 minutes instead of 7 hours with a classical use of the MLFMM. For the second one (F = 3.2 GHz), the new method requires about 13 hours instead of about 35 hours for a MLFMM. For those cases, the new method is more efficient than a classical use of the MLFMM.

## 5 Conclusion

The method we have developed couples two kinds of methods in order to speed up the solution of integral equations. Firstly, concepts of the geometrical and physical theories of diffraction enable us to reduce the size of the systems, using the microlocal discretization introduced by T. Abboud, J.-C. Nédélec and B. Zhou. Secondly, an original use of the FMM enables us to speed up the calculation of the matrices of the new systems. Such a combination has resulted in a new method that appears quite efficient. Moreover, the coupling have been performed within a new integral formulation which is suitable for iterative solution.

Now, we plan to work on non-convex objects, by considering new approximation of the phase function, and on a coupling with a finite element method for inhomogenous media. Moreover, the new scheme may have several other applications in electromagnetism.

## References

1. T. Abboud, J.-C. Nédélec, B. Zhou: 'Improvement of the Integral Equation Method for High Frequency Problems'. In: *Third International Conference on Mathematical Aspects of Wave Propagation Phenomena, SIAM, 1995*, pp. 178–187

2. E. Darrigrand: Couplage Méthodes Multipôles – Discrétisation Microlocale pour les Equations Intégrales de l'Electromagnétisme. PhD thesis, Bordeaux-1 University, Talence (2002), http://147.210.235.3/pdf/2002/DARRIGRAND_ERIC_2002.pdf
3. E. Darrigrand: *"Coupling of Fast Multipole Method and Microlocal Discretization for the 3-D Helmholtz Equation"* , J. Comput. Phys. **181**, pp. 126–154 (2002)
4. E. Darve: *"The Fast Multipole Method: Numerical Implementation"* , J. Comput. Phys. **160**, 1, pp. 195–240 (2000)
5. B. Després: *"Fonctionnelle quadratique et équations intégrales pour les problèmes d'onde harmonique en domaine extérieur"* , M2AN **31**, 6, pp. 679–732 (1997)
6. N. Engheta, W.D. Murphy, V. Rokhlin, M.S. Vassiliou: *"The Fast Multipole Method (FMM) for Electromagnetic Scattering Problems"* , IEEE Trans. Ant. and Prop. **40**, 6, pp. 634–641 (1992)
7. J.M. Song, W.C. Chew: *"Multilevel Fast Multipole Algorithm for Solving Combined Field Integral Equations of Electromagnetic Scattering"* , Microwave Opt. Tech. Letter **10**, 1, pp. 14–19 (1995)
8. B. Stupfel: *"A Hybrid Finite Element and Integral Equation Domain Decomposition Method for the Solution of the 3-D Scattering Problem"* , J. Comput. Phys. **172**, pp. 451–471 (2001)