Scattering by Black-Hole for Electromagnetic Fields

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I - Maxwell equations in Schwarzschild universe

We investigate the electromagnetic field outside spherical Black-Hole of radius $r_o > 0$, described by Schwarzschild metric

(1) $$ds^2 = \alpha^2 dt^2 - \alpha^{-2} dr^2 - r^2(d\theta^2 + sin^2\theta\, d\varphi^2) \quad , \quad r_o < r ,$$

and lapse function α is given by

(2) $$\alpha = (1 - r_o r^{-1})^{1/2} .$$

This metric is singular on the "Horizon" $\Gamma = \mathbb{R}_t \times \{r = r_o\} \times S^2$ and no radial null geodesic reaches Γ at finite time t. With Wheeler coordinate r_*, the equation of such geodesics is

(3) $$t = \pm r_* + C \quad , \quad r_* = r + r_o \ell n\,(r - r_o) .$$

In Schwarzschild vacuum, Maxwell's tensor F verifies equations :

(4) $$dF = 0 \quad , \quad d * F = 0 ,$$

where $*$ is the Hodge operator related to metric (1). We split F into electric and magnetic fields measured by an observer with four velocity u :

(5) $$E_\mu = F_{\mu,\nu}\, u^\nu \ , \quad B_\mu = -(* F)_{\mu,\nu}\, u^\nu .$$

Since we are concerned by scattering theory, we consider the Black-Hole as a perturbation and we choose an observer at rest by respect to the Black-Hole (Fiducial observer of [6]), and then

(6) $$u = \alpha^{-1} \partial_t .$$

By putting

(7) $$^tU = (E^{\hat r}, E^{\hat\theta}, E^{\hat\varphi}, B^{\hat r}, B^{\hat\theta}, B^{\hat\varphi}) = (e, b) ,$$

where

$$X = X^{\hat r}\,\alpha\,\partial_r + X^{\hat\theta}\,r^{-1}\,\partial_\theta + X^{\hat\varphi}\,(r\sin\theta)^{-1}\,\partial_\varphi\ ,\quad X = E, B\ ,$$

Maxwell's equations (4) take a familiar form

$$(8)\qquad \partial_t U = -iHU\ ,\quad \nabla_S . E = \nabla_S . B = 0\ ,$$

where

$$(9)\qquad H = i\begin{pmatrix} 0 & \nabla_S\times \\ -\nabla_S\times & 0\end{pmatrix}\ ,\ \nabla_S\times = \begin{pmatrix} 0 & -\frac{\alpha}{r\sin\theta}\partial_\varphi & \frac{\alpha}{r\sin\theta}\partial_\theta\sin\theta \\ \frac{\alpha}{r\sin\theta}\partial_\varphi & 0 & -\frac{\alpha}{r}\partial_r r\alpha \\ \frac{\alpha}{r}\partial_\theta & \frac{\alpha}{r}\partial_r r\alpha & 0\end{pmatrix}$$

$$(10)\qquad \nabla_S . X = \alpha\, r^{-2}\,\partial_r\,(r^2 X^{\hat r}) + (r\sin\theta)^{-1}\,[\partial_\theta\,(\sin\theta\, X^{\hat\theta}) + \partial_\varphi X^{\hat\varphi}\,]$$

If there is no Black-Hole, $\alpha = 1$ and we find the free dynamic in Minkowski space-time with spherical coordinates. We introduce the Hilbert space of finite redshifted energy :

$$\tilde{\mathscr{H}} = [L^2(\,]r_o\,, +\infty[_r \times S^2_\omega\,,\, r^2\,dr\,d\omega)]^6\ ,$$

and the subspace of free divergence :

$$\mathscr{H} = \{U \in \tilde{\mathscr{H}}\ ;\ \nabla_S . E = \nabla_S . B = 0\}\ .$$

THEOREM I.1 - *H is a selfadjoint operator with dense domain on $\tilde{\mathscr{H}}$ and on $\mathscr{H}$.*

Then we solve the Cauchy problem for (8) by Stone's theorem.

REMARK : We are not concerned by a mixed problem : we do not need any boundary condition on horizon Γ which is not time like.

We have a result of finite velocity dependance :

THEOREM I.2 - *Let's be U in $\tilde{\mathscr{H}}$ such that*

$$supp\ U \subset \{r^1_* \le r_* \le r^2_*\} \times S^2\ ;$$

then we have

$$supp\ e^{-itH}\ U \subset \{r^1_* - |t| \le r_* \le r^2_* + |t|\} \times S^2\ .$$

Schwarzschild metric is trapping : all great circles of sphere with radius $3r_o/2$, so called "Photons-sphere", are null geodesics ; there exist so null geodesics asymptotic to the Photons-sphere. Therefore singularities of field can be trapped and do not escape at infinity. Despite of this difficulty, there is no time-periodic solution in Schwarzschild universe, unlike the euclidian case with an obstacle, for which, the second space of cohomology yields non trivial stationary solutions :

THEOREM I.3 - *The ponctual spectrum of H on $\mathscr{H}$ is empty.*

We can deduct from this result, the decay of local energy ; but we developp here a complete scattering theory for the electromagnetic field and in particular, we find the result of Damour [3] on the behaviour of fields near the horizon. The study of scalar case was treated by Dimock and Kay [4] [5] .

II - Wave operators at infinity

Schwarzschild universe is asymptotically flat and far from the Black-Hole we compare hamiltonian H with classical electromagnetic hamiltonian H_o :

(9) $$H_o = i \begin{pmatrix} 0 & curl \\ -curl & 0 \end{pmatrix},$$

in Minkowski space-time with metric

(10) $$ds^2 = dt^2 - d\rho^2 - \rho^2 (d\theta^2 + sin^2\theta \, d\varphi^2) \ , \ 0 \leq \rho .$$

For any choice of $\rho = \rho(r)$, the difference $H - H_o$ is a long-range type perturbation but because the radial null geodesics (3) are straight like their flat analogs, we avoid long range interaction between gravitational and electromagnetic fields by choosing :

(11) $$\rho = r_* \geq 0 .$$

We introduce the usual finite energy Hilbert spaces :

$$\tilde{\mathscr{H}}_o = \{U_o = {}^t(E_o^{\hat{r}}, E_o^{\hat{\theta}}, E_o^{\hat{\varphi}}, B_o^{\hat{r}}, B_o^{\hat{\theta}}, B_o^{\hat{\varphi}}) \in [L^2(\mathbb{R}^+_{r_*} \times S^2_\omega , r_*^2 \, dr_* \, d\omega)]^6\} ,$$
$$\mathscr{H}_o = \{U_o = {}^t(E_o , B_o) \in \tilde{\mathscr{H}}_o \ ; div\, E_o = div\, B_o = 0 \} .$$

Given a cut-off function $\chi_o \in C^\infty(\mathbb{R}^+_{r_*})$ satisfying, $\chi_o(r_*) = 0$ for $0 \leq r_* < a$, and $\chi_o(r_*) = 1$ for $r_* > b$, for some $0 < a < b$, we construct an identification operator $\mathscr{I}_o : \tilde{\mathscr{H}}_o \to \tilde{\mathscr{H}}$ by putting :

$$\mathcal{J}_o U_o = \chi_o U_o \quad for \quad r_* \geq 0 \ , \quad \mathcal{J}_o U_o = 0 \quad for \quad r_* \leq 0 .$$

We define classical wave operators without Dollard's modification :

$$W_o^{\pm} U_o = s - \lim_{t \to \pm\infty} e^{itH} \mathcal{J}_o e^{-itH_o} U_o \ in \ \mathcal{H} .$$

The spherical invariance of Maxwell equations - that implies a t^{-2}decay of radial components - and our choice (11), cancel the long range effects and by Cook's method we prove the

THEOREM II.1 - $W_o^{\pm} : \mathcal{H}_o \to \mathcal{H}$ *exist, are independent of* χ_o *and* $\|W_o^{\pm}\| \leq 1$.

We deduct from this result, the existence of outgoing fields :

THEOREM II.2 - *If* $U_o \in \mathcal{H}_o$ *verifies :*

$$e^{-itH_o} U_o = 0 \quad for \quad 0 \leq r_* \leq \pm t + C ,$$

then we have

$$e^{-itH_o} W_o^{\pm} U_o = 0 \quad for \quad r_* \leq \pm t + C .$$

III - Wave operators near the black-hole

Hamiltonian H degenerates as $r \to r_o$, but $r\alpha H(r\alpha)^{-1}$ admits a formal limit H_1

(13) $$H_1 = i \begin{pmatrix} 0 & h_1 \\ -h_1 & 0 \end{pmatrix} , \ h_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\partial_{r_*} \\ 0 & \partial_{r_*} & 0 \end{pmatrix} .$$

H_1 is essentially the dynamic in Rindler metric that approximates Schwarzschild metric near the horizon. We introduce Hilbert spaces :

$$\tilde{\mathcal{H}}_1 = \{U_1 = {}^t(E_1^{\hat{r}}, E_1^{\hat{\theta}}, E_1^{\hat{\varphi}}, B_1^{\hat{r}}, B_1^{\hat{\theta}}, B_1^{\hat{\varphi}}) \in [L^2(\mathbb{R}_{r_*} \times S_\omega^2, dr_* \, d\omega)]^6\} ,$$

$$\mathcal{H}_1^{\pm} = \{U_1 \in \tilde{\mathcal{H}}_1 ; E_1^{\hat{r}} = B_1^{\hat{r}} = \pm E_1^{\hat{\theta}} + B_1^{\hat{\varphi}} = \pm E_1^{\hat{\varphi}} - B_1^{\hat{\theta}} = 0\} .$$

The fields in $\mathcal{H}_1^{+(-)}$ have an left (right) polarization and behave like a plane wave, falling into the future (coming out of the past) horizon :

$$U_1 \in \mathcal{H}_1^{\pm} \Rightarrow [e^{-itH_1} U_1](r_*, \omega) = U_1(\pm t + r_*, \omega) .$$

Given a cut-off function $\chi_1 \in C^\infty(\mathbb{R}_{r*})$ satisfying $\chi_1(r_*) = 1$ for $r_* < c$, $\chi_1(r_*) = 0$ for $r_* > d$, for some $c < d < 0$, we construct an identification operator

$\mathscr{I}_1 : \tilde{\mathscr{H}}_1 \to \tilde{\mathscr{H}}$ by putting

$$\mathscr{I}_1 U_1 = (r\alpha)^{-1} \chi_1 U_1 .$$

We define classical wave operators

$$\text{(14)} \qquad W_1^{\pm} U_1 = s - \lim_{t \to \pm\infty} e^{itH} \mathscr{I}_1 e^{-itH_1} U_1 \quad in \quad \tilde{\mathscr{H}} .$$

Because the Schwarzschild potential is exponentially decreasing as $r_* \to -\infty$, we prove easily by Cook's method the :

THEOREM III.1- *$W_1^{\pm} : \mathscr{H}_1^{\pm} \to \mathscr{H}$ exist, are independent of χ_1 and $\|W_1^{\pm}\| \leq 1$.*

We deduct from this result, the existence of infalling fields, similar to the disappearing solutions in dissipative scattering :

THEOREM III.2 - *If $U_1 \in \mathscr{H}_1^{\pm}$ verifies*

$$U_1(r_*, \omega) = 0 \quad for \quad r_* \geq c$$

then we have

$$e^{-itH} W_1^{\pm} U_1 = 0 \quad for \quad r_* \geq \pm t + c .$$

IV - Asymptotic completeness

To study the asymptotic behaviour far from the Black-Hole we introduce

$$\text{(15)} \qquad W_o U = s - \lim_{t \to +\infty} e^{itH_o} \mathscr{I}_o^* e^{-itH} U \quad in \quad \tilde{\mathscr{H}}_o .$$

At infinity of Schwarzschild universe, the electromagnetic field is asymptotic to a free field in Minkowski space-time :

THEOREM IV.1 - *$W_o : \mathscr{H} \to \mathscr{H}_o$ exists, is independent of χ_o and $\|W_o\| \leq 1$.*

To describe the field near the horizon as $t \to +\infty$ we define

$$\text{(16)} \qquad W_1 U = s - \lim_{t \to +\infty} e^{itH_1} \mathscr{I}_1^* e^{-itH} U \quad in \quad \tilde{\mathscr{H}}_1 .$$

THEOREM IV.2 - *$W_1 : \mathscr{H} \to \mathscr{H}_1^+$ exists, is independent of χ_1 and $\|W_1\| \leq 1$.*

The physical meaning of this result of completeness is the famous "impedance condition" of Damour and Znajeck [3]. More precisely the asymptotic profile of regular fields satisfies a dissipative condition or infalling left-polarization :

THEOREM IV.3 - *Let's be U in $\mathscr{H}$ such that*

(17) $$U = Hf \quad , \quad f\in[C_0^\infty(]r_o, +\infty[\times S^2)]^6 .$$

We note $e^{-itH}U = {}^t(E^{\hat{r}},..., B^{\hat{\varphi}})$. Then, for any $s\in\mathbb{R}$, there exist $e^{\hat{r}},..., b^{\hat{\varphi}}$ in $L^2(S^2)$ such that, as

(18) $$r \to r_o \quad , \quad t + r_* = s ,$$

the following limits exist in $L^2(S^2)$:

(19) $$E^{\hat{r}} \to e^{\hat{r}}, B^{\hat{r}} \to b^{\hat{r}}, \alpha E^{\hat{\theta}} \to e^{\hat{\theta}}, \alpha E^{\hat{\varphi}} \to e^{\hat{\varphi}}, \alpha B^{\hat{\theta}} \to b^{\hat{\theta}}, \alpha B^{\hat{\varphi}} \to b^{\hat{\varphi}} .$$

Moreover, we have

(20) $$e^{\hat{\theta}} = -b^{\hat{\varphi}} \quad , \quad e^{\hat{\varphi}} = b^{\hat{\theta}} ,$$

(21) $$\partial_s e^{\hat{r}} + (sin\theta)^{-1}[\partial_\theta(sin\theta\, b^{\hat{\theta}}) + \partial_\varphi b^{\hat{\varphi}}] = 0 .$$

Remark by Theorem I.3, the set of data satisfying (17) is dense in $\mathscr{H}$.
So, the horizon is rather similar to a dissipative membrane in euclidian space with surface resistivity 377 ohms : (20) is formally the impedance condition and (21) the charge conservation law ; but we emphasize that, unlike the euclidian case for which the dissipative condition is posed at each time and is necessary to solve the mixed problem, impedance property (20) is a consequence of Maxwell equations verified at infinity of infalling null geodesics.
Now, we can introduce scattering operator S by putting

$$W^- : \mathscr{H}_1^- \times \mathscr{H}_o \to \mathscr{H} \quad , \quad W^-(U_1, U_o) = W_1^- U_1 + W_o^- U_o ,$$

$$W : \mathscr{H} \to \mathscr{H}_1^+ \times \mathscr{H}_o \quad , \quad WU = (W_1 U, W_o U) , S = WW^- : \mathscr{H}_1^- \times \mathscr{H}_o \to \mathscr{H}_1^+ \times \mathscr{H}_o .$$

THEOREM IV.4 - *W^- is isometric from $\mathscr{H}_1^- \times \mathscr{H}_o$ onto $\mathscr{H}$; W is isometric from $\mathscr{H}$ onto $\mathscr{H}_1^+ \times \mathscr{H}_o$, S is isometric from $\mathscr{H}_1^- \times \mathscr{H}_o$ onto $\mathscr{H}_1^+ \times \mathscr{H}_o$.*

V - Membrane paradigm

The Membrane Paradigm [6] states that if we are concerned only by the behaviour, far from the Black-Hole, of an initially incoming field, we may approximate the Black-Hole by a dissipative spherical membrane of radius $r_o + \varepsilon$, $0 < \varepsilon$, called "streched horizon".

We consider the mixed problem for Maxwell equations (8) in $]r_o+\varepsilon,+\infty[\times S^2$ and on streched horizon $\Gamma_\varepsilon=\mathbb{R}_t\times\{r=r_o+\varepsilon\}\times S^2$ which is time like, we impose impedance condition

(22) $$E^{\hat\theta}=-B^{\hat\varphi}\ ,\ \ E^{\hat\varphi}=B^{\hat\theta}\ .$$

It is a classical dissipative hyperbolic problem of which the solution is given by a semigroup $V_\varepsilon(t)$ on Hilbert space $\tilde{\mathscr{H}}_\varepsilon=\{L^2(]r_o+\varepsilon,+\infty[_r\times S^2_\omega,r^2\,dr\,d\omega)]^6$. For $0<\varepsilon<a$ we define scattering operator

$$S_\varepsilon U_o=s-\lim_{t\to+\infty}e^{itH_o}\mathscr{I}^*_oV_\varepsilon(2t)\mathscr{I}_oe^{itH_o}U_o\ \ in\ \tilde{\mathscr{H}}_o.$$

THEOREM V.1 - *$S_\varepsilon:\mathscr{H}_o\to\mathscr{H}_o$ exists, is independant of χ_o and $\|S_\varepsilon\|\leq 1$.*

Now, in Schwarzschild universe, the asymptotic behaviour at infinity of an initially incoming field is described by operator S_{oo} defined by

(23) $$\forall U_o\in\mathscr{H}_o,\ S_{oo}U_o=\Pi_oS(0,U_o)$$

where Π_o is the projector from $\mathscr{H}^+_1\times\mathscr{H}_o$ onto $\mathscr{H}_o$. The following result is the mathematical foundation of Membrane Paradigm :

THEOREM V.2 - *For any $U_o\in\mathscr{H}_o$, $S_\varepsilon U_o$ tends to $S_{oo}U_o$ in $\mathscr{H}_o$ as $\varepsilon\to 0$.*

Of numerical analysis view point, impedance condition (22) is an absorbing boundary condition on artificial boundary Γ_ε, so called Silver-Müller radiation condition in euclidian case [2]. So, Theorem V.2 gives a method of numerical approximation, already used in [6].

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