ON THE COUPLING OF BOUNDARY ELEMENT AND FINITE ELEMENT METHODS FOR A TIME PROBLEM.

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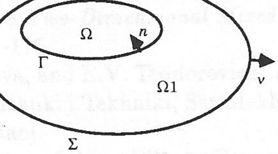
Abstract. We consider the problem of scattering transient waves, by an inhomogeneous object, in $\mathbb{R}^3$. The idea consists in coupling a finite element resolution in the volume including all inhomogeneities, with an integral equation expressing the perfectly transparent condition on the boundary. This condition, which comes from Kirchhoff's formula, introduces single and double layer retarded potentials. The integral operator can be studied, according to Ha-Duong [HD], with Fourier-Laplace transform. That leads us to the associated harmonic problem, for which we prove existence and uniqueness. We also construct another formulation of the problem which satisfies properties of coercivity. The discretization of the first time-space variational formulation conducts to a time-stepping scheme, for which we present numerical computations.

1- The Mixed Hyperbolic Problem.

We consider a three-dimensional bounded obstacle $\mathcal{O}$, composed of a perfectly conducting kernel Ω covered up with an inhomogeneous layer Ω_1.



($\mathcal{O} = \overline{\Omega \cup \Omega_1}$, $\Omega_2 = \mathbb{R}^3 \setminus \mathcal{O}$, $\Gamma = \partial\Omega$, $\Sigma = \partial\Omega_2$, $\vec{n}$ *is the unit normal to* Γ *exterior to* Ω_1, $\vec{\nu}$ *is the unit normal vector to* Σ *exterior to* Ω_1)

Given an incident wave v_{inc} solution of

$$\begin{cases} \partial_t^2 v_{inc}(t,x) - \Delta_x v_{inc}(t,x) = 0 & \forall (t,x) \in \mathbb{R}\times\mathbb{R}^3 \\ v_{inc}(t,x) = 0 & \forall t \in \mathbb{R}^-, \forall x \in \mathcal{O} \end{cases},$$

the problem consists in solving:

$$(P_0) \quad \begin{cases} r(x)\,\partial_t^2 v_1(t,x) - \Delta_x v_1(t,x) + \beta(x)\,\partial_t v_1(t,x) = 0 \quad \forall (t,x) \in \mathbb{R}_t \times \Omega_1 \\ \partial_t^2 v_2(t,x) - \Delta_x v_2(t,x) = 0 \quad \forall (t,x) \in \mathbb{R}_t \times \Omega_2 \\ v_1 = v_2 \ \ on\ \Sigma \\ \partial_\nu v_1 = \partial_\nu v_2 \ \ on\ \Sigma \\ v_1 = 0 \ \ on\ \Gamma \\ v_1(t,x) = 0 \quad \forall (t,x) \in \mathbb{R}^- \times \Omega_1 \\ v_2(t,x) = v_{inc} \quad \forall (t,x) \in \mathbb{R}^- \times \Omega_2 \end{cases}$$

where r and β are regular functions on $\overline{\Omega}_1$, $0 < r$, $0 \leq \beta$.

Let be $\chi \in \mathcal{D}(\mathbb{R}^3)$ supported in $\overline{\Omega} \cup \overline{\Omega}_1$, $\chi \equiv 1$ in a neighborhood of Ω. We denote :

$$u = \begin{cases} u_1 = v_1 + (\chi-1)v_{inc} & in\ \Omega_1 \\ u_2 = v_2 - v_{inc} & in\ \Omega_2 \end{cases}$$

$$f(t,x) = \{r(x)\,\partial_t^{\,2} - \Delta_x + \beta(x)\,\partial_t\}\,[(\chi(x)-1)v_{inc}]$$

Then u is solution of problem (P_1) :

$$(P_1)\quad \begin{cases} r(x)\,\partial_t^{\,2}u_1(t,x) - \Delta_x u_1(t,x) + \beta(x)\,\partial_t u_1(t,x) = f(t,x) \ \ \forall (t,x) \in \mathbb{R}\times\Omega_1 \\ \partial_t^{\,2}u_2(t,x) - \Delta_x u_2(t,x) = 0 \ \ \forall (t,x) \in \mathbb{R}\times\Omega_2 \\ u_1 = u_2 \ \ on\ \Sigma \\ \partial_\nu u_1 = \partial_\nu u_2 \ on\ \Sigma \\ u_1 = 0 \ \ on\ \Gamma \\ u_1(t,x) = 0 \ \ \forall (t,x) \in \mathbb{R}^-\times\Omega_1 \\ u_2(t,x) = 0 \ \ \forall (t,x) \in \mathbb{R}^-\times\Omega_2 \end{cases}$$

In order to solve numerically this problem despite the unbounded open Ω_2, we express the constraint against u_2 on Σ by an integral equation given by the Kirchhoff formula. Then we consider the following problem

$$(P_2)\quad \begin{cases} r(x)\,\partial_t^{\,2}u(t,x) - \Delta_x u(t,x) + \beta(x)\,\partial_t u(t,x) = f(t,x) \ \ \forall (t,x) \in \mathbb{R}^+\times\Omega_1 \\ u = 0 \ \ on\ \Gamma \\ u(t,x) = 0 \ \ \forall (t,x) \in \mathbb{R}^-\times\Omega_1 \\ u(t,x) = \dfrac{1}{2\pi} \displaystyle\int_{\Sigma_y} \Big[\frac{-1}{|x-y|}\,\partial_{\nu_y} u(t-|x-y|,y) \\ \quad + \dfrac{(x-y)\,.\,\nu_y}{|x-y|^2}\Big(\dfrac{1}{|x-y|}u(t-|x-y|,y) + \partial_t u(t-|x-y|,y)\Big)\Big]d\Sigma_y \end{cases}$$

The last equation is the perfectly transparent boundary condition for wave equation. This is not a Fredholm type equation; so, in order to get precise estimates, we study the harmonic associated problem via the Fourier-Laplace Transform in time.

2- The Mixed Elliptic Problem.

We suppose that f is a time-dependent Laplace-transformable distribution. Applying the Fourier-Laplace transform to (P_1), we obtain for $\omega = \alpha + i\sigma \in \mathbb{C}$, $\sigma > 0$:

$$(P_{1\omega})\quad \begin{cases} (\Delta_x + \omega^2 r(x) - i\omega\beta(x))\,\hat{u}_1(\omega,x) = -\hat{f}(\omega,x) & \forall x \in \Omega_1, \\ (\Delta_x + \omega^2)\,\hat{u}_2(\omega,x) = 0 & \forall x \in \Omega_2, \\ \hat{u}_1(\omega,x) = \hat{u}_2(\omega,x) & \forall x \in \Sigma, \\ \partial_\nu \hat{u}_1(\omega,x) = \partial_\nu \hat{u}_2(\omega,x) & \forall x \in \Sigma, \\ \hat{u}_1(\omega,x) = 0 & \forall x \in \Gamma. \end{cases}$$

where $\hat{T}(\omega)$ is the Fourier-Laplace transform in time of $T(t)$.

It is easy to show that $(P_{1\omega})$ has a unique solution for data $\hat{f}(\omega,.) \in L^2(\Omega_1)$ (see [DL]). In order to get sharp estimates, we introduce Green kernel of the Helmholtz equation

$$\forall\, x \neq y\ , \qquad G_\omega(x,y) = \frac{e^{i\omega|x-y|}}{4\pi\,|x-y|}$$

Thanks to the representation by potentials [HD], the transmission conditions lead to the following integral equation on the boundary Σ:

$$\hat{u}_1 = \left(\tfrac{1}{2}I + K'_\omega\right)\gamma_0\,\hat{u}_1 - S_\omega(\hat{\lambda}) \text{ on } \Sigma\,, \quad \text{with } \hat{\lambda} = \partial_\nu \hat{u}_1$$

where γ_0 is the trace operator on Σ, and

$$K'_\omega\gamma_0\,\hat{u}_1(\omega,x) = \int_\Sigma \gamma_0\,\hat{u}_1(\omega,y)\ \partial_{\nu_y} G_\omega(x,y)\, d\Sigma_y\ , \quad \forall\, x\in\Sigma,$$

$$S_\omega(\hat{\lambda})(\omega,x) = \int_\Sigma \hat{\lambda}(\omega,y)\ G_\omega(x,y)\, d\Sigma_y\ , \quad \forall\, x\in\Sigma\,,$$

Hence we obtain an equivalent problem:

$$(P_{2\omega})\quad \begin{cases} (\Delta_x+\omega^2 r(x)-i\omega\beta(x))\,\hat{u}_1(\omega,x) = -\hat{f}(\omega,x) & \forall\, x\in\Omega_1,\\ \hat{u}_1(\omega,x)=0 & \forall\, x\in\Gamma,\\ \partial_\nu\hat{u}_1(\omega,x)=\hat{\lambda}(\omega,x) & \forall\, x\in\Sigma,\\ \left(\tfrac{1}{2}I-K'_\omega\right)\gamma_0\,\hat{u}_1(\omega,x) + S_\omega\hat{\lambda}(\omega,x) = 0 & \forall\, x\in\Sigma. \end{cases}$$

In fact $(P_{2\omega})$ could be obtained by applying the Laplace-Fourier transform to (P_2). The main advantage of $(P_{2\omega})$ is the Fredholm property [DL] of the last equation. Now, setting

$$a_\omega(\hat{u},\hat{v}) = \int_{\Omega_1} -\nabla\hat{u}_1(\omega,x)\,\nabla\overline{\hat{v}}(\omega,x) + [\omega^2 r(x) - i\omega\beta(x)]\,\hat{u}_1(\omega,x)\,\overline{\hat{v}}(\omega,x)\,dx\ ,$$

$$<\hat{\lambda}\,,\gamma_0\hat{v}> = \int_\Sigma \hat{\lambda}(\omega,x)\,\overline{\hat{v}}(\omega,x)\,d\Sigma_x, \qquad (\hat{f}\,,\hat{v}\,) = \int_{\Omega_1} -\hat{f}(\omega,x)\,\overline{\hat{v}}(\omega,x)\,d\Sigma_x\ ,$$

and $\quad H^{1,0}(\Omega_1) = \{\,\hat{v}\in H^1(\Omega_1)\ ;\ \hat{v}=0 \text{ on } \Gamma\,\}$,

we obtain the harmonic variational formulation (PV_ω) of $(P_{2\omega})$:

$$(PV_\omega)\quad \begin{cases} a_\omega(\hat{u},\hat{v}) + <\hat{\lambda},\gamma_0\hat{v}> = (\hat{f},\hat{v}) & \forall\,\hat{v}\in H^{1,0}(\Omega_1)\\ <\left(\tfrac{1}{2}I-K'_\omega\right)\gamma_0\hat{u}\,,\hat{\chi}> + <S_\omega\hat{\lambda},\hat{\chi}> = 0 & \forall\,\hat{\chi}\in H^{-1/2}(\Sigma) \end{cases}$$

Theorem 1. *The problem (PV_ω) has a unique solution $(\hat{u},\hat{\lambda})$ and the solution satisfies*

(1) $$|\,\omega\,|\ \|\hat{u}(\omega,.)\|_{L^2(\Omega_1)} + \|\hat{u}(\omega,.)\|_{H^{1,0}(\Omega_1)} + \frac{1}{|\,\omega\,|}\|\hat{u}(\omega,.)\|_{H^2(\Omega_1)} \le \frac{C}{\sigma}\|\hat{f}(\omega,.)\|_{L^2(\Omega_1)},$$

(2) $$\|\hat{\lambda}(\omega,.)\|_{H^{-1/2}(\Sigma)} + \frac{1}{|\,\omega\,|}\|\hat{u}(\omega,.)\|_{H^{1/2}(\Sigma)} \le \frac{C}{\sigma}\|\hat{f}(\omega,.)\|_{L^2(\Omega_1)}.$$

Sketch of the proof: By putting :

$$\mathcal{A}_\omega((\hat{u},\hat{\lambda});(\hat{v},\hat{\chi})) = \mathcal{B}_\omega((\hat{u},\hat{\lambda});(\hat{v},\hat{\chi})) + \mathcal{K}_\omega((\hat{u},\hat{\lambda});(\hat{v},\hat{\chi})),$$

$$\mathcal{B}_\omega((\hat{u},\hat{\lambda});(\hat{v},\hat{\chi})) = \overline{\omega}\, a_\omega(\hat{u},\hat{v}) + \overline{\omega} < \hat{\lambda}, \gamma_0\hat{v} > + \, \omega < \gamma_0\hat{u}, \hat{\chi} > + \; 2\omega < S_\omega \hat{\lambda}, \hat{\chi} >,$$

$$\mathcal{K}_\omega((\hat{u},\hat{\lambda});(\hat{v},\hat{\chi})) = -2\omega < K'_\omega \gamma_0 \hat{u}, \hat{\chi} >, \qquad \mathcal{L}(\hat{v},\hat{\chi}) = \overline{\omega}\, (\hat{f}, \hat{v}),$$

we have to solve

$$(P_{2\omega}) \qquad \mathcal{A}_\omega((\hat{u},\hat{\lambda});(\hat{v},\hat{\chi})) = \mathcal{L}(\hat{v},\hat{\chi}).$$

We show that $\mathcal{A}_\omega$ is a continuous bilinear form satisfying Garding's inequality by proving that $\mathcal{B}_\omega$ is $H^1(\Omega_1)\times H^{-1/2}(\Sigma)$-coercive, and K_ω is a linear compact map on $H^{1,0}(\Omega_1)\times H^{-1/2}(\Sigma)$. Hence we apply Fredholm's alternative by noting the uniqueness of the solution of $(P_{1\omega})$.

3. First Space-Time Variational Formulation.

First, we recall the functional framework introduced by Ha-Duong [HD]. Let $\mathcal{E}$ be an Hilbert's space, s and σ two real numbers, $\sigma>0$.We denote

$$\mathcal{H}^s_\sigma(\mathbb{R}^+;\mathcal{E}) = \{ f : \; supp f \subset \mathbb{R}^+_t \;\; and \;\; e^{-\sigma t}\Lambda^s f \in L^2(\mathbb{R}_t;\mathcal{E}) \}$$

where

$$\mathcal{L}(\Lambda^s f)(\omega) = (-i\omega)^s \; \mathcal{L}(f)(\omega), \quad \omega = \alpha + i\sigma, \quad and \quad \mathcal{L}(f)(\omega) = \hat{f}(\omega)$$

and we define the norm:

$$|f|^2_{\sigma,s,\mathcal{E}} = \int_{-\infty}^{+\infty} e^{-2\sigma t} \; \| \Lambda^s f \|^2_{\mathcal{E}} \; dt = \frac{1}{2\pi} \int_{Im\,\omega = \sigma} |\omega|^{2s} \; \| \hat{f}(\omega) \|^2_{\mathcal{E}} \; d\omega .$$

All following time results are deduced from previous harmonic ones, by application of inverse Fourier-Laplace transform.

Theorem 2. *Given f in $\mathcal{H}^s_\sigma(\mathbb{R}^+;L^2(\Omega_1))$, $s, \sigma \in \mathbb{R}$, $\sigma > 0$, problem (P_2) has a unique solution*

$$u \in \mathcal{H}^{s+1}_\sigma(\mathbb{R}^+;L^2(\Omega_1)) \cap \mathcal{H}^s_\sigma(\mathbb{R}^+;H^{1,0}(\Omega_1)) \cap \mathcal{H}^{s-1}_\sigma(\mathbb{R}^+;H^2(\Omega_1)),$$

$$\partial_\nu u \in \mathcal{H}^s_\sigma(\mathbb{R}^+;H^{-1/2}(\Sigma)) \cap \mathcal{H}^{s-1}_\sigma(\mathbb{R}^+;H^{1/2}(\Sigma)).$$

We so obtain the first space-time variational problem (PV_t):

$$\int_0^\infty e^{-2\sigma t} \int_{\Omega_1} \{\nabla u . \nabla \overline{v} + \partial_t u \,[-r(x)\partial_t \overline{v} + 2\sigma\, r(x)\overline{v} + \beta(x)\,\overline{v}\ ]\}\ (t,\ x)\ dx\ dt =$$

$$\int_0^\infty e^{-2\sigma t} \int_\Sigma \partial_\nu u(t,x)\,\overline{v}\,(t,x)\, d\Sigma_x\ dt - \int_0^\infty e^{-2\sigma t} \int_{\Omega_1} f(t,x)\,\overline{v}\,(t,x)\,dx\ dt$$

$$\int_0^\infty e^{-2\sigma t} \iint_{\Sigma\times\Sigma} \frac{1}{|x-y|}\ \partial_\nu u\,(t-|x-y|,y)\ \overline{\varphi}(t,x)\ d\Sigma_x d\Sigma_y\, dt \quad =$$

$$\int_0^\infty e^{-2\sigma t} \iint_{\Sigma\times\Sigma} \frac{(\vec{x}-\vec{y})\,.\,\vec{\nu_y}}{|x-y|^2}\ [\frac{1}{|x-y|}\, u\,(t-|x-y|,y) + \partial_t u\,(t-|x-y|,y)]\ \overline{\varphi}(t,x)\ d\Sigma_x d\Sigma_y\, dt$$

$$-\ 2\pi \int_0^\infty e^{-2\sigma t} \int_\Sigma u\,(t,x)\ \ \overline{\varphi}\,(t,x)\ \ d\Sigma_x\ dt\,.$$

Thus we have a time-space formulation which has a unique solution, but no coercivity relation. In section 5, we will set numerically the stable behavior of this formulation. In the next paragraph, we derive another variational formulation of (P_2) for which we are able to prove a coercivity property.

4. Second Variational Formulation.

We now consider the representation formula

$$\partial_\nu \hat{u}_1 \ =\ (\tfrac{1}{2} I - K_\omega)\,\partial_\nu \hat{u}_1\ -\ D_\omega\,(\gamma_0\, \hat{u}_1) \qquad on\ \Sigma\,, \tag{3}$$

where operators K_ω et D_ω are defined by

$$K_\omega\,(\partial_\nu \hat{u}_1)\,(\omega,x) \ =\ \int_\Sigma \partial_{\nu_y} \hat{u}_1(\omega,y)\ \partial_{\nu_x} G_\omega(x,y)\ d\Sigma_y \quad ,\quad \forall\ x\in\Sigma,$$

$$D_\omega\,(\gamma_0\, \hat{u}_1)\,(\omega,x) \ =\ -\,\partial_{\nu_x} \int_\Sigma \hat{u}_1(\omega,y)\ \ \partial_{\nu_y} G_\omega(x,y)\, d\Sigma_y \quad ,\quad \forall\ x\in\Sigma\,.$$

Second variational harmonic problem (PV'_ω) is deduced from (PV_ω) by replacing $\partial_\nu \hat{u}_1$ by its representation given in (3).

$$(PV'_\omega)\ \begin{cases} a_\omega\,(\hat{u}(\omega,.),\hat{v}) + <(\tfrac{1}{2}I-K_\omega)\,\hat{\lambda}(\omega,.),\gamma_0\hat{v}> - <D_\omega(\gamma_0\hat{u}\,(\omega,.)),\gamma_0\hat{v}> = (\hat{f}\,(\omega,.),\hat{v}) \\ <(\tfrac{1}{2}I-K'_\omega)\,\gamma_0\hat{u}(\omega,.)\ ,\hat{\chi}> + <S_\omega\hat{\lambda}\,(\omega,.)\,,\hat{\chi}> \ =\ 0 \end{cases}$$

(PV_ω) and (PV'_ω) are equivalent problems, so (PV'_ω) has a unique solution.We may now consider, as before, the associated variational formulation:

$$(4)\quad \begin{cases} find\ (\hat{u},\hat{\lambda}) \in H^{1,0}(\Omega_1)\times H^{-1/2}(\Sigma)\ such\ that\ \forall\,(\hat{v},\hat{\chi})\in H^{1,0}(\Omega_1)\times H^{-1/2}(\Sigma), \\ \mathscr{A}'_{\omega}((\hat{u},\hat{\lambda});(\hat{v},\hat{\chi})) = \mathscr{L}'(\hat{v},\hat{\chi}) \end{cases}$$

where $\mathscr{A}'_{\omega}$ and $\mathscr{L}'$ are obtained by combining both equations in (PV'_{ω}):

$$\mathscr{A}'_{\omega}(\,(\hat{u},\hat{\lambda})\,;(\hat{v},\hat{\chi})\,) \;=\; \overline{\omega}\,a_{\omega}(\hat{u},\hat{v}) \;-\; \overline{\omega} < D_{\omega}(\gamma_0\hat{u}), \gamma_0\hat{v} > + \;\overline{\omega} < (\tfrac{1}{2}I - K_{\omega})\,\hat{\lambda}\,,\gamma_0\hat{v} >$$
$$+\;\omega < (\tfrac{1}{2}I - K'_{\omega})\,\gamma_0\hat{u}\;,\hat{\chi} > \;+\; \omega < S_{\omega}\,\hat{\lambda}\,,\hat{\chi} > ,$$

$$\mathscr{L}'(\hat{v},\hat{\chi}) \;=\; \overline{\omega}\;(\hat{f}\,,\hat{v}\,)\,.$$

Theorem 3. *$\mathscr{A}'_{\omega}$ is continuous on $H^{1,0}(\Omega_1)\times H^{-1/2}(\Sigma)$. Moreover, if $\omega=\alpha+i\sigma$, $\alpha\in\mathbb{R}$, $\sigma\in\mathbb{R}^{+*}$ $|\alpha| < \sigma\sqrt{3}$, then $\mathscr{A}'_{\omega}$ is $H^{1,0}(\Omega_1)\times H^{-1/2}(\Sigma)$ – coercive.*

Sketch of proof :

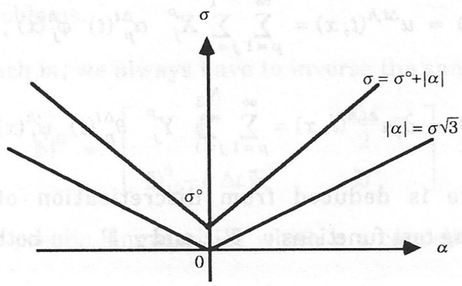
We show that $\forall\,(\hat{u},\hat{\lambda}) \in H^{1,0}(\Omega_1)\times H^{-1/2}(\Sigma)$,

$$|\,\mathscr{A}'_{\omega}(\,(\hat{u},\hat{\lambda})\,;(\hat{u},\hat{\lambda})\,)\,| \;\geq\; \sigma\;inf\,\{\,C_1\,;\,(\,1-\frac{|\omega|}{2\sigma}\,)\,C_2\,\}\;\|\,(\hat{u},\hat{\lambda})\,\|^2_{H^{1,0}(\Omega_1)\times H^{-1/2}(\Sigma)}\,,$$

where σ, C_1 and C_2 are strictly positive real numbers. In order to get a coercivity constant, it is necessary to satisfy $1-\frac{|\omega|}{2\sigma} > 0$, which is equivalent to $|\alpha| < \sigma\sqrt{3}$.

Let us return to the time-dependent problem, applying inverse Fourier-Laplace transform on path Γ_0 defined by

$$\Gamma_0 = \{\;\omega\in\mathbb{C}\;;\;\omega=\alpha+i\sigma\;;\;\;\sigma=\;\sigma_0+|\alpha|\;\}\;\subset\;\{\;\omega\in\mathbb{C}\;;\;\omega=\alpha+i\sigma\;;\;\;|\alpha|<\sigma\sqrt{3}\;\}$$



By noting that for regular functions F, G, compactly supported in $t > 0$, we have

$$\int_{\omega\in\Gamma_0} \hat{F}(\omega)\;\overline{\hat{G}(\omega)}\,d\omega \;=\; \iint_{\mathbb{R}^+\times\mathbb{R}^+} e^{-\sigma_0(t_1+t_2)}\;\frac{t_1+t_2}{t_1^{\,2}+t_2^{\,2}}\;F(t_1)\,\overline{G(t_2)}\,dt_1\,dt_2\,,$$

we deduce the variational space-time formulation by integrating $\mathscr{A}'_{\omega}$ along path Γ_0 :

$$\mathbb{A}[\,(u,\lambda);(v,\Psi)\,] = \mathbb{L}\,(v,\Psi),$$

$$\mathbb{A}[\,(u,\lambda);(v,\chi)\,] = \int_{\omega\in\Gamma_0} \mathscr{A}'_{\omega}(\,(\hat{u},\hat{\lambda})\,;(\hat{v},\hat{\chi})\,)\;d\omega,\qquad \mathbb{L}\,(v,\chi) = \int_{\omega\in\Gamma_0} \mathscr{L}'(\hat{v},\hat{\chi})\,d\omega\,.$$

For $u \in C^{\infty}(\mathbb{R}_t\,;\mathcal{H})$ compactly supported in $]0,\infty[_t$ where $\mathcal{H}$ is a Hilbert's space, we note

$$\|u\|_{m,\mathcal{H},\Gamma_0} = [\sum_{j=0}^{m} \iint_{\mathbb{R}^+\times\mathbb{R}^+} e^{-\sigma_0(t_1+t_2)} \frac{t_1+t_2}{t_1^{\,2}+t_2^{\,2}} < \partial_t^j u(t_1)\,;\,\partial_t^j u(t_2)>_{\mathcal{H}} dt_1 dt_2\,]^{1/2}.$$

Theorem 4. *There exists* $C>0$ *such that*

$$\mathbb{A}[\,(u,\lambda);(u,\lambda)\,] \geq C\,(\|u\|_{0,H^{1,0}(\Omega_1),\Gamma_0} + \|\lambda\|_{0,H^{-1/2}(\Sigma),\Gamma_0}\,)^2 .$$

Thus we found a space-time variational formulation satisfying a relation of coercivity, for a scattering acoustic problem with inhomogeneous obstacle. Unfortunately, this formulation is highly time-unlocal, and seems very hard to compute. That is the reason why we return to the first time-space formulation to set numerical results.

5. Discretization of the First Variational Formulation.

We give an approximation of variational problem (PV_t) thanks to a finite element method in both space and time. The straight line $\mathbb{R}_t$ is divided into regular spaces of time of lenght Δt. The volume Ω_1 is approximated with regular tetrahedra shaping Ω_{1h} , whose boundary Σ_h is made with triangular faces of volume elements. We use two kinds of basis functions: $\mathbb{P}_0$ functions are piecewise constant, and $\mathbb{P}_1$ functions are piecewise linear and continuous. u is choosen $\mathbb{P}_1$, and $\partial_\nu u$ is $\mathbb{P}_0$ in both space and time. Unknows $(u,\partial_\nu u)$ are then represented as

$$u(t,x) \approx u^{\Delta t,h}(t,x) = \sum_{p=1}^{\infty}\sum_{j=1}^{N_1} X_j^p\ \ \alpha_p^{\Delta t}(t)\ \ \varphi_j^h(x)\ ,\quad t\geq 0\ ,\ \ x\in\Omega_1\ ,$$

$$\partial_\nu u(t,x) \approx \partial_\nu u^{\Delta t,h}(t,x) = \sum_{p=1}^{\infty}\sum_{j=1}^{N_3} Y_j^p\ \ \theta_p^{\Delta t}(t)\ \ \psi_j^h(x)\ ,\quad t\geq 0\ ,\ \ x\in\Sigma\ ,$$

The scheme to solve is deduced from discretization of approached time-space formulation, by choosing test functions $v\ \ \mathbb{P}_1$, and $\chi\ \ \mathbb{P}_0$, in both space and time.

First step :
$$\begin{cases} \mathbb{C}^+X^1 - \dfrac{\Delta t}{2}\ \mathbb{E}\ Y^1 = F^0, \\ \mathbb{D}^0\ Y^1 - (\,\mathbb{D}_{12}^0 - \pi\,\Delta t\,\mathbb{E}^{\mathrm{T}}\,)X^1 = 0 \end{cases}$$

step $m\geq 2$:
$$\begin{cases} \mathbb{C}^+X^m - \dfrac{\Delta t}{2}\ \mathbb{E}\ Y^m = -\mathbb{C}^-X^{m-1} + \mathbb{C}^+X^{m-2} + \dfrac{\Delta t}{2}\ \mathbb{E}\ Y^{m-1} + F^{m-1} \\ \mathbb{D}^0\ Y^m - (\,\mathbb{D}_{12}^0 - \pi\,\Delta t\mathbb{E}^{\mathrm{T}}\,)X^m = \displaystyle\sum_{k=1}^{m} (-\mathbb{D}^k\,Y^{m-k} + \mathbb{D}_{12}^k\,X^{m-k}\,) - \pi\,\Delta t\,\mathbb{E}^{\mathrm{T}}\,X^{m-1} \end{cases}$$

where (X^m,Y^m) symbolizes double unknown $(u,\partial_\nu u)$,

$$(\mathbb{C}^{+}_{ij})_{1\le i,j\le N_1} = \frac{\Delta t}{6} \left(\int_{\Omega_{1h}} \nabla\varphi^h_i(x)\, \nabla\varphi^h_j(x)\; dx \right) + \frac{1}{\Delta t} \left(\int_{\Omega_{1h}} \varphi^h_i(x)\, \varphi^h_j(x)\; dx \right),$$

$$(\mathbb{C}_{ij})_{1\le i,j\le N_1} = \frac{2\Delta t}{3} \left(\int_{\Omega_{1h}} \nabla\varphi^h_i(x)\, \nabla\varphi^h_j(x)\; dx \right) - \frac{1}{2\Delta t} \left(\int_{\Omega_{1h}} \varphi^h_i(x)\, \varphi^h_j(x)\; dx \right),$$

$$(\mathbb{E}_{i,j})_{\substack{1\le i\le N_1\\ 1\le j\le N_3}} = \int_{\Sigma_h} \psi^h_j(x)\;\; \gamma_0\varphi^h_i(x)\, d\Sigma_x$$

$$F^m_i = \int_0^\infty \int_{\Omega_{1h}} \varphi^h_i(x)\; \alpha^{\Delta t}_m(t)\; f^{\Delta t,h}(t,x)\; dx\;\; dt,$$

$$(\mathbb{D}^k)_{i,j})_{1\le i,j\le N_3} = \int_0^\infty \iint_{\Sigma_h\times\Sigma_h} \frac{1}{|x-y|}\;\; \theta^{\Delta t}_{m-k}(t-|x-y|)\;\; \theta^{\Delta t}_m(t)\;\; \psi^h_i(x)\; \psi^h_j(y)\; d\Sigma_x\, d\Sigma_y\;\; dt$$

$$(\mathbb{D}^k_{12})_{ij})_{\substack{1\le i\le N_3\\ 1\le j\le N_1}} = \int_0^\infty \iint_{\Sigma_h\times\Sigma_h} \partial_\nu\!\left(\frac{1}{|x-y|}\right)\; [\alpha^{\Delta t}_{m-k}(t-|x-y|)+ |x-y|\partial_t\alpha^{\Delta t}_{m-k}(t-|x-y|)\,]\;\; \theta^{\Delta t}_m(t)\;\; \gamma_0\varphi^h_j(x)\;\; \psi^h_i(y)\; d\Sigma_x\, d\Sigma_y\;\; dt$$

On condition that X^{-1}, X^0, Y^0 are initialized in the volume equation, we obtain a marching-in-time scheme. The sought solution is causal so we take the natural choice $X^{-1}= X^0= Y^0= 0$. Hence unknowns (X^m,Y^m) depend on previous (X^k,Y^k) ($k \le m-1$).

Remarks:

- In our numerical implementation, we take $\sigma=0$: this choice is reasonable because we will work on time finite problems.

- We also notice that for each m, we always have to inverse the same matrix

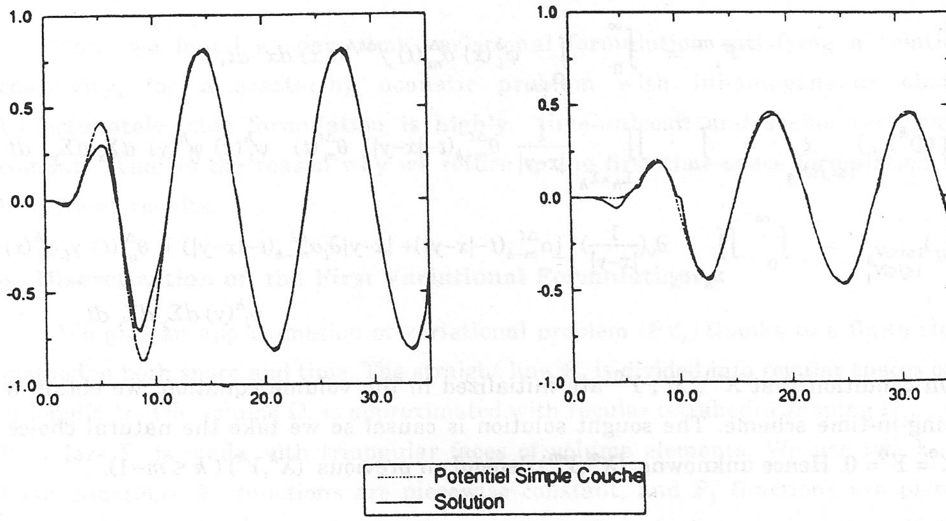
$$\mathbb{M}^0 = \begin{bmatrix} \mathbb{C}^+ & -\dfrac{\Delta t}{2}\,\mathbb{E} \\ \mathbb{D}^0_{12}-\pi\,\Delta t\,\mathbb{E}^T & \mathbb{D}^0 \end{bmatrix}$$

- Matrices $\mathbb{D}$ and $\mathbb{D}_{12}$ have pseudo-singularities which are cancelled, thanks to suitable changes of variable.

- $\sum_{k=1}^{m} (-\mathbb{D}^k\, Y^{m-k} + \mathbb{D}^k_{12}\, X^{m-k})$ is actually a finite sum, because for $k > [\max_{x,y\in\Sigma\times\Sigma} |x-y| \,/\Delta t]$ we have $\mathbb{D}^k = \mathbb{D}_{12}{}^k = 0$.

6. Numerical example.

In the following result, the computation is performed for the case of a spherical object lightened with an incident plane sinusoidal wave. This sphere is centered in origin, with radius 1, holds a hole of radius 0.7 also centered in origin, and is approximated with tetrahedra. For instance, we can observe the time-dependent solution in two points of the boundary Σ: the first figure represents the solution taken on the lightened zone, and the second figure describes the solution at a shaded point. The domain Ω_1 is supposed to be transparent, so we can compare our results with solution obtained by solving an exterior Dirichlet problem, using a retarded single layer representation (for more details, see [LU]). We have a good accuracy of results, wich can improve by increasing number of integrating points in computation of matrices $\mathbb{D}^k$, $\mathbb{D}_{12}{}^k$.



Conclusion.

We present a coupling of boundary element and finite element method to solve wave equation in 3D+1. The exterior time-dependent problem is phrased into a mixed problem for wave equation in a bounded domain $\mathcal{O}$, with unlocal boundary conditions on $\mathbb{R}_t\times\partial\mathcal{O}$. Fourier-Laplace transform allows to set an equivalent harmonic problem which is solved thanks to Fredholm's alternative. The inverse Fourier-Laplace transform gives then a space-time variational formulation, which the only solution provides the looked after field. The discretization of this variational formulation leads to a time-stepping scheme on which we obtain numerical stable results.

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