PROPAGATION OF MASSIVE SCALAR FIELDS IN PRE-BIG BANG COSMOLOGIES

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Abstract. We investigate the linear and semilinear massive Klein-Gordon equations in geometrical frameworks of type "Conformal Cyclic Cosmology" of R. Penrose, or "Singular Bouncing Scenario" as well. We give sufficient conditions on the decay of the mass to the fields be able to propagate across the Big-Bang.

I. Introduction

In this work we investigate the propagation of a *massive* scalar field in simple models of the pre-big bang cosmology, in particular for the Conformal Cyclic Cosmology (CCC). In this theory developped by R. Penrose [15] (see also [11], [23]), we consider a $n+1$ dimensional C^2, time oriented, Lorentzian manifold $(\mathcal{M}, g)$. We assume that this spacetime is globally hyperbolic. Given a Cauchy hypersurface Σ we split $\mathcal{M}$ as

$$\mathcal{M} = \hat{\mathcal{M}} \cup \Sigma \cup \check{\mathcal{M}}, \quad \hat{\mathcal{M}} := I^-(\Sigma), \quad \check{\mathcal{M}} := I^+(\Sigma), \tag{I.1}$$

where $I^-(\Sigma)$ (resp. $I^+(\Sigma)$) is the chronological past (resp. chronological future) of Σ. The past (resp. future) set $\hat{\mathcal{M}}$ (resp. $\check{\mathcal{M}}$) is an open subset called *Previous Aeon* (resp. *Present Aeon*). The metric g is just a mathematical tool, called *bridging metric*, and the physical metrics $\hat{g}$ and $\check{g}$ on the Aeons are solutions of the Einstein equations that are conformally equivalent to g,

$$\hat{g} = \hat{\Omega}^2 g, \quad \check{g} = \check{\Omega}^2 g, \tag{I.2}$$

where $\bar{\Omega}$ is a non-vanishing C^2 function on $\bar{\mathcal{M}}$ (throughout the paper, the bar stands for the hat or the check). In the CCC we assume that $\check{\Omega}$ can be continuously extended by zero to Σ. Therefore the Present Aeon is an universe beginning by a Big Bang, and Σ is called the *Bang Surface*. In contrast we assume that $\hat{\Omega}$ tends to the infinity at Σ that is the Future Infinity of the Previous Aeon. In this paper we consider the simple but important case, of the static metric g, therefore the Aeons are generalized FLRW spacetimes. Our results can also be applied to the *Singular Bouncing Scenario* for which $\hat{\Omega}$ and $\check{\Omega}$ tend to zero at Σ, and even for the rather unphysical case where Σ is the Future Infinity of $\hat{\mathcal{M}}$ and the Past Infinity of $\check{\mathcal{M}}$ if $\hat{\Omega}$ and $\check{\Omega}$ tend to the infinity at Σ. We adress the fundamental issue : to find sufficient conditions on the decay of the mass near Σ to that a massive scalar field in $\hat{\mathcal{M}}$ can be extended in $\check{\mathcal{M}}$ despite the severe singularity on Σ that leads to the blow-up or the vanishing of the fields on it. The idea is following: we renormalize the fields $\bar{u}$ by using the Liouville transforms associated to the conformal factors

$$\bar{\varphi} := \bar{\Omega}^{\frac{n-1}{2}} \bar{u}, \tag{I.3}$$

then we investigate the wave equation satisfied by the renormalized fields $\bar{\varphi}$. We impose some constraints on the mass that allow to obtain the existence of trace of $\bar{\varphi}$ and a suitable time-derivative, on the bang surface Σ. We conclude that the asymptotic behaviour of $\hat{u}$ at Σ in the Previous Aeon, leads to a Cauchy data that determines $\check{u}$ in the Present Aeon.

A free massive scalar field obeys to the Klein-Gordon equation satisfied in the interior $\mathring{\bar{\mathcal{M}}}$ of each Aeon $\bar{\mathcal{M}}$,

$$\left[\Box_{\bar{g}} + \bar{\xi} R_{\bar{g}} + \bar{m}^2\right] \bar{u} = \bar{f} \ \ in \ \ \mathring{\bar{\mathcal{M}}}. \tag{I.4}$$

Here $\Box_h$ and R_h are respectively the D'Alembert operator and the Ricci scalar associated to a metric h,

$$\Box_h u := \frac{1}{\sqrt{|\det(h)|}}\partial_\mu\left(h^{\mu\nu}\sqrt{|\det(h)|}\partial_\nu u\right), \tag{I.5}$$

$\bar{\xi}$ is a constant describing the coupling with the geometry, the mass $\bar{m}$ is a non negative function defined on $\bar{\mathcal{M}}$ and the source term $\bar{f}$ is given. The fundamental issue concerns the propagation of the field from the Previous Aeon to the Present Aeon across the Bang Surface. The true problem is fully non linear and we should deal with the coupled Einstein-Scalar Field system. A very important result in this domain is due to H. Friedrich [7] who investigated the 1+3 dimensional case with a cosmological constant $\Lambda > 0$ and

$$\hat{\xi} = 0, \;\; \hat{m}^2 = \frac{2}{3}\Lambda. \tag{I.6}$$

Our aim is more modest : we study the linear or semilinear Klein-Gordon equation in a fixed geometrical framework of the CCC or the Bouncing Scenario (semi-classical approximation for the weak fields). In this context, there is a trivial situation, called the *conformal invariant massless case*, defined by

$$\bar{\xi} = \frac{n-1}{4n}, \;\; \bar{m} = 0. \tag{I.7}$$

Using the Liouville transform (I.3) we can easily check that $\bar{u}$ is a solution of (I.4) iff $\bar{\varphi}$ is solution of the Klein-Gordon equation with the variable effective mass $|\,\bar{m}\bar{\Omega}\,|$, associated to the bridging metric g,

$$\left[\Box_g + \xi R_g + \bar{\Omega}^{-1}\left(\xi - \frac{n-1}{4n}\right)\left(2\bar{\Omega}_{;\mu\nu} + (n-3)\bar{\Omega}^{-1}\bar{\Omega}_{;\mu}\bar{\Omega}_{;\nu}\right)g^{\mu\nu} + \bar{m}^2\bar{\Omega}^2\right]\bar{\varphi} = \bar{\Omega}^{\frac{n+3}{2}}\bar{f}. \tag{I.8}$$

Hence for the conformal invariant massless case (I.7) with $\bar{f} = 0$, the propagation of the field is just described by the wave equation

$$\left[\Box_g + \frac{n-1}{4n}R_g\right]\bar{\varphi} = 0, \tag{I.9}$$

and since $(\mathcal{M}, g)$ is globally hyperbolic, the global Cauchy problem is well posed for this equation by the Leray theorem : given a Cauchy hypersurface, any initial data on it determines a unique solution defined on the whole manifold. In this sense, the conformal invariant massless case without source term is trivial: the field freely propagates from the Previous Aeon to the Present Aeon.

The situation drastically changes if $\bar{\xi} \neq (n-1)/4n$ or $\bar{m} \neq 0$ since $\bar{\Omega}^{-1}$ (resp $\bar{\Omega}$) blows up on Σ for a Big Bang (resp. an expanding universe[1]) and the Klein-Gordon equation (I.8) is highly singular. As a consequence $\bar{\varphi}$ can diverge at Σ and the possibility of a propagation from the Previous Aeon to the Present Aeon seems to be doubtful in general. For instance if $\hat{\xi} = 0$, $\hat{m}$ is a strictly positive constant and $\hat{g} = \tau^{-2}\left(d\tau^2 - g_{ij}dx^i dx^j\right)$ is De-Sitter like, the results established by A. Vasy in [24] show that $\hat{\varphi}(\tau, .) \sim \tau^{\frac{1}{2}-\sqrt{\frac{n^2}{4}+\hat{m}^2}}\psi(\tau, .)$ near $\Sigma = \{\tau = 0\}$ with $\psi \in C^0(\hat{\mathcal{M}} \cup \Sigma)$ (see also [2] for $\hat{\xi} = \hat{m} = 0$). In this paper we follow the ideas of Penrose and we assume that $\bar{\xi} = (n-1)/4n$ and the mass of the field is a function that is decaying to zero as $\tau \to 0$. Therefore we investigate the

[1]However when $\hat{g}$ is a solution of the Vacuum Einstein Equations $R_{\mu\mu} - \frac{1}{2}Rg_{\mu\mu} + \Lambda g_{\mu\mu} = 0$, we have $R_{\hat{g}} = 2\frac{n+1}{n-1}\Lambda$, and the case $(\xi = 0, m^2 = \frac{n+1}{2n})$ is equivalent to the conformal invariant massless case. In particular the Klein-Gordon equation studied by Friedrich in [7] is a non-linear perturbation of (I.9).

both Klein-Gordon equations

$$\left[\Box_g + \frac{n-1}{4n}R_g + \bar{m}^2\bar{\Omega}^2\right]\bar{\varphi} = \bar{\Omega}^{\frac{n+3}{2}}\bar{f} \quad in \quad \mathring{\bar{\mathcal{M}}}, \tag{I.10}$$

and we look for sufficient conditions on $\bar{m}$ to that $\hat{\varphi}$ determines $\check{\varphi}$ despite the singularity on Σ. An obvious way could consist in introducing

$$\tilde{m} = \hat{m}, \;\; \tilde{\Omega} = \hat{\Omega}, \;\; \tilde{f} = \hat{f} \;\; on \;\; \hat{\mathcal{M}}, \;\; \tilde{m} = \check{m}, \;\; \tilde{\Omega} = \check{\Omega}, \;\; \tilde{f} = \check{f} \;\; on \;\; \check{\mathcal{M}}, \tag{I.11}$$

and solving on the whole manifold $\mathcal{M}$ the equation

$$\left[\Box_g + \frac{n-1}{4n}R_g + \tilde{m}^2\tilde{\Omega}^2\right]\tilde{\varphi} = \tilde{\Omega}^{\frac{n+3}{2}}\tilde{f} \quad in \quad \mathring{\mathcal{M}}. \tag{I.12}$$

Then $\check{\varphi}$ could be defined by $\tilde{\varphi}$ on $\check{\mathcal{M}}$ if $\tilde{\varphi} = \hat{\varphi}$ on $\hat{\mathcal{M}}$. This elementary approach could work if the equation (I.12) makes sense. In particular $\tilde{m}^2\tilde{\Omega}^2\tilde{\varphi}$ has to be well defined as a distribution in $\mathcal{M}$. Unfortunately $\bar{m}^2\bar{\Omega}^2$ is not in $L^1_{loc}(\bar{\mathcal{M}} \cup \Sigma)$ in general, unless we add unreasonably strong conditions on the decay of the mass. In fact (I.12) holds for the Singular Bouncing Scenario for which $\tilde{m}^2\tilde{\Omega}^2 \in L^1_{loc}(\mathcal{M})$ (see Theorem III.2 and Theorem V.2), but this very simple method fails for the CCC for which $\tilde{m}^2\tilde{\Omega}^2 \notin L^1_{loc}(\mathcal{M})$ and equation (I.12) does not make sense. To overcome this difficulty and to obtain the main results of this paper (Theorem IV.4 and Theorem V.3), we adapt a method used in [1] and [4] based on the solving of the Riccati equation

$$\bar{A}' - \bar{A}^2 = \bar{m}^2\bar{\Omega}^2$$

that allows to transform equation (I.10) into an new equation in which all the coefficients are integrable functions (see (IV.42) and V.24)). From a mathematical point of view, this work deals with a linear scalar wave equation with a time-dependent mass, or in a time-dependent background, a topic for which a lot of papers have been devoted, see *e.g.* [5] and the references therein, and in the framework of General Relativity [1], [2], [18], [19], [24]. We also investigate the semi-linear Klein-Gordon equation for $n = 3$. This important model has been investigated in the case of the smooth Lorentzian manifolds and fixed mass, in particular in [3], [6], [8], [9], [10], [12], [13],[14]; here we consider the semi-linear equation (I.10) with $\bar{f} = -\kappa \mid \bar{u} \mid^2 \bar{u}$, and the strongly singular effective mass $\bar{m}\bar{\Omega}$, and we prove that, like for the linear case, the fields propagate across the Bang Surface. Ultimately, our results are just a preliminary step toward the fundamental issue of the stability of the Einstein-scalar field system for a cosmological scenario with a Big Bang (in this contexte we have to mention [7], [16], [17], [20] and [21]).

II. Geometrical and Functional frameworks

We suppose that there exists a n-dimensional C^∞ Riemannian manifold $(\mathbf{K}, \gamma)$ and τ_-, τ_+, $-\infty < \tau_- < 0 < \tau_+ < \infty$ such that

$$\mathcal{M} = [\tau_-, \tau_+]_\tau \times \mathbf{K}_{\mathrm{x}}, \;\; g_{\mu\nu}dx^\mu dx^\nu = d\tau^2 - \gamma_{ij}d\mathrm{x}^i d\mathrm{x}^j, \tag{II.1}$$

and we take

$$\Sigma := \{\tau = 0\} \times \mathbf{K}, \;\; \hat{\mathcal{M}} := [\tau_-, 0) \times \mathbf{K}, \;\; \check{\mathcal{M}} := (0, \tau_+] \times \mathbf{K}. \tag{II.2}$$

Given two non-vanishing functions $\hat{\Omega} \in C^2(\hat{\mathcal{M}})$, $\check{\Omega} \in C^2(\check{\mathcal{M}})$ we endowe $\bar{\mathcal{M}}$ with the metrics defined by (I.2). Some of our results are obtained without supplement assumption on $\bar{\Omega}$. Nevertheless the cases of physical interest are (1) the *Singular Bouncing Scenario* with

$$\forall \mathrm{x} \in \mathbf{K}, \;\; \hat{\Omega}(\tau, \mathrm{x}) \xrightarrow[\tau\to 0^-]{} 0, \;\; \check{\Omega}(\tau, \mathrm{x}) \xrightarrow[\tau\to 0^+]{} 0, \tag{II.3}$$

($\tau=0$ is a Big Crunch for $\hat{\mathcal{M}}$ and a Big Bang for $\check{\mathcal{M}}$), and (2) the CCC of Penrose for which

$$\forall \mathrm{x}\in\mathbf{K},\ \ \int_{\tau_-}^{0} \mid \hat{\Omega}(\tau,\mathrm{x})\mid d\tau=\infty,\ \ \check{\Omega}(\tau,\mathrm{x})\xrightarrow[\tau\to 0^+]{} 0, \tag{II.4}$$

($\tau=0$ is the Future Infinity for $\hat{\mathcal{M}}$ and a Big Bang for $\check{\mathcal{M}}$). We could also consider more exotic cases such as

$$\forall \mathrm{x}\in\mathbf{K},\ \ \int_{\tau_-}^{0} \mid \hat{\Omega}(\tau,\mathrm{x})\mid d\tau=\int_{0}^{\tau_+} \mid \check{\Omega}(\tau,\mathrm{x})\mid d\tau=\infty, \tag{II.5}$$

for which $\tau=0$ is the Future Infinity for $\hat{\mathcal{M}}$ and the Past Infinity for $\check{\mathcal{M}}$, or a Previous Aeon ending with a *Big Rip* at $\tau=0$,

$$\forall \mathrm{x}\in\mathbf{K},\ \ \int_{\tau_-}^{0} \mid \hat{\Omega}(\tau,\mathrm{x})\mid d\tau<\infty,\ \ \hat{\Omega}(\tau,\mathrm{x})\xrightarrow[\tau\to 0^-]{} \infty. \tag{II.6}$$

We shall be able to apply our results to a very important example of CCC that is associated to generalized FLRW universes:

Example II.1. *Given $\hat{t}_-\in\mathbb{R}$, we introduce*

$$\hat{\mathcal{M}}:=[\hat{t}_-,\infty)_{\hat{t}}\times\mathbf{K}_{\mathrm{x}}, \tag{II.7}$$

$$\hat{g}_{\mu\nu}dx^{\mu}dx^{\nu}=d\hat{t}^2-\hat{a}(\hat{t})^2\gamma_{ij}d\mathrm{x}^i d\mathrm{x}^j. \tag{II.8}$$

Here the scale factor $\hat{a}$ is a strictly positive function in $C^2([\hat{t}_-,\infty))$. We assume that this space-time is sufficiently accelerating to that

$$\hat{a}^{-1}\in L^1(\hat{t}_-,\infty). \tag{II.9}$$

Now we introduce the conformal time τ that is defined as

$$\tau(\hat{t}):=-\int_{\hat{t}}^{\infty}\frac{1}{\hat{a}(s)}ds \tag{II.10}$$

and in the (τ,x) coordinates, the Previous Aeon is described as

$$\hat{\mathcal{M}}=[\tau_-,0)_{\tau}\times\mathbf{K}_{\mathrm{x}},\ \ \tau_-:=-\int_{\hat{t}_-}^{\infty}\frac{1}{\hat{a}(s)}ds\in(-\infty,0), \tag{II.11}$$

$$\hat{g}_{\mu\nu}dx^{\mu}dx^{\nu}=\hat{\Omega}^2(\tau)\left[d\tau^2-\gamma_{ij}d\mathrm{x}^i d\mathrm{x}^j\right],\ \ \hat{\Omega}(\tau):=\pm\hat{a}(\hat{t}). \tag{II.12}$$

An important particular case is the De-Sitter like metric for which

$$\hat{a}(\hat{t})\sim\hat{C}e^{\hat{H}\hat{t}},\ \ \hat{t}\to\infty,\ \ \hat{C},\hat{H}>0. \tag{II.13}$$

We easily check that

$$\hat{\Omega}(\tau)\sim\mp\frac{1}{\hat{H}\tau},\ \ \tau\to 0^-. \tag{II.14}$$

In a similar way, given $\check{t}_+\in(0,\infty)$ we introduce

$$\check{\mathcal{M}}:=(0,\check{t}_+]_{\check{t}}\times\mathbf{K}_{\mathrm{x}}, \tag{II.15}$$

$$\check{g}_{\mu\nu}dx^{\mu}dx^{\nu}=d\check{t}^2-\check{a}(\check{t})^2\gamma_{ij}d\mathrm{x}^i d\mathrm{x}^j. \tag{II.16}$$

Now the scale factor $\check{a}$ is a strictly positive function in $C^2((0,\check{t}_+])$ that tends to zero as $\check{t}\to 0$ sufficiently slowly to that

$$\check{a}^{-1}\in L^1(0,\check{t}_+). \tag{II.17}$$

The conformal time τ *is defined as*

$$\tau(\check{t}) := \int_0^{\check{t}} \frac{1}{\check{a}(s)} ds \tag{II.18}$$

and in the (τ, x) *coordinates, the Present Aeon is described as*

$$\check{\mathcal{M}} = (0, \tau_+]_\tau \times \mathbf{K}_{\mathrm{x}}, \quad \tau_+ := \int_0^{\check{t}_+} \frac{1}{\check{a}(s)} ds \in (0, \infty), \tag{II.19}$$

$$\check{g}_{\mu\nu} dx^\mu dx^\nu = \check{\Omega}^2(\tau) \left[d\tau^2 - \gamma_{ij} d\mathrm{x}^i d\mathrm{x}^j \right], \quad \check{\Omega}(\tau) := \check{a}(\check{t}). \tag{II.20}$$

An interesting case is the C^0 *Big Bang studied in* [1] *for which*

$$\check{a}(\check{t}) \sim \check{C} \check{t}^{\check{\eta}}, \quad \check{t} \to 0^+, \quad 0 < \check{C}, \quad \check{\eta} \in (0,1). \tag{II.21}$$

Then we have

$$\check{\Omega}(\tau) \sim \check{C}^{\frac{1}{1-\check{\eta}}} (1-\check{\eta})^{\frac{\check{\eta}}{1-\check{\eta}}} \tau^{\frac{\check{\eta}}{1-\check{\eta}}}, \quad \tau \to 0^+. \tag{II.22}$$

We remark that in the cases (II.13), (II.21), with $\tau_- = -\tau_+$*, the Penrose's "reciprocal proposal"* [15]

$$\hat{\Omega}(-\tau)\check{\Omega}(\tau) = -1, \quad \tau \in (0, \tau_+), \tag{II.23}$$

can be satisfied when

$$\hat{\Omega}(\tau) = -\hat{a}(\hat{t}), \quad \check{\eta} = \frac{1}{2}, \quad \check{C} = \sqrt{2\hat{H}}. \tag{II.24}$$

As regards the manifold $(\mathbf{K}, \gamma)$ we assume it is complete and the scalar curvature R_γ is bounded

$$R_\gamma \in L^\infty(\mathbf{K}). \tag{II.25}$$

Here we denote $L^p(\mathbf{K})$ the L^p-Lebesgue space on $\mathbf{K}$ endowed with the volume measure associated to the metric γ. It is well known the the Laplace-Beltrami operator

$$\Delta_{\mathbf{K}} := \frac{1}{\sqrt{|\det(\gamma)|}} \partial_i \left(\gamma^{ij} \sqrt{|\det \gamma)|} \partial_j u \right),$$

is essentially selfadjoint on $C_0^\infty(\mathbf{K})$. Hence we can introduce the Sobolev spaces $H^s(\mathbf{K})$, $s \in \mathbb{R}$, defined as the closure of $C_0^\infty(\mathbf{K})$ for the norm

$$\|f\|_{H^s(\mathbf{K})} := \left\| (-\Delta_{\mathbf{K}} + 1)^{\frac{s}{2}} f \right\|_{L^2(\mathbf{K})}. \tag{II.26}$$

To investigate the semi-linear Klein-Gordon equation, we need the Sobolev embedding, therefore we have to strengthen the assumptions on the metric. Thus to study the self-interacting fields, we shall assume that $(\mathbf{K}, \gamma)$ is a C^∞ *bounded geometry manifold*, *i.e.* the following conditions are satisfied: (1) the injectivity radius is strictly positive, (2) every covariant derivative of the Riemannian curvature tensor is bounded, or in an equivalent way,

$$\forall \alpha \in \mathbb{N}^d, \quad D^\alpha \gamma_{ij} \in L^\infty(\mathbf{K}), \quad \gamma^{ij} \in L^\infty(\mathbf{K}), \tag{II.27}$$

where D represents coordinate derivatives in any *normal* coordinate system. With these hypotheses, if $d = 3$ we have the continuous embedding

$$H^1(\mathbf{K}) \subset L^6(\mathbf{K}). \tag{II.28}$$

III. Free Scalar Fields in the Aeons

We investigate the propagation of a scalar field in the Aeon $\bar{\mathcal{M}}$, that obeys to the Klein-Gordon equation with a variable mass $\bar{m}$ that is a measurable function on $\bar{\mathcal{M}}$:

$$\left[\Box_{\bar{g}} + \frac{n-1}{4n} R_{\bar{g}} + \bar{m}^2\right] \bar{u} = \bar{f} \;\; in \;\; \mathring{\bar{\mathcal{M}}}. \tag{III.1}$$

By the Liouville transform (I.3), this equation is equivalent to the equation (I.10) in $\bar{I} \times \mathbf{K}$ where

$$\hat{I} := [\tau_-, 0), \quad \check{I} := (0, \tau_+]. \tag{III.2}$$

Since $(\mathcal{M}, g)$ satisfies (II.1), this equation has the very simple form

$$\left[\partial_\tau^2 - \Delta_{\mathbf{K}} + \frac{n-1}{4n} R_\gamma + \bar{m}^2 \bar{\Omega}^2\right] \bar{\varphi} = \bar{\Omega}^{\frac{n+3}{2}} \bar{f}, \quad (\tau, \mathrm{x}) \in \mathring{\bar{I}} \times \mathbf{K}. \tag{III.3}$$

We suppose that the mass and the conformal factor satisfy

$$\bar{m}^2 \bar{\Omega}^2 \in L^1_{loc}\left(\bar{I}; L^\infty(\mathbf{K})\right). \tag{III.4}$$

The global Cauchy problem is easily solved:

Proposition III.1. *We assume that (II.25) and (III.4) hold. Then given $\tau_0 \in \bar{I}$, $s \in [0,1]$, $\bar{\varphi}_0 \in H^s(\mathbf{K})$, $\bar{\varphi}_1 \in H^{s-1}(\mathbf{K})$, $\bar{f} \in L^1_{loc}\left(\bar{I}; H^{s-1}(\mathbf{K})\right)$, the equation (III.3) has a unique solution $\bar{\varphi}$ satisfying*

$$\bar{\varphi} \in L^\infty_{loc}\left(\bar{I}; H^s(\mathbf{K})\right), \quad \partial_\tau \bar{\varphi} \in L^\infty_{loc}\left(\bar{I}; H^{s-1}(\mathbf{K})\right), \tag{III.5}$$

$$\bar{\varphi}(\tau_0) = \bar{\varphi}_0, \quad \partial_\tau \bar{\varphi}(\tau_0) = \bar{\varphi}_1. \tag{III.6}$$

Moreover we have

$$\bar{\varphi} \in C^0\left(\bar{I}; H^s(\mathbf{K})\right) \cap C^1\left(\bar{I}; H^{s-1}(\mathbf{K})\right) \tag{III.7}$$

and there exists $C > 0$ such that any solution satisfies

(III.8)
$$\|\bar{\varphi}(\tau)\|_{H^s(\mathbf{K})} + \|\partial_\tau \bar{\varphi}(\tau)\|_{H^{s-1}(\mathbf{K})} \leqslant C\left(\|\bar{\varphi}_0\|_{H^s(\mathbf{K})} + \|\bar{\varphi}_1\|_{H^{s-1}(\mathbf{K})} + \left|\int_{\tau_0}^{\tau} \left\|\bar{\Omega}^{\frac{n+3}{2}}(\sigma) \bar{f}(\sigma)\right\|_{H^{s-1}(\mathbf{K})} d\sigma\right|\right)$$
$$\times \exp\left(\left|\int_{\tau_0}^{\tau} \|\bar{m}(\sigma)\bar{\Omega}(\sigma)\|^2_{L^\infty(\mathbf{K})} d\sigma\right|\right).$$

If we assume that

$$\bar{m}^2 \bar{\Omega}^2 \in L^1\left(\bar{I}; L^\infty(\mathbf{K})\right), \tag{III.9}$$

$$\bar{\Omega}^{\frac{n+3}{2}} \bar{f} \in L^1\left(\bar{I}; H^{s-1}(\mathbf{K})\right), \tag{III.10}$$

then the following limits exist:

$$\bar{\psi}_0 := \lim_{\tau \to 0} \bar{\varphi}(\tau) \;\; in \;\; H^s(\mathbf{K}), \tag{III.11}$$

$$\bar{\psi}_1 := \lim_{\tau \to 0} \partial_\tau \bar{\varphi}(\tau) \;\; in \;\; H^{s-1}(\mathbf{K}). \tag{III.12}$$

Furthermore, (III.9) and (III.10) assure that given $\bar{\psi}_0 \in H^s(\mathbf{K})$, $\bar{\psi}_1 \in H^{s-1}(\mathbf{K})$, there exists a unique solution $\bar{\varphi}$ of (III.3) satisfying (III.7), (III.11), (III.12) and the map $(\bar{\varphi}_0, \bar{\varphi}_1) \mapsto (\bar{\psi}_0, \bar{\psi}_1)$ is a homeomorphism on $H^s(\mathbf{K}) \times H^{s-1}(\mathbf{K})$.

The initial conditions (III.6) make sense thanks to the Strauss theorem [22]: (III.5) implies that $\bar{\varphi}$ is a weakly continuous function with values in $H^s(\mathbf{K})$ and since $\bar{m}^2\bar{\Omega}^2\bar{\varphi} \in L^1_{loc}(\bar{I}; L^2(\mathbf{K}))$, the equation (III.3) assures that $\partial_\tau^2\bar{\varphi} \in L^1_{loc}(\bar{I}; H^{s-2}(\mathbf{K}))$ and then $\partial_\tau\bar{\varphi}$ is a weakly continuous function with values in $H^{s-1}(\mathbf{K})$.

Proof. Due to the weak regularity of the coefficients, this proposition is not a straight consequence of the classic results on the hyperbolic Cauchy problem, but we can establish the existence of the solution using a standard method. We introduce the operator

$$\mathcal{A} := \begin{pmatrix} 0 & 1 \\ -\Delta_{\mathbf{K}} + 1 & 0 \end{pmatrix}, \quad Dom(\mathcal{A}) = H^s(\mathbf{K}) \times H^{s-1}(\mathbf{K}), \tag{III.13}$$

that is selfadjoint in $H^{s-1}(\mathbf{K}) \times H^{s-2}(\mathbf{K})$ and we solve the integral equation

$$\begin{aligned} \begin{pmatrix} \bar{\varphi}(\tau) \\ \bar{\psi}(\tau) \end{pmatrix} = \mathcal{F}\begin{pmatrix} \bar{\varphi} \\ \bar{\psi} \end{pmatrix}(\tau) :=& e^{i(\tau-\tau_0)\mathcal{A}} \begin{pmatrix} \bar{\varphi}_0 \\ \bar{\varphi}_1 \end{pmatrix} + \int_{\tau_0}^{\tau} e^{i(\tau-\sigma)\mathcal{A}} \begin{pmatrix} 0 \\ \bar{\Omega}^{\frac{n+3}{2}}(\sigma)\bar{f}(\sigma) \end{pmatrix} d\sigma \\ &+ \int_{\tau_0}^{\tau} e^{i(\tau-\sigma)\mathcal{A}} \begin{pmatrix} 0 \\ \left[1 - \frac{n-1}{4n}R_\gamma - \bar{m}^2(\sigma)\bar{\Omega}^2(\sigma)\right]\bar{\varphi}(\sigma) \end{pmatrix} d\sigma. \end{aligned} \tag{III.14}$$

Since $s \in [0,1]$, we have for any $\Phi, \Psi \in X_h := C^0\left([\tau_0 - h, \tau_0 + h]; H^s(\mathbf{K}) \times H^{s-1}(\mathbf{K})\right)$ and $h > 0$,

$$\left\|\mathcal{F}(\Phi) - \mathcal{F}(\Psi)\right\|_{X_h} \leqslant \left[(1 + \|R_\gamma\|_{L^\infty})\,h + \int_{\tau_0-h}^{\tau_0+h} \|\bar{m}(\sigma)\bar{\Omega}(\sigma)\|_{L^\infty} d\sigma\right] \|\Phi - \Psi\|_{X_h}.$$

Therefore $\mathcal{F}$ is a contraction on X_h for h small enough, and its unique fixed point $\bar{\Phi} := (\bar{\varphi}, \bar{\psi})$ is a local solution of (III.14). Moreover we have

$$\begin{aligned} \left\|\begin{pmatrix} \bar{\varphi}(\tau) \\ \bar{\psi}(\tau) \end{pmatrix}\right\|_{H^s\times H^{s-1}} \leqslant& \left\|\begin{pmatrix} \bar{\varphi}_0 \\ \bar{\varphi}_1 \end{pmatrix}\right\|_{H^s\times H^{s-1}} + \left|\int_{\tau_0}^{\tau} \left\|\bar{\Omega}^{\frac{n+3}{2}}(\sigma)\bar{f}(\sigma)\right\|_{H^{s-1}} d\sigma\right| \\ &+ \left|\int_{\tau_0}^{\tau} \left(1 + \|R_\gamma\|_{L^\infty} + \|\bar{m}^2(\sigma)\bar{\Omega}^2(\sigma)\|_{L^\infty}\right) \|\bar{\varphi}(\sigma)\|_{H^s} d\sigma\right|. \end{aligned}$$

Therefore the Gronwall Lemma implies that

$$\begin{aligned} \left\|\begin{pmatrix} \bar{\varphi}(\tau) \\ \bar{\psi}(\tau) \end{pmatrix}\right\|_{H^s\times H^{s-1}} \leqslant& C\left(\left\|\begin{pmatrix} \bar{\varphi}_0 \\ \bar{\varphi}_1 \end{pmatrix}\right\|_{H^s\times H^{s-1}} + \left|\int_{\tau_0}^{\tau} \left\|\bar{\Omega}^{\frac{n+3}{2}}(\sigma)\bar{f}(\sigma)\right\|_{H^{s-1}} d\sigma\right|\right) \\ &\times \exp\left(\left|\int_{\tau_0}^{\tau} \|\bar{m}^2(\sigma)\bar{\Omega}^2(\sigma)\|_{L^\infty} d\sigma\right|\right), \end{aligned} \tag{III.15}$$

and the principle of the unique continuation assures that the local solution $\bar{\Phi}$ can be extended to a global solution $\bar{\Phi} \in C^0\left(\bar{I}; H^s(\mathbf{K}) \times H^{s-1}(\mathbf{K})\right)$ of the integral equation (III.14) that obviously satisfies also (III.15) on $\bar{I}$. We warn that in general $\bar{\Phi} \notin C^1\left([\tau_0 - h, \tau_0 + h]; H^{s-1} \times H^{s-2}\right)$ since we do not assume $\bar{m}$ to be continuous. Nevertheless $\bar{\psi} = \partial_\tau\bar{\varphi}$ and $\bar{\varphi}$ is a global solution of (III.3) satisfying (III.6), (III.7) and (III.8).

To establish the uniqueness, we recall (see *e.g.* [22]) that any $\varphi \in L^\infty_{loc}(\bar{I}; H^1(\mathbf{K}))$ with $\partial_\tau\varphi \in L^\infty_{loc}(\bar{I}; L^2(\mathbf{K}))$ solution in the sense of the distributions of $(\partial_\tau^2 - \Delta_{\mathbf{K}} + 1)\varphi = F \in L^1_{loc}(\bar{I}; L^2(\mathbf{K}))$ satisfies

$$\|\varphi(\tau)\|^2_{H^1} + \|\partial_\tau\varphi(\tau)\|^2_{L^2} = \|\varphi(\tau_0)\|^2_{H^1} + \|\partial_\tau\varphi(\tau_0)\|^2_{L^2} + 2\Re\int_{\tau_0}^{\tau}\int_{\mathbf{K}} F(\sigma, \mathrm{x})\overline{\partial_\tau\varphi(\sigma, \mathrm{x})}d\mathrm{x}d\sigma, \tag{III.16}$$

hence,

$$\|\varphi(\tau)\|^2_{H^1} + \|\partial_\tau\varphi(\tau)\|^2_{L^2} \leqslant \|\varphi(\tau_0)\|^2_{H^1} + \|\partial_\tau\varphi(\tau_0)\|^2_{L^2} + 2\left|\int_{\tau_0}^{\tau} \|F(\sigma)\|_{L^2}\|\partial_\tau\varphi(\sigma)\|_{L^2} d\sigma\right|.$$

We can apply this inequality to $(-\Delta_{\mathbf{K}}+1)^{-\frac{1-s}{2}}\varphi$ where $\varphi\in L^\infty_{loc}(\bar I;H^s(\mathbf{K}))$ with $\partial_\tau\varphi\in L^\infty_{loc}(\bar I;H^{s-1}(\mathbf{K}))$ is a solution in the sense of the distributions of $(\partial_\tau^2-\Delta_{\mathbf{K}}+1)\varphi=F\in L^1_{loc}(\bar I;H^{s-1}(\mathbf{K}))$ to obtain

$$\|\varphi(\tau)\|^2_{H^s}+\|\partial_\tau\varphi(\tau)\|^2_{H^{s-1}}\leqslant\|\varphi(\tau_0)\|^2_{H^s}+\|\partial_\tau\varphi(\tau_0)\|^2_{H^{s-1}}+2\left|\int_{\tau_0}^{\tau}\|F(\sigma)\|_{H^{s-1}}\|\partial_\tau\varphi(\sigma)\|_{H^{s-1}}d\sigma\right|.$$

Taking $F=\left(1-\frac{n-1}{4n}R_\gamma-\bar m^2\bar\Omega^2\right)\bar\varphi$ where $\bar\varphi$ is a solution of (III.3) with $\bar f=0$, this inequality implies

$$\begin{aligned}\|\bar\varphi(\tau)\|^2_{H^s}+\|\partial_\tau\bar\varphi(\tau)\|^2_{H^{s-1}}\leqslant&\|\bar\varphi(\tau_0)\|^2_{H^s}+\|\partial_\tau\bar\varphi(\tau_0)\|^2_{H^{s-1}}\\&+\left|\int_{\tau_0}^{\tau}\left(1+\|R_\gamma\|_{L^\infty}+\|\bar m^2(\sigma)\bar\Omega^2(\sigma)\|_{L^\infty}\right)\left(\|\bar\varphi(\sigma)\|^2_{H^s}+\|\partial_\tau\bar\varphi(\sigma)\|^2_{H^{s-1}}\right)d\sigma\right|\end{aligned}$$

hence the uniqueness follows from the Gronwall lemma.

Now (III.8), (III.9) and (III.10) imply that $\bar\varphi\in L^\infty(\bar I;H^s(\mathbf{K}))$, then we have by (III.14)

$$\text{(III.17)}\quad\begin{aligned}\lim_{\tau\to0}\begin{pmatrix}\bar\varphi(\tau)\\ \partial_\tau\bar\phi(\tau)\end{pmatrix}=&e^{-i\tau_0\mathcal{A}}\begin{pmatrix}\bar\varphi_0\\ \bar\varphi_1\end{pmatrix}+\int_{\tau_0}^0e^{-i\sigma\mathcal{A}}\begin{pmatrix}0\\ \bar\Omega^{\frac{n+3}{2}}(\sigma)\bar f(\sigma)\end{pmatrix}d\sigma\\&+\int_{\tau_0}^0e^{-i\sigma\mathcal{A}}\begin{pmatrix}0\\ \left[1-\frac{n-1}{4n}R_\gamma-\bar m^2(\sigma)\bar\Omega^2(\sigma)\right]\bar\varphi(\sigma)\end{pmatrix}d\sigma.\end{aligned}$$

Furthermore, assumptions (III.9) and (III.10) allow to solve the initial value problem with data given at $\tau=0$. Thanks to the integrability of $\|\bar m\bar\Omega\|^2_{L^\infty}$, we can mimick the previous proof and solve the integral equation (III.14) with $\tau_0=0$. We conclude that for any $\tau_1,\tau_2\in\bar I\cup\{0\}$, the map $(\bar\varphi(\tau_1),\partial_\tau\bar\varphi(\tau_1))\mapsto(\bar\varphi(\tau_2),\partial_\tau\bar\varphi(\tau_2))$ is a homeomorphism on $H^s(\mathbf{K})\times H^{s-1}(\mathbf{K})$.

Q.E.D.

We immediately deduce the following:

Theorem III.2. *We assume (II.25) and*

$$\text{(III.18)}\quad\int_{\tau_-}^{\tau_+}\left\|\tilde\Omega^{\frac{n+3}{2}}(\sigma)\tilde f(\sigma)\right\|_{H^{s-1}(\mathbf{K})}+\|\tilde m(\tau)\tilde\Omega(\tau)\|^2_{L^\infty(\mathbf{K})}d\tau<\infty,$$

where $\tilde m$, $\tilde\Omega$ and $\tilde f$ are defined by (I.11). Then given $u_0\in H^s(\mathbf{K})$, $u_1\in H^{s-1}(\mathbf{K})$, $s\in[0,1]$, there exist unique solutions $\bar u\in C^0\left(\bar I;H^s(\mathbf{K})\right)\cap C^1\left(\bar I;H^{s-1}(\mathbf{K})\right)$ of (III.1) satisfying

$$\text{(III.19)}\quad\hat u(\tau_-)=u_0,\quad\partial_\tau\hat u(\tau_-)=u_1,$$

$$\text{(III.20)}\quad\lim_{\tau\to0^+}\check\Omega(\tau)^{\frac{n-1}{2}}\check u(\tau)=\lim_{\tau\to0^-}\hat\Omega^{\frac{n-1}{2}}(\tau)\hat u(\tau)\ \ in\ \ H^s(\mathbf{K}),$$

$$\text{(III.21)}\quad\lim_{\tau\to0^+}\partial_\tau\left[\check\Omega^{\frac{n-1}{2}}\check u\right](\tau)=\lim_{\tau\to0^-}\partial_\tau\left[\hat\Omega^{\frac{n-1}{2}}\hat u\right](\tau)\ \ in\ \ H^{s-1}(\mathbf{K}).$$

The function $\tilde u$ defined by

$$\text{(III.22)}\quad\tau\in[\tau_-,0)\Rightarrow\tilde u(\tau)=\hat u(\tau),\quad\tau\in(0,\tau_+]\Rightarrow\tilde u(\tau)=\check u(\tau),$$

satisfies

$$\text{(III.23)}\quad\tilde\Omega^{\frac{n-1}{2}}\tilde u\in C^0\left([\tau_-,\tau_+];H^s(\mathbf{K})\right)\cap C^1\left([\tau_-,\tau_+];H^{s-1}(\mathbf{K})\right),$$

$$\text{(III.24)}\quad\left[\partial_\tau^2-\Delta_{\mathbf{K}}+\frac{n-1}{4n}R_\gamma+\tilde m^2\tilde\Omega^2\right]\left(\tilde\Omega^{\frac{n-1}{2}}\tilde u\right)=\tilde\Omega^{\frac{n+3}{2}}\tilde f,\quad(\tau,\mathrm{x})\in(\tau_-,\tau_+)\times\mathbf{K},$$

The linear map $(\hat u(\tau_-),\partial_\tau\hat u(\tau_-))\mapsto(\check u(\tau_+),\partial_\tau\check u(\tau_+))$ is a homeomorphism on $H^s(\mathbf{K})\times H^{s-1}(\mathbf{K})$.

Proof. We introduce the linear maps

$$\text{(III.25)}\qquad \hat{\mathfrak{L}}(\tau)=\begin{pmatrix}\hat{\Omega}^{\frac{n-1}{2}}(\tau) & 0\\ \partial_\tau\left[\hat{\Omega}^{\frac{n-1}{2}}\right](\tau) & \hat{\Omega}^{\frac{n-1}{2}}(\tau)\end{pmatrix},\quad \check{\mathfrak{L}}(\tau)=\begin{pmatrix}\check{\Omega}^{\frac{n-1}{2}}(\tau) & 0\\ \partial_\tau\left[\check{\Omega}^{\frac{n-1}{2}}\right](\tau) & \check{\Omega}^{\frac{n-1}{2}}(\tau)\end{pmatrix}$$

that are isomorphisms on $H^s(\mathbf{K})\times H^{s-1}(\mathbf{K})$. We apply the previous proposition. First we solve the Cauchy problem (III.3), (III.6) on $\hat{I}$ with $(\hat{\varphi}_0,\hat{\varphi}_1)=\hat{\mathfrak{L}}(\tau_-)(u_0,u_1)$ and we put $\hat{u}(\tau)=\hat{\Omega}^{\frac{1-n}{2}}(\tau)\hat{\varphi}(\tau)$. Then we consider the solution $\check{\varphi}$ of (III.3) on $\check{I}$ satisfying $\check{\varphi}(0)=\hat{\psi}_0$, $\partial_\tau\check{\varphi}(0)=\hat{\psi}_1$, and we put $\check{u}(\tau)=\check{\Omega}^{\frac{1-n}{2}}(\tau)\check{\varphi}(\tau)$. Then $\tilde{u}$ satisfies (III.19), (III.20), (III.21), and these transmission conditions imply that $\tilde{u}$ is solution of (III.23) and (III.24). Finally the maps $(\hat{u}_0,\hat{u}_1)\mapsto\hat{\mathfrak{L}}(\tau_-)(\hat{u}_0,\hat{u}_1)=(\hat{\varphi}_0,\hat{\varphi}_1)\mapsto(\hat{\psi}_0,\hat{\psi}_1)=(\check{\psi}_0,\check{\psi}_1)\mapsto(\check{\varphi}(\tau_+),\partial_\tau\check{\varphi}(\tau_+))\mapsto\left[\check{\mathfrak{L}}(\tau_+)\right]^{-1}(\check{\varphi}(\tau_+),\partial_\tau\check{\varphi}(\tau_+))=(\check{u}(\tau_+),\partial_\tau\check{u}(\tau_+))$ are homeomorphisms on $H^s(\mathbf{K})\times H^{s-1}(\mathbf{K})$.

Q.E.D.

This result allows to treat the situation of the Singular Bouncing Scenario since the assumption (III.18) is very weak in this case: if the source $\tilde{f}=0$, it is sufficient that the mass is bounded and the convergences (II.3) are uniform. Then we have a natural propagation from the Previous Aeon to the Present Aeon despite the blow-up of the fields $\hat{u}$ and $\check{u}$ at the Bang Surface, since the normalized field $\tilde{\varphi}$ is a solution, continuous in time, of the equation (I.12). In contrast (III.18) is a very strong constraint for the expanding Aeons: for instance for the De Sitter like metric (II.13) the mass has to decay exponentially to that

$$\text{(III.26)}\qquad \int_{\tau_-}^0\hat{m}^2(\tau)\frac{d\tau}{\tau^2}\sim\int_{\hat{t}_-}^\infty\hat{m}^2(\hat{t})e^{\hat{H}\hat{t}}d\hat{t}<\infty.$$

In the next section we relax this assumption to be able to treat the CCC scenario with a much less drastic constraint on the decay of the mass.

IV. Asymptotics at the Bang Surface for a Slow Mass Decay

In this part, we assume that the mass and the conformal factor depend only on the time coordinate:

$$\text{(IV.1)}\qquad \bar{m}\in C^0\left(\bar{I}\right),\quad \bar{\Omega}\in C^2(\bar{I}).$$

In this framework the hypothesis (III.18) was

$$\text{(IV.2)}\qquad \int_{\bar{I}}\bar{m}^2(\tau)\bar{\Omega}^2(\tau)d\tau<\infty,$$

and we now investigate the asymptotic behaviour of $\bar{\varphi}$, $\partial_\tau\bar{\varphi}$ under the weaker assumption

$$\text{(IV.3)}\qquad \int_{\bar{I}}\bar{m}^2(\tau)\bar{\Omega}^2(\tau)\mid\tau\mid d\tau<\infty.$$

In particular for the De Sitter like metric (II.13) this constraint is

$$\text{(IV.4)}\qquad \int_{\tau_-}^0\hat{m}^2(\tau)\frac{d\tau}{\mid\tau\mid}\sim\int_{\hat{t}_-}^\infty\hat{m}^2(\hat{t})d\hat{t}<\infty$$

that is much more weaker than (III.26). Our strategy is based on the expression of $\bar{m}^2\bar{\Omega}^2$ that, *a priori*, does not belong to L^1, with an auxiliary fonction $\bar{A}$ that belongs to L^1. These functions are linked by the Riccati equation

$$\text{(IV.5)}\qquad \bar{A}'-\bar{A}^2=\bar{m}^2\bar{\Omega}^2.$$

This method has been initiated in [4] and used in [1]. The main motivation of this approach is the following fundamental result that treats the hard case of the blowing-up $\partial_\tau\bar{\varphi}$ and describes its asymptotic behaviour as $\tau \to 0$. To study the wave equation near $\tau = 0$, we introduce for $h > 0$ small enough, the interval

$$\bar{I}_h := \bar{I} \cap [-h, h]. \tag{IV.6}$$

Lemma IV.1. *Assume there exist $h > 0$ and $\bar{A} \in C^1 \cap L^1(\bar{I}_h)$ satisfying (IV.5) on $\bar{I}_h$. Then there exists $C > 0$ such that for the solution $\bar{\varphi} \in C^0\left(\bar{I}; H^s(\mathbf{K})\right) \cap C^1\left(\bar{I}; H^{s-1}(\mathbf{K})\right)$ of (III.3), (III.6) with $s \in [0,1]$ and $\bar{f}$ satisfying (III.10), we have for any $\tau \in \bar{I}_h$*

$$\begin{aligned} \|\bar{\varphi}(\tau)\|_{H^s(\mathbf{K})} + \|\partial_\tau \bar{\varphi}(\tau) + &\bar{A}(\tau)\bar{\varphi}(\tau)\|_{H^{s-1}(\mathbf{K})} \leqslant \\ & C\left(\|\bar{\varphi}_0\|_{H^s(\mathbf{K})} + \|\bar{\varphi}_1\|_{H^{s-1}(\mathbf{K})} + \left|\int_{\tau_0}^{\tau} \bar{\Omega}^{\frac{n+3}{2}}(\sigma)\|\bar{f}(\sigma)\|_{H^{s-1}(\mathbf{K})} d\sigma\right|\right), \end{aligned} \tag{IV.7}$$

and the following limits exist:

$$\bar{\psi}_0 := \lim_{\tau \to 0} \bar{\varphi}(\tau) \ \text{ in } \ H^s(\mathbf{K}), \tag{IV.8}$$

$$\bar{\psi}_1 := \lim_{\tau \to 0} \left(\partial_\tau \bar{\varphi}(\tau) + \bar{A}(\tau)\bar{\varphi}(\tau)\right) \ \text{ in } \ H^{s-1}(\mathbf{K}). \tag{IV.9}$$

Moreover the map $W_{\bar{A}} : (\bar{\varphi}_0, \bar{\varphi}_1) \mapsto (\bar{\psi}_0, \bar{\psi}_1)$ is a homeomorphism on $H^s(\mathbf{K}) \times H^{s-1}(\mathbf{K})$.

We emphasize that $\bar{A}(\tau)$ is allowed to blow-up as $\tau \to 0$.

Proof. Given $\bar{\tau}_h \in \bar{I}_h$, we put

$$\bar{\psi}(\bar{\tau}_h; \tau) := \bar{\varphi}(\tau) \exp\left(\int_{\bar{\tau}_h}^{\tau} \bar{A}(\sigma) d\sigma\right). \tag{IV.10}$$

Then $\bar{\psi}(\bar{\tau}_h; .)$ belongs to $C^0\left(\bar{I}_h; H^s(\mathbf{K})\right) \cap C^1\left(\bar{I}_h; H^{s-1}(\mathbf{K})\right)$ and is solution of

$$\left(\partial_\tau^2 - \Delta_{\mathbf{K}} + \frac{n-1}{4n} R_\gamma - 2\bar{A}\partial_\tau\right)\bar{\psi} = \bar{F}, \quad \bar{F} := \bar{\Omega}^{\frac{n+3}{2}} \bar{f} e^{\int_{\bar{\tau}_h}^{\tau} \bar{A}(\sigma)d\sigma}, \tag{IV.11}$$

$$\bar{\psi}(\bar{\tau}_h; \bar{\tau}_h) = \bar{\varphi}(\bar{\tau}_h), \quad \partial_\tau \bar{\psi}(\bar{\tau}_h; \bar{\tau}_h) = \partial_\tau \bar{\varphi}(\bar{\tau}_h) + \bar{A}(\bar{\tau}_h)\bar{\varphi}(\bar{\tau}_h). \tag{IV.12}$$

Following the proof of the Proposition III.1 , $\bar{\psi}(\bar{\tau}_h; .)$ is the solution integral equation

$$\begin{aligned} \begin{pmatrix} \bar{\psi}(\tau) \\ \partial_\tau \bar{\psi}(\tau) \end{pmatrix} =& e^{i(\tau - \bar{\tau}_h)\mathcal{A}} \begin{pmatrix} \bar{\psi}(\bar{\tau}_h; \bar{\tau}_h) \\ \partial_\tau \bar{\psi}(\bar{\tau}_h; \bar{\tau}_h) \end{pmatrix} \\ &+ \int_{\bar{\tau}_h}^{\tau} e^{i(\tau-\sigma)\mathcal{A}} \begin{pmatrix} 0 \\ \left[1 - \frac{n-1}{4n} R_\gamma\right] \bar{\psi}(\sigma) + 2\bar{A}(\sigma)\partial_\tau \bar{\psi}(\sigma) + \bar{F}(\sigma) \end{pmatrix} d\sigma, \end{aligned} \tag{IV.13}$$

and we have the energy inequality

$$\begin{aligned} \left\| \begin{pmatrix} \bar{\psi}(\tau) \\ \partial_\tau \bar{\psi}(\tau) \end{pmatrix} \right\|_{H^s \times H^{s-1}} \leqslant & \left\| \begin{pmatrix} \bar{\psi}(\bar{\tau}_h; \bar{\tau}_h) \\ \partial_\tau \bar{\psi}(\bar{\tau}_h; \bar{\tau}_h) \end{pmatrix} \right\|_{H^s \times H^{s-1}} + \left| \int_{\bar{\tau}_h}^{\tau} \|\bar{F}(\sigma)\|_{H^{s-1}} d\sigma \right| \\ & + \left| \int_{\bar{\tau}_h}^{\tau} \left(1 + \|R_\gamma\|_{L^\infty} + 2|\bar{A}(\sigma)|\right) \left\| \begin{pmatrix} \bar{\psi}(\sigma) \\ \partial_\tau \bar{\psi}(\sigma) \end{pmatrix} \right\|_{H^s \times H^{s-1}} d\sigma \right|. \end{aligned}$$

Since $\bar{A} \in L^1(\bar{I}_h)$, the Gronwall Lemma assures that there exists $C > 0$ independent of τ, $\bar{\tau}_h$ such that

$$\left\| \begin{pmatrix} \bar{\psi}(\tau) \\ \partial_\tau \bar{\psi}(\tau) \end{pmatrix} \right\|_{H^s \times H^{s-1}} \leqslant C \left(\left\| \begin{pmatrix} \bar{\psi}(\bar{\tau}_h; \bar{\tau}_h) \\ \partial_\tau \bar{\psi}(\bar{\tau}_h; \bar{\tau}_h) \end{pmatrix} \right\|_{H^s \times H^{s-1}} + \left| \int_{\bar{\tau}_h}^{\tau} \bar{\Omega}^{\frac{n+3}{2}}(\sigma) \|\bar{f}(\sigma)\|_{H^{s-1}} d\sigma \right| \right). \tag{IV.14}$$

Using (III.8) we have the estimate

$$\left(\begin{array}{c}\bar{\psi}(\bar{\tau}_h;\bar{\tau}_h)\\ \partial_\tau\bar{\psi}(\bar{\tau}_h;\bar{\tau}_h)\end{array}\right) \leqslant C'\left(\left\|\left(\begin{array}{c}\bar{\varphi}_0\\ \bar{\varphi}_1\end{array}\right)\right\|_{H^s\times H^{s-1}}+\left|\int_{\tau_0}^{\bar{\tau}_h}\bar{\Omega}^{\frac{n+3}{2}}(\sigma)\|\bar{f}(\sigma)\|_{H^{s-1}}d\sigma\right|\right), \tag{IV.15}$$

hence (IV.7) is a consequence of (IV.14) and (IV.15).

Using the integrability of $\bar{A}$ again, we can do $\tau\to 0$ in (IV.13) and we obtain the existence of the limits (IV.8), (IV.9) by putting

(IV.16)

$$\left(\begin{array}{c}\bar{\psi}_0\\ \bar{\psi}_1\end{array}\right):=e^{-i\bar{\tau}_h\mathcal{A}}\left(\begin{array}{c}\bar{\psi}(\bar{\tau}_h;\bar{\tau}_h)\\ \partial_\tau\bar{\psi}(\bar{\tau}_h;\bar{\tau}_h)\end{array}\right)+\int_{\bar{\tau}_h}^{0}e^{-i\sigma\mathcal{A}}\left(\begin{array}{c}0\\ \left[1-\frac{n-1}{4n}R_\gamma\right]\bar{\psi}(\sigma)+2\bar{A}(\sigma)\partial_\tau\bar{\psi}(\sigma)+\bar{F}(\sigma)\end{array}\right)d\sigma,$$

moreover (IV.14) holds with $\tau=0$, hence $W_{\bar{A}}$ is a well defined continuous map on $H^s\times H^{s-1}$.

Conversely, given $(\bar{\psi}_0,\bar{\psi}_1)\in H^s\times H^{s-1}$, we can solve, as for Proposition III.1, the integral equation

$$\left(\begin{array}{c}\bar{\psi}(\tau)\\ \partial_\tau\bar{\psi}(\tau)\end{array}\right)=e^{i\tau\mathcal{A}}\left(\begin{array}{c}\bar{\psi}_0\\ \partial_\tau\bar{\psi}_1\end{array}\right)+\int_0^{\tau}e^{i(\tau-\sigma)\mathcal{A}}\left(\begin{array}{c}0\\ \left[1-\frac{n-1}{4n}R_\gamma\right]\bar{\psi}(\sigma)+2\bar{A}(\sigma)\partial_\tau\bar{\psi}(\sigma)+\bar{F}(\sigma)\end{array}\right)d\sigma$$

and we obtain a unique solution $\bar{\psi}(0;.)\in C^0\left(\bar{I}_h\cup\{0\};H^s(\mathbf{K})\right)\cap C^1\left(\bar{I}_h\cup\{0\};H^{s-1}(\mathbf{K})\right)$, and we have

$$\left(\begin{array}{c}\bar{\psi}(0;\tau)\\ \partial_\tau\bar{\psi}(0;\tau)\end{array}\right)\leqslant C'\left(\left\|\left(\begin{array}{c}\bar{\psi}_0\\ \bar{\psi}_1\end{array}\right)\right\|_{H^s\times H^{s-1}}+\left|\int_0^{\tau}\bar{\Omega}^{\frac{n+3}{2}}(\sigma)\|\bar{f}(\sigma)\|_{H^{s-1}}d\sigma\right|\right). \tag{IV.17}$$

Finally we solve the Cauchy problem for (III.3) with initial data $\bar{\varphi}(\bar{\tau}_h)=\bar{\psi}(0;\bar{\tau}_h)$, $\partial_\tau\bar{\varphi}(\bar{\tau}_h)=\partial_\tau\bar{\psi}(0;\bar{\tau}_h)-\bar{A}(\bar{\tau}_h)\bar{\psi}(0;\bar{\tau}_h)$, and we can invert $W_{\bar{A}}$ by putting $\bar{\varphi}_0:=\bar{\varphi}(\tau_0)$, $\bar{\varphi}_1:=\partial_\tau\bar{\varphi}(\tau_0)$. The bi-continuity of $W_{\bar{A}}:\ (\bar{\varphi}_0,\bar{\varphi}_1)\mapsto(\bar{\psi}_0,\bar{\psi}_1)$ is assured by the estimates (IV.15) and IV.17).

Q.E.D.

Lemma IV.2. *Assume there exist $h>0$, $\hat{A}\in C^1\cap L^1([-h,0))$ and $\check{A}\in C^1\cap L^1((0,h])$ satisfying (IV.5). Then given $\hat{\varphi}_0\in H^s(\mathbf{K})$, $\hat{\varphi}_1\in H^{s-1}(\mathbf{K})$ and $\tilde{f}$ satisfying (III.10), there exists a unique $\tilde{\varphi}\in C^0\left([\tau_-,\tau_+];H^s(\mathbf{K})\right)\cap C^1\left([\tau_-,0)\cup(0,\tau_+];H^{s-1}(\mathbf{K})\right)$ such that*

$$\tilde{\varphi}(\tau_-)=\hat{\varphi}_0,\quad \partial_\tau\tilde{\varphi}(\tau_-)=\hat{\varphi}_1, \tag{IV.18}$$

$$\left[\partial_\tau^2-\Delta_{\mathbf{K}}+\frac{n-1}{4n}R_\gamma+\tilde{m}^2\tilde{\Omega}^2\right]\tilde{\varphi}=\tilde{\Omega}^{\frac{n+3}{2}}\tilde{f},\quad (\tau,\mathrm{x})\in(\tau_-,\tau_+)\backslash\{0\}\times\mathbf{K} \tag{IV.19}$$

and the function $\tilde{\psi}$ defined by

$$\tilde{\psi}(\tau):=\tilde{\varphi}(\tau)\exp\left(\int_0^{\tau}\tilde{A}(\sigma)d\sigma\right) \tag{IV.20}$$

belongs to $C^0\left([-h,h];H^s(\mathbf{K})\right)\cap C^1\left([-h,h];H^{s-1}(\mathbf{K})\right)$, and satisfies

$$\left(\partial_\tau^2-\Delta_{\mathbf{K}}+\frac{n-1}{4n}R_\gamma-2\tilde{A}\partial_\tau\right)\tilde{\psi}=\tilde{\Omega}^{\frac{n+3}{2}}\tilde{f}e^{\int_0^{\tau}\tilde{A}(\sigma)d\sigma}\ \ in\ \ (-h,h)\times\mathbf{K}, \tag{IV.21}$$

where $\tilde{m}$, $\tilde{\Omega}$ and $\tilde{f}$ are defined by (I.11) and $\tilde{A}\in L^1(-h,h)$ is given by

$$\tau\in(-h,0)\Rightarrow\tilde{A}(\tau)=\hat{A}(\tau),\quad \tau\in(0,h)\Rightarrow\tilde{A}(\tau)=\check{A}(\tau). \tag{IV.22}$$

The map $S_{\tilde{A}}:\ (\tilde{\varphi}(\tau_-),\partial_\tau\tilde{\varphi}(\tau_-))\mapsto(\tilde{\varphi}(\tau_+),\partial_\tau\tilde{\varphi}(\tau_+))$ is a homeomorphism of $H^s(\mathbf{K})\times H^{s-1}(\mathbf{K})$.

Proof. For $\tau \in \hat{I}$ we define $\tilde{\varphi}(\tau,.) = \hat{\varphi}(\tau,.)$ where $\hat{\varphi}$ is the solution on $\hat{I}$ of (III.3), (III.6) with $\tau_0 = \tau_-$. Using the notations of Lemma IV.1, we introduce $(\check{\varphi}_0, \check{\varphi}_1) := W_{\check{A}}^{-1} W_{\hat{A}}(\hat{\varphi}_0, \hat{\varphi}_1)$, and for $\tau \in \check{I}$ we define $\tilde{\varphi}(\tau,.) = \check{\varphi}(\tau,.)$ where $\check{\varphi}$ is the solution on $\check{I}$ of (III.3), (III.6) with $\tau_0 = \tau_+$. Since $\lim_{\tau\to 0^-} \hat{\varphi}(\tau) = \lim_{\tau \to 0^+} \check{\varphi}(\tau)$, $\tilde{\varphi}$ belongs to $C^0\left([\tau_-, \tau_+]; H^s(\mathbf{K})\right) \cap C^1\left([\tau_-, 0) \cup (0, \tau_+]; H^{s-1}(\mathbf{K})\right)$ and it is the solution of (IV.18) and (IV.19). Moreover $\tilde{\psi}$ defined by (IV.20) satisfies

$$\tau \in (-h,0) \Rightarrow \hat{\psi}(\tau) = \hat{\psi}(\hat{\tau}_h;\tau) \exp\left(\int_0^{\hat{\tau}_h} \hat{A}(\sigma)d\sigma\right), \ \tau \in (0,h) \Rightarrow \check{\psi}(\tau) = \check{\psi}(\check{\tau}_h;\tau) \exp\left(\int_0^{\check{\tau}_h} \check{A}(\sigma)d\sigma\right),$$

$$\lim_{\tau\to 0^-} \hat{\psi}(\tau) = \lim_{\tau\to 0^+} \check{\psi}(\tau), \quad \lim_{\tau\to 0^-} \partial_\tau \hat{\psi}(\tau) = \lim_{\tau\to 0^+} \partial_\tau \check{\psi}(\tau),$$

therefore $\tilde{\psi} \in C^0\left([-h,h]; H^s(\mathbf{K})\right) \cap C^1\left([-h,h]; H^{s-1}(\mathbf{K})\right)$, and satisfies (IV.21). Finally $S_{\tilde{A}} = W_{\check{A}}^{-1} W_{\hat{A}}$ is a homeomorphism.

Q.E.D.

Now given the mass and the conformal factor, we have to solve the Riccati equation near $\tau = 0$.

Proposition IV.3. *1) If $\bar{m}$ and $\bar{\Omega}$ satisfy (IV.2), then given $\bar{\alpha} \in \mathbb{R}$, there exist $h > 0$ and a unique $\bar{A} \in C^0(\{0\} \cup \bar{I}_h) \cap C^1(\bar{I}_h)$ solution of (IV.5) with $\bar{A}(0) = \bar{\alpha}$.*

2) We assume that $\bar{m}$ and $\bar{\Omega}$ satisfy (IV.3). Then for any $\epsilon > 0$, there exist $h_\epsilon > 0$, $\tau_\epsilon \in \bar{I}_{h_\epsilon}$ and $\bar{A}_\epsilon \in C^1(\bar{I}_{h_\epsilon})$ a real solution de (IV.5) satisfying

$$\bar{A}_\epsilon \in L^1(\bar{I}_{h_\epsilon}), \tag{IV.23}$$

$$\bar{\eta} \int_{\tau_\epsilon}^{\tau} \bar{m}^2(\sigma)\bar{\Omega}^2(\sigma)d\sigma \leqslant \bar{\eta}\bar{A}_\epsilon(\tau) \leqslant \bar{\eta}(1+\epsilon)\int_{\tau_\epsilon}^{\tau} \bar{m}^2(\sigma)\bar{\Omega}^2(\sigma)d\sigma, \ \ \hat{\eta} := +, \ \ \check{\eta} := -, \tag{IV.24}$$

$$\bar{\eta} \int_{\tau_\epsilon}^{0} \mid \bar{A}_\epsilon(\tau) \mid d\tau \leqslant \frac{\epsilon}{1+\epsilon}. \tag{IV.25}$$

The solutions $\bar{A} \in C^1(\bar{I}_h)$ of (IV.5) for some $h > 0$ are, either integrable, or satisfying

$$\bar{A}(\tau) - \bar{A}_\epsilon(\tau) \sim -\tau^{-1}, \ \ \tau \to 0. \tag{IV.26}$$

Moreover if $(\bar{m}, \bar{\Omega})$ do not satisfy (IV.2), then any real solution $\bar{A} \in C^1 \cap L^1(\bar{I}_h)$ solution of (IV.5) for some $h > 0$, satisfies

$$\bar{A}(\tau) \longrightarrow \bar{\eta}\infty, \ \ \tau \to 0, \tag{IV.27}$$

$$\exists \bar{\alpha} := \lim_{\tau\to 0} \bar{A}(\tau) - \bar{A}_\epsilon(\tau) \in \mathbb{R}. \tag{IV.28}$$

Furthermore, for any $\bar{\alpha} \in \mathbb{R}$ there exists a unique $\bar{A} \in L^1 \cap C^1(\bar{I}_h)$ solution of (IV.5) for some $h > 0$, satisfying (IV.28).

Proof. 1) Since $\bar{m}\bar{\Omega}$ can be unbounded as $\tau \to 0$, the first assertion is not a direct consequence of the Cauchy-Lipschitz theorem but it proved by the usual way. If $\bar{m}\bar{\Omega} \in L^2(\bar{I})$, we have to solve

$$\bar{A}(\tau) = \mathcal{G}(\bar{A})(\tau) := \bar{\alpha} + \int_0^\tau \bar{m}^2(\sigma)\bar{\Omega}^2(\sigma)d\sigma + \int_0^\tau \bar{A}^2(\sigma)d\sigma.$$

We take $h > 0$ small enough to that

$$\int_{\bar{I}\cap[-h,h]} \bar{m}^2(\sigma)\bar{\Omega}^2(\sigma)d\sigma \leqslant 1, \ \ 0 < h < \frac{1}{(2+\mid \bar{\alpha} \mid)^2}.$$

We can easily check that $\mathcal{G}$ is a strict contraction on $\left\{A \in C^0(\{0\} \cup \bar{I}_h), \ \ \|A - \bar{\alpha}\|_\infty \leqslant 2\right\}$. Therefore its unique fixed point $\bar{A}$ is the wanted solution.

2) We first construct $\hat{A}_\epsilon \in C^1([\tau_\epsilon, 0))$ solution of

$$\hat{A}_\epsilon(\tau) = \int_{\tau_\epsilon}^{\tau} \bar{m}^2(\sigma)\bar{\Omega}^2(\sigma)d\sigma + \int_{\tau_\epsilon}^{\tau} \hat{A}_\epsilon^2(\sigma)d\sigma \tag{IV.29}$$

Thanks to the Fubini theorem and (IV.3) we choose $\tau_\epsilon = -h_\epsilon \in (\tau_-, 0)$ such that

$$\int_{\tau_\epsilon}^{0}\left(\int_{\tau_\epsilon}^{\tau} \bar{m}^2(\sigma)\bar{\Omega}^2(\sigma)d\sigma\right)d\tau = \int_{\tau_\epsilon}^{0} \bar{m}^2(\tau)\bar{\Omega}^2(\tau) \mid \tau \mid d\tau \leqslant \frac{\epsilon}{(1+\epsilon)^2}. \tag{IV.30}$$

We define a sequence $A_n \in C^1([\tau_\epsilon, 0))$, $n \in \mathbb{N}$, by

$$\forall \tau \in [\tau_\epsilon, 0),\ \ A_0(\tau) = 0,\ \ A_{n+1}(\tau) := \int_{\tau_\epsilon}^{\tau} \bar{m}^2(\sigma)\bar{\Omega}^2(\sigma)d\sigma + \int_{\tau_\epsilon}^{\tau} A_n^2(\sigma)d\sigma. \tag{IV.31}$$

We have $A_n \geqslant 0$ and

$$A_1 - A_0 \geqslant 0,\ \ A_{n+1}(\tau) - A_n(\tau) = \int_{\tau_\epsilon}^{\tau} \left(A_n(\sigma) + A_{n-1}(\sigma)\right)\left((A_n(\sigma) - A_{n-1}(\sigma)\right),$$

hence by recurrence we deduce that

$$\int_{\tau_\epsilon}^{\tau} \bar{m}^2(\sigma)\bar{\Omega}^2(\sigma)d\sigma \leqslant A_n(\tau) \leqslant A_{n+1}(\tau). \tag{IV.32}$$

Assume that

$$A_n(\tau) \leqslant (1+\epsilon)\int_{\tau_\epsilon}^{\tau} \bar{m}^2(\sigma)\bar{\Omega}^2(\sigma)d\sigma. \tag{IV.33}$$

Then we have with (IV.30):

$$\begin{aligned}
A_{n+1}(\tau) &\leqslant \int_{\tau_\epsilon}^{\tau} \bar{m}^2(\sigma)\bar{\Omega}^2(\sigma)d\sigma + (1+\epsilon)^2 \int_{\tau_\epsilon}^{\tau}\left(\int_{\tau_\epsilon}^{\sigma} \bar{m}^2(\zeta)\bar{\Omega}^2(\zeta)d\zeta\right)^2 d\sigma \\
&\leqslant \int_{\tau_\epsilon}^{\tau} \bar{m}^2(\sigma)\bar{\Omega}^2(\sigma)d\sigma + (1+\epsilon)^2 \int_{\tau_\epsilon}^{\tau} \bar{m}^2(\sigma)\bar{\Omega}^2(\sigma)d\sigma \int_{\tau_\epsilon}^{\tau}\left(\int_{\tau_\epsilon}^{\sigma} \bar{m}^2(\zeta)\bar{\Omega}^2(\zeta)d\zeta\right) d\sigma \\
&\leqslant (1+\epsilon)\int_{\tau_\epsilon}^{\tau} \bar{m}^2(\sigma)\bar{\Omega}^2(\sigma)d\sigma,
\end{aligned}$$

hence (IV.33) is established for any n. Taking advantage of (IV.32) and (IV.33) we can introduce the measurable function

$$\hat{A}_\epsilon(\tau) := \lim_{n\to\infty} A_n(\tau) \tag{IV.34}$$

That satisfies (IV.24). Thanks to the Beppo Levi theorem we deduce from (IV.31) that $\hat{A}_\epsilon$ satisfies also (IV.29) and then $\hat{A}_\epsilon \in C^1([\tau_\epsilon, 0))$. Now (IV.23) and (IV.25) follow from (IV.24) and (IV.30) by integration. The construction of $\check{A}_\epsilon$ is similar: we apply the previous procedure to $-\check{A}(-\tau)$.

Now given a real solution $\bar{A} \in C^1(\bar{I}_h))$ of (IV.5), we have

$$-h \leqslant \tau < 0 \Rightarrow \hat{A}(\tau) \geqslant A(-h) + \int_{-h}^{\tau} \bar{m}^2\bar{\Omega}^2 d\sigma,\ \ 0 < \tau \leqslant h \Rightarrow \check{A}(\tau) \leqslant \check{A}(h) + \int_{h}^{\tau} \bar{m}^2(\sigma)\bar{\Omega}^2(\sigma)d\sigma,$$

hence (IV.27) is satisfied if $(\bar{m}, \bar{\Omega})$ do not satisfy (IV.2). Now $\delta := \bar{A} - \bar{A}_\epsilon$ is solution of the equation

$$\delta'(\tau) = \delta(\tau)\left[\bar{A}(\tau) + \bar{A}_\epsilon(\tau)\right],$$

hence given $\tau_0 \in \bar{I}_h \cap \bar{I}_{h_\epsilon}$ we have

$$\delta(\tau) = \delta(\tau_0)\exp\left(\int_{\tau_0}^{\tau} \bar{A}(\sigma) + \bar{A}_\epsilon(\sigma)d\sigma\right) \longrightarrow \bar{\alpha} := \delta(\tau_0)\exp\left(\int_{\tau_0}^{0} \bar{A}(\sigma) + \bar{A}_\epsilon(\sigma)d\sigma\right),\ \ \tau \to 0,$$

and (IV.28) is proved. Moreover δ is also a solution of the Bernouilli equation

$$\delta'(\tau) - 2\bar{A}_\epsilon(\tau)\delta(\tau) = \delta^2(\tau),$$

hence if $\bar{A} \neq \bar{A}_\epsilon$, δ^{-1} is a solution of the linear equation

$$\zeta'(\tau) + 2\bar{A}_\epsilon(\tau)\zeta(\tau) = -1,$$

then we conclude that all the real solutions $\bar{A} \neq \bar{A}_\epsilon$ of the Riccati equation are given near $\tau = 0$ by

$$\bar{A}(\tau) = \bar{A}_\epsilon + e^{2\int_{\tau_0}^{\tau} \bar{A}_\epsilon(\sigma)d\sigma} \left(\delta(\tau_0) - \int_{\tau_0}^{\tau} e^{2\int_{\tau_0}^{\sigma} \bar{A}_\epsilon(s)ds} d\sigma\right)^{-1}, \tag{IV.35}$$

and these solutions are in L^1 near zero iff $\delta(\tau_0) \neq \int_{\tau_0}^0 e^{2\int_{\tau_0}^{\sigma} \bar{A}_\epsilon(s)ds} d\sigma$ and satisfy (IV.26) if $\delta(\tau_0) = \int_{\tau_0}^0 e^{2\int_{\tau_0}^{\sigma} \bar{A}_\epsilon(s)ds} d\sigma$. Finally given $\bar{\alpha} \in \mathbb{R}$, we put

$$\bar{A}(\tau) := \bar{A}_\epsilon(\tau) + \bar{\alpha} e^{2\int_0^{\tau} \bar{A}_\epsilon(\sigma)d\sigma} \left(1 + \bar{\alpha} \int_{\tau}^{0} e^{2\int_{\tau}^{\sigma} \bar{A}_\epsilon(s)ds} d\sigma\right)^{-1} \tag{IV.36}$$

that is a solution of (IV.5) that is well defined in $C^1 \cap L^1$ near $\tau = 0$ and satisfies (IV.28). To prove the uniqueness, we consider two solutions $\bar{A}, \bar{A}_* \in C^1 \cap L^1(\bar{I}_h)$ satisfying (IV.28). Then given $\tau_0 \in \bar{I}_h$ we have

$$\left(\bar{A}(\tau_0) - \bar{A}_*(\tau_0)\right) exp\left(\int_{\tau_0}^{\tau} \bar{A}(\sigma) + \bar{A}_*(\sigma)d\sigma\right) \to 0, \quad \tau \to 0.$$

We conclude that $\bar{A}(\tau_0) = \bar{A}_*(\tau_0)$ and therefore $\bar{A} = \bar{A}_*$.

Q.E.D.

Since the existence of the solutions of the Riccati equation is established, we can state the main result of this part.

Theorem IV.4. *We assume (II.25) and*

$$\int_{\tau_-}^{0} \hat{m}^2(\tau)\hat{\Omega}^2(\tau) \mid \tau \mid d\tau + \int_0^{\tau_+} \check{m}^2(\tau)\check{\Omega}^2(\tau)\tau d\tau < \infty. \tag{IV.37}$$

Then given $u_0 \in H^s(\mathbf{K})$, $u_1 \in H^{s-1}(\mathbf{K})$, $s \in [0,1]$, and $\bar{f}$ satisfying (III.10), given real solutions $\bar{A} \in C^1 \cap L^1(\bar{I}_h))$ of (IV.5) for some $h > 0$, there exist unique solutions $\bar{u} \in C^0\left(\bar{I}; H^s(\mathbf{K})\right) \cap C^1\left(\bar{I}; H^{s-1}(\mathbf{K})\right)$ of (III.1) satisfying

$$\hat{u}(\tau_-) = u_0, \quad \partial_\tau \hat{u}(\tau_-) = u_1, \tag{IV.38}$$

$$\lim_{\tau\to 0^+} \check{\Omega}^{\frac{n-1}{2}}(\tau)\check{u}(\tau) = \lim_{\tau\to 0^-} \hat{\Omega}^{\frac{n-1}{2}}(\tau)\hat{u}(\tau) \;\; in \;\; H^s(\mathbf{K}), \tag{IV.39}$$

$$\begin{aligned} &\lim_{\tau\to 0^+} \left(\partial_\tau\left[\check{\Omega}^{\frac{n-1}{2}}\check{u}\right](\tau) + \check{A}(\tau)\check{\Omega}^{\frac{n-1}{2}}(\tau)\check{u}(\tau)\right) \\ &\quad = \lim_{\tau\to 0^-} \left(\partial_\tau\left[\hat{\Omega}^{\frac{n-1}{2}}\hat{u}\right](\tau) + \hat{A}(\tau)\hat{\Omega}^{\frac{n-1}{2}}(\tau)\hat{u}(\tau)\right) \;\; in \;\; H^{s-1}(\mathbf{K}). \end{aligned} \tag{IV.40}$$

With the notations (I.11), (III.22), (IV.22), we have

$$\tilde{\Omega}^{\frac{n-1}{2}} e^{\int_0^{\tau} \tilde{A}(\sigma)d\sigma}\tilde{u} \in C^0\left([-h,h]; H^s(\mathbf{K})\right) \cap C^1\left([-h,h]; H^{s-1}(\mathbf{K})\right), \tag{IV.41}$$

$$\left(\partial_\tau^2 - \Delta_{\mathbf{K}} + \frac{n-1}{4n}R_\gamma - 2\tilde{A}\partial_\tau\right)\left(\tilde{\Omega}^{\frac{n-1}{2}} e^{\int_0^{\tau} \tilde{A}(\sigma)d\sigma}\tilde{u}\right) = \tilde{\Omega}^{\frac{n+3}{2}} e^{\int_0^{\tau} \tilde{A}(\sigma)d\sigma}\tilde{f} \;\; in \;\; (-h,h) \times \mathbf{K}, \tag{IV.42}$$

The linear map $\mathfrak{S}: \;(\hat{u}(\tau_-), \partial_\tau\hat{u}(\tau_-)) \mapsto (\check{u}(\tau_+), \partial_\tau\check{u}(\tau_+))$ is a continuous homeomorphism on $H^s(\mathbf{K}) \times H^{s-1}(\mathbf{K})$.

Remark IV.5. *If (III.9) is satisfied, the transmissions conditions (III.20), (III.21) correspond to (IV.39), (IV.40) by choosing the solutions $\bar{A}$ of the Riccati equation with $\bar{A}(0) = 0$. There is less real freedoms in (IV.40) than is apparent from the two arbitrary functions $\hat{A}$ and $\check{A}$. In fact these transmission conditions (IV.39), (IV.40) form a one-real-parameter family: if we fix two solutions $\hat{A}_\epsilon$, $\check{A}_\epsilon$, then (IV.28) and (IV.39) assure that putting $\delta := \hat{\alpha} - \check{\alpha}$, (IV.40) is equivalent to*

$$\text{(IV.43)}\quad \begin{aligned}&\lim_{\tau\to 0^+}\left(\partial_\tau\left[\check{\Omega}^{\frac{n-1}{2}}\check{u}\right](\tau) + \check{A}_\epsilon(\tau)\check{\Omega}^{\frac{n-1}{2}}(\tau)\check{u}(\tau)\right)\\ &= \lim_{\tau\to 0^-}\left(\partial_\tau\left[\hat{\Omega}^{\frac{n-1}{2}}\hat{u}\right](\tau) + \left(\hat{A}_\epsilon(\tau)+\delta\right)\hat{\Omega}^{\frac{n-1}{2}}(\tau)\hat{u}(\tau)\right) \quad in \quad H^{s-1}(\mathbf{K}).\end{aligned}$$

Conversely, given $\delta \in \mathbb{R}$, the last assertion of Proposition IV.3 assures that there exists $\bar{A}$ satisfying (IV.39). We conclude that the whole family of the conditions (IV.39), (IV.40) indexed by the integrable solutions $\bar{A}$ of the Riccati equation, is reduced to the one-parameter family of transmission conditions (IV.39), (IV.43) indexed by the real parameter $\delta \in \mathbb{R}$.

Proof. The existence of $\hat{u}$ is given by Proposition III.1 by taking $\hat{u}(\tau) := \hat{\Omega}^{\frac{1-n}{2}}(\tau)\hat{\varphi}(\tau)$ where $\hat{\varphi}$ is the solution of (III.3), (III.6) with the initial data $(\hat{\varphi}_0, \hat{\varphi}_1) = \hat{\mathfrak{L}}(\tau_-)(u_0, u_1)$ given at $\tau_0 = \tau_-$ and $\hat{\mathfrak{L}}$ is defined by (III.25) . Lemma IV.1 assures that the following limits exist

$$\hat{\psi}_0 := \lim_{\tau\to 0^-}\hat{\Omega}^{\frac{n-1}{2}}(\tau)\hat{u}(\tau) \quad in \quad H^s(\mathbf{K}),$$

$$\hat{\psi}_1 := \lim_{\tau\to 0^-}\left(\partial_\tau\left[\hat{\Omega}^{\frac{n-1}{2}}\hat{u}\right](\tau) + \hat{A}(\tau)\hat{\Omega}^{\frac{n-1}{2}}(\tau)\hat{u}(\tau)\right) \quad in \quad H^{s-1}(\mathbf{K}),$$

and we can define

$$(\check{\varphi}_0, \check{\varphi}_1) := W_{\check{A}}^{-1}(\hat{\psi}_0, \hat{\psi}_1).$$

We now define $\check{u}(\tau) := \check{\Omega}^{\frac{1-n}{2}}(\tau)\check{\varphi}(\tau)$ with the solution $\check{\varphi}$ of of (III.3), (III.6) with $\tau_0 = \tau_+$. Then (IV.39) and (IV.40) are direct consequences of this construction. Finally we invoke Lemma IV.2 to conclude that (IV.42) is deduced from (IV.21) and

$$\mathfrak{S} = \left[\check{\mathfrak{L}}(\tau_+)\right]^{-1} S_{\tilde{A}}\hat{\mathfrak{L}}(\tau_-)$$

is a homeomorphism on $H^s(\mathbf{K}) \times H^{s-1}(\mathbf{K})$.

Q.E.D.

It is well known that the Riccati equations cannot be solved by quadrature nevertheless we can give an explicit formulation of the transmission conditions in the following important case.

Example IV.6. *We consider the case where*

$$\text{(IV.44)}\qquad \bar{m}^2(\tau)\bar{\Omega}^2(\tau) = \bar{c}^2 \mid \tau \mid^{-1} + \bar{F}(\tau)$$

with $\bar{c} > 0$ and $\bar{F}$ is a holomorphic function on a neigborhood of zero. We take a (generalized) eigenfunction $\Phi_\lambda \in L^2_{loc}(\mathbf{K})$ solution of $\left(-\Delta_{\mathbf{K}} + \frac{n-1}{4n}R_\gamma\right)\Phi_\lambda = \lambda\Phi_\lambda$, $\lambda \in \mathbb{R}$. The solutions $\bar{\varphi}$ of (III.3) with $\bar{f} = 0$, of type $\bar{\varphi}(\tau, \mathrm{x}) = \bar{\phi}(\tau)\Phi_\lambda(\mathrm{x})$, are defined by the solutions $\bar{\phi}$ of the ODE

$$\text{(IV.45)}\qquad \bar{\phi}''(\tau) + \left(\lambda + \frac{\bar{c}^2}{\mid \tau \mid} + \bar{F}(\tau)\right)\bar{\phi}(\tau) = 0, \quad \tau \in \bar{I}.$$

We can solve this ODE by the Frobenius method and since $\tau = 0$ is a regular singular point, the Fuchs theorem assures that there exist two functions $\bar{h}_1$, $\bar{h}_2$ which are analytic near zero with $\bar{h}_1(0) = \bar{h}_2(0) = 1$, and the general solution of (IV.45) can be written as

$$\text{(IV.46)}\qquad \bar{\phi}(\tau) = \bar{C}_1\tau\bar{h}_1(\tau) + \bar{C}_2\left[\bar{h}_2(\tau) - \bar{c}^2 \mid \tau \mid \bar{h}_1(\tau)\ln(\mid \tau \mid)\right], \quad \bar{C}_j \in \mathbb{C}.$$

We deduce that

$$\text{(IV.47)}\qquad \lim_{\tau\to 0}\bar{\phi}(\tau) = \bar{C}_2, \quad \bar{\phi}'(\tau) = \bar{\eta}\bar{C}_2\bar{c}^2\ln(\mid \tau \mid) + \bar{C}_1 + \bar{C}_2\left(\bar{h}_2'(0) + \bar{\eta}\bar{c}^2\right) + o(1), \ \tau \to 0,$$

with $\hat{\eta}=+$, $\check{\eta}=-$. Now the Riccati equation (IV.5) is reduced to a linear second order ODE by the usual way, by putting $\bar{A}=-\frac{\bar{\alpha}'}{\bar{\alpha}}$ where $\bar{\alpha}$ is a solution of

$$\bar{\alpha}''(\tau)+\left(\frac{\bar{c}^2}{\mid\tau\mid}+\bar{F}(\tau)\right)\bar{\alpha}(\tau)=0.$$

We apply the Fuchs theorem again: there exist two functions $\bar{k}_1$, $\bar{k}_2$, holomorphic near zero, with $\bar{k}_j(0)=1$ and

$$\bar{\alpha}(\tau)=\bar{D}_1\tau\bar{k}_1(\tau)+\bar{D}_2\left[\bar{k}_2(\tau)-\bar{c}^2\mid\tau\mid\bar{k}_1(\tau)\ln(\mid\tau\mid)\right],\quad \bar{D}_j\in\mathbb{C}.$$

We get that the solutions of the Riccati equation are given by

$$\bar{A}(\tau)=-\frac{\bar{D}_1\left[\bar{k}_1(\tau)+\tau\bar{k}_1'(\tau)\right]+\bar{D}_2\left[\bar{k}_2'(\tau)+\bar{\eta}\bar{c}^2\bar{k}_1(\tau)\ln(\mid\tau\mid)-\bar{c}^2\mid\tau\mid\bar{k}_1'(\tau)\ln(\mid\tau\mid)+\bar{\eta}\bar{c}^2\bar{k}_1(\tau)\right]}{\bar{D}_1\tau\bar{k}_1(\tau)+\bar{D}_2\left[\bar{k}_2(\tau)-\bar{c}^2\mid\tau\mid\bar{k}_1(\tau)\ln(\mid\tau\mid)\right]}.$$

If $\bar{D}_2=0$ we have $\bar{A}(\tau)\sim-\tau^{-1}$ as $\tau\to 0$. We conclude that the solutions of the Riccati equation that are integrable near zero satisfy $\bar{D}_2\neq 0$, and then

$$\bar{A}(\tau)=-\bar{\eta}\bar{c}^2\ln(\mid\tau\mid)-\frac{\bar{D}_1}{\bar{D}_2}-\bar{k}_2'(0)-\bar{\eta}\bar{c}^2+o(1),\quad \tau\to 0.$$

Then putting

$$\delta:=\hat{h}_2'(0)-\check{h}_2'(0)+\check{k}_2'(0)-\hat{k}_2'(0)+\frac{\check{D}_1}{\check{D}_2}-\frac{\hat{D}_1}{\hat{D}_2},$$

the family of transmission conditions (IV.39), (IV.40) has the form

$$\check{C}_2=\hat{C}_2, \tag{IV.48}$$

$$\check{C}_1=\hat{C}_1+\delta\hat{C}_2 \tag{IV.49}$$

where the real parameter δ can be arbitrarily choosen in $\mathbb{R}$.

V. SELF-INTERACTING SCALAR FIELD

In this section we assume that the complete manifold $\mathbf{K}$ is a 3-dimensional C^∞ *bounded geometry manifold*, and we investigate the massive semilinear Klein-Gordon equation

$$\left[\Box_{\bar{g}}+\frac{1}{6}R_{\bar{g}}+\bar{m}^2\right]\bar{u}=-\kappa\mid u\mid^2 u\ \ in\ \ \bar{\mathcal{M}}, \tag{V.1}$$

where $\kappa>0$ is a coupling constant. The Liouville transform $\bar{\varphi}:=\bar{\Omega}\bar{u}$ leads to the equivalent equation

$$\left[\Box_g+\frac{1}{6}R_\gamma+\bar{m}^2\bar{\Omega}^2\right]\bar{\varphi}=-\kappa\mid\bar{\varphi}\mid^2\bar{\varphi}\quad in\ \ \bar{\mathcal{M}}. \tag{V.2}$$

First we suppose that the mass and the conformal factor satisfy (III.4) and we solve the global Cauchy problem.

Proposition V.1. *We assume that (III.4) holds. Then given $\tau_0\in\bar{I}$, $\bar{\varphi}_0\in H^1(\mathbf{K})$, $\bar{\varphi}_1\in L^2(\mathbf{K})$, the equation (V.2) has a unique solution $\bar{\varphi}$ satisfying*

$$\bar{\varphi}\in C^0\left(\bar{I};H^1(\mathbf{K})\right)\cap C^1\left(\bar{I};L^2(\mathbf{K})\right) \tag{V.3}$$

$$\bar{\varphi}(\tau_0)=\bar{\varphi}_0,\quad \partial_\tau\bar{\varphi}(\tau_0)=\bar{\varphi}_1. \tag{V.4}$$

Moreover there exists $C > 0$ such that any solution satisfies

$$\|\bar{\varphi}(\tau)\|_{H^1(\mathbf{K})} + \|\partial_\tau\bar{\varphi}(\tau)\|_{L^2(\mathbf{K})} \leqslant C\left(\|\bar{\varphi}_0\|_{H^1(\mathbf{K})} + \|\bar{\varphi}_0\|^2_{H^1(\mathbf{K})} + \|\bar{\varphi}_1\|_{L^2(\mathbf{K})}\right) \times \exp\left(\frac{1}{2}\left|\int_{\tau_0}^{\tau}\left\|\bar{m}(\sigma)\bar{\Omega}(\sigma)\right\|^2_{L^\infty(\mathbf{K})}d\sigma\right|\right), \tag{V.5}$$

and the map $(\bar{\varphi}_0, \bar{\varphi}_1) \in H^1 \times L^2 \mapsto \bar{\varphi} \in C^0(\bar{I}; H^1) \cap C^1(\bar{I}; L^2)$ is continuous.

If we assume (III.9), then the following limits exist:

$$\bar{\psi}_0 := \lim_{\tau\to 0}\bar{\varphi}(\tau) \;\; in \;\; H^1(\mathbf{K}), \tag{V.6}$$

$$\bar{\psi}_1 := \lim_{\tau\to 0}\partial_\tau\bar{\varphi}(\tau) \;\; in \;\; L^2(\mathbf{K}). \tag{V.7}$$

Furthermore, (III.9) assures that given $\bar{\psi}_0 \in H^1(\mathbf{K})$, $\bar{\psi}_1 \in L^2(\mathbf{K})$, there exists a unique solution $\bar{\varphi}$ of (V.2) satisfying (V.3), (V.6), (V.7) and the map $(\bar{\varphi}_0, \bar{\varphi}_1) \mapsto (\bar{\psi}_0, \bar{\psi}_1)$ is a bi-Lipschitz bijection on $H^1(\mathbf{K}) \times L^2(\mathbf{K})$.

Proof. First we prove the local existence of the mild solutions by a classic way. We solve the integral equation

$$\begin{pmatrix}\bar{\varphi}(\tau)\\ \bar{\psi}(\tau)\end{pmatrix} = \mathcal{F}\begin{pmatrix}\bar{\varphi}\\ \bar{\psi}\end{pmatrix}(\tau), \tag{V.8}$$

with

$$\mathcal{F}\begin{pmatrix}\bar{\varphi}\\ \bar{\psi}\end{pmatrix}(\tau) := e^{i(\tau-\tau_0)\mathcal{A}}\begin{pmatrix}\bar{\varphi}_0\\ \bar{\varphi}_1\end{pmatrix} + \int_{\tau_0}^{\tau} e^{i(\tau-\sigma)\mathcal{A}}\begin{pmatrix}0\\ \left[1-\frac{1}{6}R_\gamma - \bar{m}^2(\sigma)\bar{\Omega}^2(\sigma) - \kappa\mid\varphi(\sigma)\mid^2\right]\bar{\varphi}(\sigma)\end{pmatrix}d\sigma \tag{V.9}$$

where $\mathcal{A}$ is defined by (III.13). Using the Sobolev inequality

$$\|u\|_{L^6(\mathbf{K})} \leqslant K\|u\|_{H^1(\mathbf{K})}, \tag{V.10}$$

we get

$$\left\|\mathcal{F}\begin{pmatrix}\bar{\varphi}\\ \bar{\psi}\end{pmatrix}(\tau)\right\|_{H^1\times L^2} \leqslant \left\|\begin{pmatrix}\bar{\varphi}_0\\ \bar{\varphi}_1\end{pmatrix}\right\|_{H^1\times L^2} + \left|\int_{\tau_0}^{\tau}\|\bar{m}(\sigma)\bar{\Omega}(\sigma)\|^2_{L^\infty}d\sigma\right|\sup_{(\tau_0,\tau)}\|\bar{\varphi}(\sigma)\|_{H^1}$$
$$+\mid\tau-\tau_0\mid\left(1+\|R_\gamma\|_{L^\infty}+\kappa K^3\sup_{(\tau_0,\tau)}\|\bar{\varphi}(\sigma)\|^2_{H^1}\right)\sup_{(\tau_0,\tau)}\|\bar{\varphi}(\sigma)\|_{H^1}.$$

Putting

$$\rho := \|(\bar{\varphi}_0, \bar{\phi}_1)\|_{H^1\times L^2}, \quad J_\epsilon := \{\tau \in \bar{I};\;\; \mid\tau-\tau_0\mid\leqslant\epsilon\} \;\; (\epsilon > 0), \tag{V.11}$$

$$B_\rho := \left\{(\bar{\varphi}, \bar{\psi}) \in C^0\left(J_\epsilon; H^1(\mathbf{K})\times L^2(\mathbf{K})\right);\;\; \sup_{\tau\in J_\epsilon}\|(\bar{\varphi}(\tau), \bar{\psi}(\tau))\|_{H^1\times L^2} \leqslant 2\rho\right\}, \tag{V.12}$$

we deduce that $\mathcal{F}$ is a map from B_ρ into B_ρ if

$$2\int_{J_\epsilon}\|\bar{m}(\sigma)\bar{\Omega}(\sigma)\|^2_{L^\infty}d\sigma + \epsilon\left(1+\|R_\gamma\|_{L^\infty}+4\kappa K^3\rho^2\right) \leqslant 1.$$

We have also:

$$\left\|\mathcal{F}\begin{pmatrix}\bar{\varphi}\\ \bar{\psi}\end{pmatrix}(\tau) - \mathcal{F}\begin{pmatrix}\bar{\varphi}_*\\ \bar{\psi}_*\end{pmatrix}(\tau)\right\|_{H^1\times L^2} \leqslant \mathfrak{K}(\tau)\sup_{(\tau_0,\tau)}\|\bar{\varphi}(\sigma)-\bar{\varphi}_*(\sigma)\|_{H^1},$$

where

$$\mathfrak{K}(\tau):=\left|\int_{\tau_0}^{\tau}\|\bar{m}(\sigma)\bar{\Omega}(\sigma)\|^2_{L^\infty}d\sigma\right|+\mid\tau-\tau_0\mid\left[1+\|R_\gamma\|_{L^\infty}+2\kappa K^3\left(\sup_{(\tau_0,\tau)}\|\bar{\varphi}(\sigma)\|^2_{H^1}+\sup_{(\tau_0,\tau)}\|\bar{\varphi}_*(\sigma)\|^2_{H^1}\right)\right].$$

Hence $\mathcal{F}$ is a strict contraction on B_ρ if

$$2\int_{J_\epsilon}\|\bar{m}(\sigma)\bar{\Omega}(\sigma)\|^2_{L^\infty}d\sigma+\epsilon\left(1+\|R_\gamma\|_{L^\infty}+16\kappa K^3\rho^2\right)<1.$$

Its unique fixed point satisfies $\bar{\psi}=\partial_\tau\bar{\varphi}$ and it is a local solution of (V.2), (V.4) in $C^0\left(J_\epsilon;H^1(\mathbf{K})\right)\cap C^1\left(J_\epsilon;L^2(\mathbf{K})\right)$. Now we deduce from (III.16) that this local solution satisfies

(V.13)

$$\|\bar{\varphi}(\tau)\|^2_{H^1}+\|\partial_\tau\bar{\varphi}(\tau)\|^2_{L^2}+\frac{\kappa}{2}\|\bar{\varphi}(\tau)\|^4_{L^4}\leqslant\|\bar{\varphi}_0\|^2_{H^1}+\|\bar{\varphi}_1\|^2_{L^2}+\frac{\kappa}{2}\|\bar{\varphi}_0\|^4_{L^4}$$
$$+\left|\int_{\tau_0}^{\tau}\left(1+\|R_\gamma\|_{L^\infty}+\|\bar{m}(\sigma)\bar{\Omega}(\sigma)\|^2_{L^\infty}\right)\left(\|\bar{\varphi}(\sigma)\|^2_{L^2}+\|\partial_\tau\bar{\varphi}(\sigma)\|^2_{L^2}\right)d\sigma\right|,$$

hence we get by the Gronwall Lemma,

$$\|\bar{\varphi}(\tau)\|^2_{H^1}+\|\partial_\tau\bar{\varphi}(\tau)\|^2_{L^2}+\frac{\kappa}{2}\|\bar{\varphi}(\tau)\|^4_{L^4}\leqslant\left(\|\bar{\varphi}_0\|^2_{H^1}+\|\bar{\varphi}_1\|^2_{L^2}+\frac{\kappa}{2}\|\bar{\varphi}_0\|^4_{L^4}\right)$$
$$\times\exp\left|\int_{\tau_0}^{\tau}\left(1+\|R_\gamma\|_{L^\infty}+\|\bar{m}(\sigma)\bar{\Omega}(\sigma)\|^2_{L^\infty}\right)d\sigma\right|,\tag{V.14}$$

and (V.5) is proved for $\tau\in J_\epsilon$. Now the global existence on the whole interval $\bar{I}$ follows from the principle of unique continuation. We can easily prove the continuous dependence of the solution with respect to the initial data. Given another solution $\bar{\varphi}_*$ we have

$$\left\|\begin{pmatrix}\bar{\varphi}\\ \partial_\tau\bar{\varphi}\end{pmatrix}(\tau)-\begin{pmatrix}\bar{\varphi}_*\\ \partial_\tau\bar{\varphi}_*\end{pmatrix}(\tau)\right\|_{H^1\times L^2}\leqslant\left\|\begin{pmatrix}\bar{\varphi}\\ \partial_\tau\bar{\varphi}\end{pmatrix}(\tau_0)-\begin{pmatrix}\bar{\varphi}_*\\ \partial_\tau\bar{\varphi}_*\end{pmatrix}(\tau_0)\right\|_{H^1\times L^2}$$
$$+C\left|\int_{\tau_0}^{\tau}\|\bar{m}(\sigma)\bar{\Omega}(\sigma)\|^2_{L^\infty}\|\bar{\varphi}(\sigma)-\bar{\varphi}_*(\sigma)\|_{H^1}d\sigma\right|$$
$$+C\left|\int_{\tau_0}^{\tau}\left(1+\|\bar{\varphi}(\sigma)\|^2_{H^1}+\|\bar{\varphi}_*(\sigma)\|^2_{H^1}\right)\|\bar{\varphi}(\sigma)-\bar{\varphi}_*(\sigma)\|_{H^1}d\sigma\right|$$

hence the Lipschitz property of the map $(\bar{\varphi}_0,\bar{\varphi}_1)\mapsto\bar{\varphi}$ follows from the Gronwall Lemma and (V.5).

Finally we assume that (III.9) is satisfied. Then (V.5) implies that $\bar{\varphi}(\tau)$ is bounded in $H^1(\mathbf{K})$ hence we can take the limit of the integral in (V.9) as $\tau\to 0$, and we obtain (V.6), (V.7). Furthermore we can take $\tau_0=0$ in (V.9) and repeating the previous arguments, we can solve the global Cauchy problem with initial data specified at $\tau=0$. The proof is complete.

Q.E.D.

As a consequence we obtain directly the following:

Theorem V.2. *We assume (III.9). Then given $u_0\in H^1(\mathbf{K})$, $u_1\in L^2(\mathbf{K})$, there exist unique solutions $\bar{u}\in C^0\left(\bar{I};H^1(\mathbf{K})\right)\cap C^1\left(\bar{I};L^2(\mathbf{K})\right)$ of (V.1) satisfying*

$$\hat{u}(\tau_-)=u_0,\quad\partial_\tau\hat{u}(\tau_-)=u_1,\tag{V.15}$$

$$\lim_{\tau\to0^+}\check{\Omega}(\tau)\check{u}(\tau)=\lim_{\tau\to0^-}\hat{\Omega}(\tau)\hat{u}(\tau)\;\;in\;\;H^1(\mathbf{K}),\tag{V.16}$$

$$\lim_{\tau\to0^+}\partial_\tau\left[\check{\Omega}\check{u}\right](\tau)=\lim_{\tau\to0^-}\partial_\tau\left[\hat{\Omega}\hat{u}\right](\tau)\;\;in\;\;L^2(\mathbf{K}).\tag{V.17}$$

Moreover $\tilde u$ defined by (III.22) satisfies

$$\tilde\Omega\tilde u \in C^0\left([\tau_-,\tau_+];H^1(\mathbf{K})\right)\cap C^1\left([\tau_-,\tau_+];L^2(\mathbf{K})\right), \tag{V.18}$$

$$\left[\Box_g+\frac{1}{6}R_\gamma+\tilde m^2\tilde\Omega^2\right]\left[\tilde\Omega\tilde u\right]=-\kappa\tilde\Omega^3\mid\tilde u\mid^2\tilde u\quad in\ \ \mathcal{M}. \tag{V.19}$$

The map $(\hat u(\tau_-),\partial_\tau\hat u(\tau_-))\mapsto(\check u(\tau_+),\partial_\tau\check u(\tau_+))$ is a bi-Lipschitz bijection on $H^1(\mathbf{K})\times L^2(\mathbf{K})$.

Proof. We apply the previous proposition with $\tau_0=\tau_-$ and $(\hat\varphi_0,\hat\varphi_1)=\hat{\mathfrak{L}}(\tau_-)(u_0,u_1)$, and we put $\hat u(\tau)=\hat\Omega^{-1}(\tau)\hat\varphi(\tau)$. Then we consider the solution $\check\varphi$ of (V.2) on $\check I$ satisfying $(\check\psi_0,\check\psi_1)=(\hat\psi_0,\hat\psi_1)$ and we put $\check u(\tau)=\check\Omega^{-1}(\tau)\check\varphi(\tau)$. Therefore $\bar u$ are solutions of (V.1) and satisfy (V.16), (V.17), and these transmission conditions imply that $\tilde u$ satisfies also (V.18) and (V.19). Finally the maps $(u_0,u_1)\mapsto(\hat\varphi_0,\hat\varphi_1)\mapsto(\hat\psi_0,\hat\psi_1)=(\check\psi_0,\check\psi_1)\mapsto(\check\varphi(\tau_+),\partial_\tau\check\varphi(\tau_+))\mapsto(\check u(\tau_+),\partial_\tau\check u(\tau_+))$ are bi-Lipschitz bijections of $H^1(\mathbf{K})\times L^2(\mathbf{K})$.

Q.E.D.

Like for the linear case, this result is suitable to treat the case of the Singular Bouncing Scenario but (III.9) is a much too strong assumption for the CCC. We now consider the more reasonnable assumption (IV.37).

Theorem V.3. *We assume (IV.37). Then given $u_0\in H^1(\mathbf{K})$, $u_1\in L^2(\mathbf{K})$, given real solutions $\bar A\in C^1\cap L^1(\bar I_h)$ of (IV.5) for some $h>0$, there exist unique solutions $\bar u\in C^0\left(\bar I;H^1(\mathbf{K})\right)\cap C^1\left(\bar I;L^2(\mathbf{K})\right)$ of (V.1) satisfying*

$$\hat u(\tau_-)=u_0,\quad \partial_\tau\hat u(\tau_-)=u_1, \tag{V.20}$$

$$\lim_{\tau\to0^+}\check\Omega(\tau)\check u(\tau)=\lim_{\tau\to0^-}\hat\Omega(\tau)\hat u(\tau)\quad in\ \ H^1(\mathbf{K}), \tag{V.21}$$

$$\lim_{\tau\to0^+}\left(\partial_\tau\left[\check\Omega\check u\right](\tau)+\check A(\tau)\check\Omega(\tau)\check u(\tau)\right)=\lim_{\tau\to0^-}\left(\partial_\tau\left[\hat\Omega\hat u\right](\tau)+\hat A(\tau)\hat\Omega(\tau)\hat u(\tau)\right)\quad in\ \ L^2(\mathbf{K}). \tag{V.22}$$

With the notations (I.11), (III.22), (IV.22), we have

$$\tilde\psi:=\tilde\Omega e^{\int_0^\tau\tilde A(\sigma)d\sigma}\tilde u\in C^0\left([-h,h];H^1(\mathbf{K})\right)\cap C^1\left([-h,h];L^2(\mathbf{K})\right), \tag{V.23}$$

$$\left(\partial_\tau^2-\Delta_{\mathbf{K}}+\frac{1}{6}R_\gamma-2\tilde A\partial_\tau\right)\tilde\psi=-\kappa\mid\tilde\psi\mid^2\tilde\psi e^{-2\int_0^\tau\tilde A(\sigma)d\sigma}\quad in\ \ (-h,h)\times\mathbf{K}, \tag{V.24}$$

The map $\mathfrak{S}:\ (\hat u(\tau_-),\partial_\tau\hat u(\tau_-))\mapsto(\check u(\tau_+),\partial_\tau\check u(\tau_+))$ is a bi-Lipschitz bijection on $H^1(\mathbf{K})\times L^2(\mathbf{K})$.

Proof. $\hat u$ is given by Proposition V.1 by putting for $\tau\in\hat I$, $\hat u(\tau)=\hat\Omega^{-1}(\tau)\hat\varphi(\tau)$ where $\hat\varphi$ is the solution of (V.2), (V.4) with $\tau_0=\tau_-$, $(\hat\varphi_0,\hat\varphi_1)=\hat{\mathfrak{L}}(\tau_-)(u_0,u_1)$. Now for $\tau\in\hat I_h$ we introduce

$$\hat\psi(\tau):=e^{\int_0^\tau\hat A(\sigma)d\sigma}\hat\varphi(\tau)$$

that is a solution of

$$\left(\partial_\tau^2-\Delta_{\mathbf{K}}+1\right)\hat\psi=\left(1+2\hat A\partial_\tau-\frac{1}{6}R_\gamma\right)\hat\psi-\kappa\mid\hat\psi\mid^2\hat\psi e^{-2\int_0^\tau\hat A(\sigma)d\sigma}\quad in\ \ \hat I_h\times\mathbf{K}.$$

Since $\hat\psi\in C^0(\hat I_h;H^1(\mathbf{K}))\cap C^1(\hat I_h;L^2(\mathbf{K}))$, we deduce from (III.16) that

$$\begin{aligned}\|\hat\psi(\tau)\|_{H^1}^2+\|\partial_\tau\hat\psi(\tau)\|_{L^2}^2\leqslant &C\left[\|u_0\|_{H^1}^2+\|u_1\|_{L^2}^2+\int_{\tau_-}^\tau(1+\mid\hat A(\sigma)\mid)\left(\|\hat\psi(\sigma)\|_{H^1}^2+\|\partial_\sigma\hat\psi(\sigma)\|_{L^2}^2\right)d\sigma\right]\\&-\frac{\kappa}{2}\int_{\tau_-}^\tau\frac{d}{d\sigma}\left(\|\hat\psi(\sigma)\|_{L^4}^4\right)e^{-2\int_0^\sigma\hat A(s)ds}d\sigma,\end{aligned}$$

hence

$$\|\hat{\psi}(\tau)\|_{H^1}^2+\|\partial_\tau\hat{\psi}(\tau)\|_{L^2}^2+\frac{\kappa}{2}\|\hat{\psi}(\tau)\|_{L^4}^4e^{-2\int_0^\tau\hat{A}(s)ds}\leqslant C'\left[\|u_0\|_{H^1}^2+\|u_0\|_{H^1}^4+\|u_1\|_{L^2}^2\right]$$
$$+C''\int_{\tau_-}^{\tau}(1+\mid\hat{A}(\sigma)\mid)\left(\|\hat{\psi}(\sigma)\|_{H^1}^2+\|\partial_\sigma\hat{\psi}(\sigma)\|_{L^2}^2+\frac{\kappa}{2}\|\hat{\psi}(\sigma)\|_{L^4}^4e^{-2\int_0^\sigma\hat{A}(s)ds}\right)d\sigma$$

therefore we conclude with the Gronwall Lemma that

$$\|\hat{\psi}(\tau)\|_{H^1}^2+\|\partial_\tau\hat{\psi}(\tau)\|_{L^2}^2\leqslant C'''\left[\|u_0\|_{H^1}^2+\|u_0\|_{H^1}^4+\|u_1\|_{L^2}^2\right]\exp\left(\int_{\tau_-}^{\tau}(1+\mid\hat{A}(\sigma)\mid)d\sigma\right)\tag{V.25}$$

and finally since $\hat{A}\in L^1(\hat{I}_h)$,

$$\hat{\psi}\in L^\infty(\hat{I}_h;H^1(\mathbf{K})),\quad\partial_\tau\hat{\psi}\in L^\infty(\hat{I}_h;L^2(\mathbf{K})).\tag{V.26}$$

Now we have

$$\begin{pmatrix}\hat{\psi}(\tau)\\\partial_\tau\hat{\psi}(\tau)\end{pmatrix}=e^{i(\tau-\tau_-)\mathcal{A}}\begin{pmatrix}\hat{\psi}(\tau_-)\\\partial_\tau\hat{\psi}(\tau_-)\end{pmatrix}$$
$$+\int_{\tau_-}^{\tau}e^{i(\tau-\sigma)\mathcal{A}}\begin{pmatrix}0\\\hat{\psi}(\sigma)-\frac{1}{6}R_\gamma\hat{\psi}(\sigma)+2\hat{A}(\sigma)\partial_\sigma\hat{\psi}(\sigma)-\kappa\mid\hat{\psi}(\sigma)\mid^2\hat{\psi}(\sigma)e^{-2\int_0^\sigma\hat{A}(s)ds}\end{pmatrix}d\sigma$$

We deduce from (V.26) that the following limits exist

$$\lim_{\tau\to0^-}\begin{pmatrix}\hat{\Omega}(\tau)\hat{u}(\tau)\\\partial_\tau\left[\hat{\Omega}\hat{u}\right](\tau)+\hat{A}(\tau)\hat{u}(\tau)\end{pmatrix}=\lim_{\tau\to0^-}\begin{pmatrix}\hat{\psi}(\tau)\\\partial_\tau\hat{\psi}(\tau)\end{pmatrix}$$
$$=e^{-i\tau_-\mathcal{A}}\begin{pmatrix}\hat{\psi}(\tau_-)\\\partial_\tau\hat{\psi}(\tau_-)\end{pmatrix}\tag{V.27}$$
$$+\int_{\tau_-}^{0}e^{-i\sigma\mathcal{A}}\begin{pmatrix}0\\\hat{\psi}(\sigma)-\frac{1}{6}R_\gamma\hat{\psi}(\sigma)+2\hat{A}(\sigma)\partial_\sigma\hat{\psi}(\sigma)-\kappa\mid\hat{\psi}(\sigma)\mid^2\hat{\psi}(\sigma)e^{-2\int_0^\sigma\hat{A}(s)ds}\end{pmatrix}d\sigma.$$

Now we define $\hat{\psi}_0:=\lim_{\tau\to0^-}\hat{\psi}(\tau)$, $\hat{\psi}_1:=\lim_{\tau\to0^-}\partial_\tau\hat{\psi}(\tau)$, and we look for $\check{\psi}\in C^0\left(\check{I}_h;H^1(\mathbf{K})\right)\cap C^1\left(\check{I}_h;L^2(\mathbf{K})\right)$ solution of

$$\left(\partial_\tau^2-\Delta_{\mathbf{K}}+1\right)\check{\psi}=\left(1+2\check{A}\partial_\tau-\frac{1}{6}R_\gamma\right)\check{\psi}-\kappa\mid\check{\psi}\mid^2\check{\psi}e^{-2\int_0^\tau\check{A}(\sigma)d\sigma}\quad in\quad\check{I}_h\times\mathbf{K},$$

satisfying

$$\lim_{\tau\to0^+}\check{\psi}(\tau)=\hat{\psi}_0,\quad\lim_{\tau\to0^+}\partial_\tau\check{\psi}(\tau)=\hat{\psi}_1.$$

It is sufficient to establish that the Cauchy problem is well posed in $C^0\left((-h,h);H^1(\mathbf{K})\right)\cap C^1\left((-h,h);L^2(\mathbf{K})\right)$ for equation (V.24) with an initial data $(\tilde{\psi}_0,\tilde{\psi}_1)$ given at any given time $\tau_0\in[-h,h]$. In an equivalent way, we have to solve the integral equation

(V.28)

$$\begin{pmatrix}\tilde{\psi}(\tau)\\\tilde{\chi}(\tau)\end{pmatrix}=\mathcal{G}\begin{pmatrix}\tilde{\psi}\\\tilde{\chi}\end{pmatrix}(\tau)$$
$$:=e^{i(\tau-\tau_0)\mathcal{A}}\begin{pmatrix}\tilde{\psi}_0\\\tilde{\psi}_1\end{pmatrix}$$
$$+\int_{\tau_0}^{\tau}e^{i(\tau-\sigma)\mathcal{A}}\begin{pmatrix}0\\\tilde{\psi}(\sigma)-\frac{1}{6}R_\gamma\tilde{\psi}(\sigma)+2\tilde{A}(\sigma)\tilde{\chi}(\sigma)-\kappa\mid\tilde{\psi}(\sigma)\mid^2\tilde{\psi}(\sigma)e^{-2\int_0^\sigma\tilde{A}(s)ds}\end{pmatrix}d\sigma.$$

We define

$$\rho := \|(\tilde{\psi}_0, \tilde{\psi}_1)\|_{H^1\times L^2}, \quad M := \max\left(1+\frac{1}{6}\|R_\gamma\|_{L^\infty}, \kappa \exp\left(2\|\tilde{A}\|_{L^1(-h,h)}\right)\right),$$

then using (V.10) and the notations (V.11) and (V.12), we get for any $(\tilde{\psi}, \tilde{\chi}) \in B_\rho$

$$\sup_{\tau\in J_\epsilon}\left\|\mathcal{G}\left(\begin{array}{c}\tilde{\psi}\\ \tilde{\chi}\end{array}\right)(\tau)\right\|_{H^1\times L^2} \leqslant \rho\left[1+2M(1+4K^3\rho^2)\epsilon + 4\int_{J_\epsilon} \mid \tilde{A}(\tau)\mid d\tau\right],$$

hence $\mathcal{G}$ is a map from B_ρ into B_ρ if $\epsilon > 0$ is small enough to that

$$2M(1+4K^3\rho^2)\epsilon + 4\int_{J_\epsilon}\mid \tilde{A}(\tau)\mid d\tau \leqslant 1.$$

Moreover, given $(\tilde{\psi}, \tilde{\chi}), (\tilde{\psi}_*, \tilde{\chi}_*) \in B_\rho$, we have

$$\begin{aligned}\sup_{\tau\in J_\epsilon}\left\|\mathcal{G}\left(\begin{array}{c}\tilde{\psi}\\ \tilde{\chi}\end{array}\right)(\tau)-\mathcal{G}\left(\begin{array}{c}\tilde{\psi}_*\\ \tilde{\chi}_*\end{array}\right)(\tau)\right\|_{H^1\times L^2} &\leqslant \epsilon M(1+12K^3\rho^2)\sup_{\tau\in J_\epsilon}\|\tilde{\psi}(\tau)-\tilde{\psi}_*(\tau)\|_{H^1}\\ &\quad + 2\left(\int_{J_\epsilon}\mid \tilde{A}(\tau)\mid dt\right)\sup_{\tau\in J_\epsilon}\|\tilde{\chi}(\tau)-\tilde{\chi}_*(\tau)\|_{L^2}.\end{aligned}$$

We conclude that if

$$2M(1+6K^3\rho^2)\epsilon + 4\int_{J_\epsilon}\mid \tilde{A}(\tau)\mid d\tau < 1,$$

then $\mathcal{G}$ is a strict contraction on B_ρ and its fixed point satisfies $\tilde{\chi} = \partial_\tau\tilde{\psi}$ and $\tilde{\psi} \in C^0\left(J_\epsilon; H^1(\mathbf{K})\right) \cap C^1\left(J_\epsilon; L^2(\mathbf{K})\right)$ is solution of (V.24). Therefore the Cauchy problem for (V.24) is locally well posed. To obtain the global existence it is sufficient to prove that there exists $C > 0$ such that
(V.29)

$$\|\tilde{\psi}(\tau)\|^2_{H^1} + \|\partial_\tau\tilde{\psi}(\tau)\|^2_{L^2} \leqslant C\left[\|\tilde{\psi}(\tau_0)\|^2_{H^1} + \|\tilde{\psi}(\tau_0)\|^4_{H^1} + \|\partial_\tau\tilde{\psi}(\tau_0)\|^2_{L^2}\right]\exp\left(\left|\int_{\tau_0}^{\tau}(1+\mid \tilde{A}(\sigma)\mid)d\sigma\right|\right)$$

This energy estimate is easily obtained by repeating the proof of (V.25). Now we can define $\breve{u}(\tau)$ for $\tau \in [0,h]$ by the formula

$$\breve{u}(\tau) = \breve{\Omega}^{-1}(\tau)e^{-\int_0^\tau \breve{A}(s)ds}\tilde{\psi}(\tau).$$

Finally we extend $\breve{u}(\tau)$ for $\tau \in [h, \tau_+]$ by solving, with Proposition V.1, the Cauchy problem for (V.1) with initial data given at $\tau_0 = h$.

We now show that given $\tau_1, \tau_2 \in [-h,h]$, the bijection $(\tilde{\psi}(\tau_1), \partial_\tau\tilde{\psi}(\tau_1)) \mapsto (\tilde{\psi}(\tau_2), \partial_\tau\tilde{\psi}(\tau_2))$ is Lipschitz. We deduce from (V.28) that given two solutions $\tilde{\psi}$ and $\tilde{\psi}_*$, we have

$$\begin{aligned}\left\|\left(\begin{array}{c}\tilde{\psi}\\ \partial_\tau\tilde{\psi}\end{array}\right)(\tau_2)-\left(\begin{array}{c}\tilde{\psi}_*\\ \partial_\tau\tilde{\psi}_*\end{array}\right)(\tau_2)\right\|_{H^1\times L^2} &\leqslant \left\|\left(\begin{array}{c}\tilde{\psi}\\ \partial_\tau\tilde{\psi}\end{array}\right)(\tau_1)-\left(\begin{array}{c}\tilde{\psi}_*\\ \partial_\tau\tilde{\psi}_*\end{array}\right)(\tau_1)\right\|_{H^1\times L^2}\\ &\quad + 2\left|\int_{\tau_1}^{\tau_2} M\left(1+\|\tilde{\psi}(\tau)\|^2_{H^1}+\|\tilde{\psi}_*(\tau)\|^2_{H^1}\right)\|\tilde{\psi}(\tau)-\tilde{\psi}_*(\tau)\|_{H^1}d\tau\right|\\ &\quad + 2\left|\int_{\tau_1}^{\tau_2}\mid \tilde{A}(\tau)\mid \|\partial_\tau\tilde{\psi}(\tau)-\partial_\tau\tilde{\psi}_*(\tau)\|_{L^2}d\tau\right|.\end{aligned}$$

We apply the Gronwall lemma and (V.29) to conclude there exists a continuous function $\mathfrak{K}$ independent of $\tilde{\psi}$ and $\tilde{\psi}_*$, such that for any $\tau_j \in [-h,h]$ we have

$$\begin{aligned}\left\|\left(\begin{array}{c}\tilde{\psi}\\ \partial_\tau\tilde{\psi}\end{array}\right)(\tau_2)-\left(\begin{array}{c}\tilde{\psi}_*\\ \partial_\tau\tilde{\psi}_*\end{array}\right)(\tau_2)\right\|_{H^1\times L^2} &\leqslant \mathfrak{K}\left(\left\|\left(\begin{array}{c}\tilde{\psi}\\ \partial_\tau\tilde{\psi}\end{array}\right)(\tau_1)\right\|_{H^1\times L^2}, \left\|\left(\begin{array}{c}\tilde{\psi}_*\\ \partial_\tau\tilde{\psi}_*\end{array}\right)(\tau_1)\right\|_{H^1\times L^2}\right)\\ &\quad \times\left\|\left(\begin{array}{c}\tilde{\psi}\\ \partial_\tau\tilde{\psi}\end{array}\right)(\tau_1)-\left(\begin{array}{c}\tilde{\psi}_*\\ \partial_\tau\tilde{\psi}_*\end{array}\right)(\tau_1)\right\|_{H^1\times L^2}\end{aligned}$$

Finally we can see that $\mathfrak{S}$ is a bi-Lipschitz bijection as a composition of bi-Lipschitz bijections: $(\hat{u}(\tau_-), \partial_\tau \hat{u}(\tau_-)) \mapsto (\hat{u}(-h), \partial_\tau \hat{u}(-h)) \mapsto \left(\tilde{\psi}(-h), \partial_\tau \tilde{\psi}(-h)\right) \mapsto \left(\tilde{\psi}(h), \partial_\tau \tilde{\psi}(h)\right) \mapsto (\check{u}(h), \partial_\tau \check{u}(h)) \mapsto (\check{u}(\tau_+), \partial_\tau \check{u}(\tau_+))$.

Q.E.D.

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