Centre de Mathématiques Laurent Schwartz
Ecole Polytechnique

SEMINAIRE

Equations aux Dérivées Partielles

2001-2002

Alain Bachelot

Wave Equation and Causality Violation

Séminaire É. D. P. (2001-2002), Exposé n° VII, 13 p.

<http://sedp.cedram.org/item?id=SEDP_2001-2002____A7_0>

U.M.R. 7640 du C.N.R.S.
F-91128 PALAISEAU CEDEX
Fax : 33 (0)1 69 33 49 49
Tél : 33 (0)1 69 33 49 99

WAVE EQUATION AND CAUSALITY VIOLATION

ALAIN BACHELOT

I. Introduction

The theory of the linear waves equations on globally hyperbolic manifolds has a long history since M. Riesz and J. Hadamard. It is impossible to cite all the important authors in the area, but we mention the fundamental works related to our study: the Cauchy problem investigated by J. Leray [26] and Y. Choquet-Bruhat [6] (see e.g. the excellent monograph [11] by F.G. Friedlander), the scattering theory for a compactly supported perturbation by P. Lax and R. Phillips [25], the microlocal analysis of the solutions by L. Hörmander [19] and J-M. Bony [4].

In opposite there are few works on the global hyperbolic problems on the *non* globally hyperbolic spacetimes. Nevertheless the global hyperbolicity is an extremely strong hypothesis, which is not satisfied by a lot of solutions of the (in)homogeneous Einstein equations. The origin of the loss of global hyperbolicity can be a non trivial topology, an elementary example is $S^1_t \times \mathbb{R}^3_x$ endowed with the Minkowski metric. Other examples are the lorentzian wormholes [12], [36], but since they lead to violations of the local energy conditions, these models are somewhat exotic. A deeper raison is linked with the non linearity of the Einstein equations that can create some singularities of curvature, and also some closed time-like geodesics. In particular, the violation of the causality can be due to a fast rotation of the space-time that tilts over the light cones so strongly that some closed causal curves appear. This phenomenon is present in several important Einstein manifolds: the Van Stockum space-time [33], the Gödel universe [15], the Kerr black-hole (third Boyer-Lindquist block and fast Kerr) [24], the spinning cosmic string [9]. These lorentzian manifolds belong to a wide range of stationnary, axisymmetric spacetimes that are described by the Papapetrou metric [29]

$$g_{\mu,\nu}dx^\mu dx^\nu = A(r,z)\left[dt - C(r,z)d\varphi\right]^2 - \frac{1}{A(r,z)}\left[r^2 d\varphi^2 + B(r,z)\left(dr^2+dz^2\right)\right],\ \ 0<A,B,\ \ 0\le C, \tag{I.1}$$

on some 3D+1 manifold $\mathcal{M}$.

Our model consists by choosing $\mathcal{M} = \mathbb{R}^4$, $A = B = 1$, and for simplicity we assume that C is compactly supported. When we allow that $C(r,z) > r$ (resp. $C(r,z) = r$) for some (r,z), some *closed* time-like (resp. null) curves appear and this spacetime has the same properties that the previous Einstein manifolds of point of view of the causality. We investigate the wave equation

$$\mid \det g \mid^{-\frac{1}{2}} \partial_\mu \left(\mid \det g \mid^{\frac{1}{2}} g^{\mu,\nu}\partial_\nu\right) u = \left(1-\frac{C^2}{r^2}\right)\partial_t^2 u - \Delta_x u - 2\frac{C}{r^2}\partial_t\partial_\varphi u = 0. \tag{I.2}$$

We also consider the zero-order perturbation of the D'Alembertian by a potential, for instance the conformally invariant wave equation. Obviously the study of the solutions is difficult because of the presence of closed timelike/null curves: there exists no global Cauchy hypersurface. We can see how much intricated is the situation by formally expanding a solution of (I.2) in Fourier series with respect to φ:

$$u(t,\varphi,r,z) = \sum_{m\in\mathbb{Z}} r^{-\frac{1}{2}} u_m(t,r,z)e^{im\varphi}.$$

Then u_m is solution of a changing type equation:

$$\left(1-\frac{C^2}{r^2}\right)\partial_t^2 u_m - \left(\partial_r^2+\partial_z^2\right)u_m - 2im\frac{C}{r^2}\partial_t u_m + \frac{m^2}{r^2}u_m = 0,$$

which is hyperbolic on $\{C<r\}$, elliptic on $\mathbb{T} := \{C>r\}$, and of Schrödinger type on $\Sigma := \{C=r\}$. In particular, $M_{t_0} := \{t=t_0\}\times\mathbb{R}^3_x$ is not a Cauchy hypersurface for (I.2) when Σ is not empty. Another crucial point is that since ∂_t is a Killing vector field, there exists a conserved current for the sufficiently smooth solutions of (I.2):

$$E(u) := \frac{1}{2}\int_{\mathbb{R}^3}\left(1-\frac{C^2}{r^2}\right) \mid \partial_t u(t,x) \mid^2 + \mid \nabla u(t,x) \mid^2 dx.$$

But this energy is *not* a positive form when the manifold is not chronological ($\mathbb{T} \neq \emptyset$).

II. Geometrical Framework

We consider the topologically trivial manifold

$$\mathcal{M} := \mathbb{R}^4_{(x^0,x^1,x^2,x^3)} = \mathbb{R}_t \times \mathbb{R}^3_x \tag{II.1}$$

endowed with a lorentzian metric g which is equal to the Minkowski metric outside a torus

$$\mathbb{R}_t \times \left\{(x^1,x^2,x^3); 0 < r_-^2 < \mid x^1 \mid^2 + \mid x^2 \mid^2 < r_+^2,\ z_- < x^3 < z_+\right\}.$$

We choose a particular case of the Papapetrou metric:

$$g_{\mu,\nu}dx^\mu dx^\nu = dt^2 - \left[r^2 - C^2(r,z)\right]d\varphi^2 - 2C(r,z)dtd\varphi - dr^2 - dz^2, \tag{II.2}$$

where we have used the cylindrical coordinates $(t,\varphi,r,z) \in \mathbb{R} \times [0,2\pi[\times[0,\infty[\times\mathbb{R}$ given by

$$x^1 = r\cos\varphi,\ x^2 = r\sin\varphi,\ x^3 = z. \tag{II.3}$$

We assume that C satisfies

$$0 \le C(r,z),\ \ C \in C^2(\mathbb{R}^2),\ \ (r,z) \notin [r_-,r_+] \times [z_-,z_+] \Rightarrow C(r,z) = 0, \tag{II.4}$$

and our geometrical framework is given by (II.1), (II.2), (II.4).

We note that t is a timelike coordinate and $(\mathcal{M},g)$ is naturally time oriented by the continuous, nowhere vanishing, timelike (and Killing) vector field ∂_t. Moreover r and z are spacelike coordinates. The interesting fact is that the nature of the Killing vector field ∂_φ is ambiguous: the crucial point is that φ is a *timelike* coordinate when $C > r$, thus we introduce

$$\mathcal{T} := \mathbb{R}_t \times \mathcal{T}_0,\ \ \mathcal{T}_0 := S^1 \times \{(r,z);\ \ C(r,z) > 0\}, \tag{II.5}$$

$$\mathbb{T} := \mathbb{R}_t \times \mathbb{T}_0,\ \ \mathbb{T}_0 := S^1 \times \{(r,z);\ \ C(r,z) > r\}, \tag{II.6}$$

$$\Sigma := \mathbb{R}_t \times \Sigma_0,\ \ \Sigma_0 := S^1 \times \{(r,z);\ \ C(r,z) = r > 0\}. \tag{II.7}$$

We shall need the hypersurfaces

$$M_t := \{t\} \times \mathbb{R}^3. \tag{II.8}$$

Its causal structure is complex. Since its normal is dt, the nature of M_t is locally given by the sign of

$$g^{tt} = 1 - \frac{C^2}{r^2},$$

hence $M_t \cap (\mathbb{R}^4 \setminus (\mathbb{T} \cup \Sigma))$ is spacelike, $M_t \cap \Sigma$ is null, and $M_t \cap \mathbb{T}$ is timelike.

We shall be mainly concerned by the case where Σ is not empty. In this situation the causality is violated in a severe way: given $m_0 = (t_0,\varphi_0,r_0,z_0)$, the path

$$\tau \in \mathbb{R} \longmapsto m(\tau) = (t_0,\varphi_0 - \tau, r_0, z_0) \in \mathcal{M}, \tag{II.9}$$

is a future directed closed null curve if $m_0 \in \Sigma$, and a future directed closed timelike curve if $m_0 \in \mathbb{T}$ since:

$$g\left(\frac{dm}{d\tau},\frac{dm}{d\tau}\right) = C^2(r_0,z_0) - r_0^2,\ \ g\left(\frac{dm}{d\tau},\frac{\partial}{\partial t}\right) = 2C(r_0,z_0) > 0.$$

More precisely, the causal structure of $\mathcal{M}$ is described by the following:

Proposition II.1. *Let $(\mathcal{M},g)$ be the lorentzian manifold defined by (II.1), (II.2), (II.4).*

(1) *If $\Sigma = \emptyset$, $(\mathcal{M},g)$ is globally hyperbolic: M_t is a Cauchy hypersurface for any $t \in \mathbb{R}$.*

(2) *If $\mathbb{T}=\emptyset$ and $\Sigma\neq\emptyset$, $(\mathcal{M},g)$ is chronological but non causal: there exists no closed timelike curve, but there exists a closed null geodesic.*
(3) *If $\mathbb{T}\neq\emptyset$, $(\mathcal{M},g)$ is totally vicious i.e. given $m_0,m_1\in\mathcal{M}$, there exists a timelike future-pointing curve from m_0 to m_1.*

The previous proposition explains why, in the physical litterature (see e.g. [14], [36]), $\mathbb{T}$ and Σ are respectively called, *time machine*, and *velocity-of-light surface*. This last term is somewhat misleading since $\partial(\mathcal{M}\setminus\mathbb{T})\subset\Sigma$, but it can happen that $\partial(\mathcal{M}\setminus\mathbb{T})\neq\Sigma$ and Σ is not necessarily a hypersurface. If there exists no (r_0,z_0) satisfying (II.11), the theorem of implicit functions immediately assures that Σ is a C^2-hypersurface that is timelike because its normal $N=(\partial_rC-1)dr+\partial_zCdz$ is spacelike since $g^{\mu,\nu}N_{\mu,\nu}=-(\partial_rC-1)^2-(\partial_zC)^2<0$. Moreover, this is a sufficient and necessary condition on C for a geometrical property of non-trapping type:

Proposition II.2. *Let $m\in C^2\left(\mathbb{R}_\tau;\mathcal{M}\right)$ be a path. Then the following assertions are equivalent:*

(i) m is a null geodesic and for some $T>0$:

$$m(\mathbb{R})\subset[-T,+T]_t\times\Sigma_0, \tag{II.10}$$

(ii) there exists (t_0,φ_0,r_0,z_0), $\lambda\in\mathbb{R}^*$, such that:

$$\left\{\begin{array}{l} C(r_0,z_0)=r_0>0,\quad \partial_rC(r_0,z_0)=1,\quad \partial_zC(r_0,z_0)=0,\\ m(\tau)=(t_0,\varphi_0+\lambda\tau,r_0,z_0).\end{array}\right. \tag{II.11}$$

We say that Σ_0 is *Non-Confining* if there exists no null geodesic included in $\{t_0\}\times\Sigma_0$ for some t_0. Following the previous result, a necessary and sufficient condition is

$$C(r_0,z_0)=r_0>0\Longrightarrow(\partial_rC(r_0,z_0),\partial_zC(r_0,z_0))\neq(1,0), \tag{II.12}$$

and in this case Σ is a C^2 timelike hypersurface.

III. The Wave Equation

The D'Alembertian on a Lorentzian manifold $(\mathcal{M},g)$ is defined by

$$\Box_g:=\mid\det g\mid^{-\frac{1}{2}}\partial_\mu\left(\mid\det g\mid^{\frac{1}{2}}g^{\mu,\nu}\partial_\nu\right).$$

For the space-time given by (II.1), (II.2), we obtain:

$$\Box_g=\left(1-\frac{C^2}{r^2}\right)\partial_t^2-\Delta_x-2\frac{C}{r^2}\partial_t\partial_\varphi, \tag{III.1}$$

with

$$r^2=\mid x^1\mid^2+\mid x^2\mid^2,\ \ \Delta_x:=\partial_{x^1}^2+\partial_{x^3}^2+\partial_{x^3}^2=\partial_r^2+\partial_z^2+r^{-2}\partial_\varphi^2+r^{-1}\partial_r,\ \ \partial_\varphi=x^1\partial_{x^2}-x^2\partial_{x^1}.$$

More generally we consider the scalar perturbations of the massless wave equation, compactly supported in x, invariant with respect to the both Killing vector fields ∂_t, ∂_φ:

$$L:=\Box_g+V, \tag{III.2}$$

where

$$V\in C_0^0(\mathbb{R}_x^3;\mathbb{R}),\ \ \partial_\varphi V=0. \tag{III.3}$$

These assumptions are fulfilled in the important case of the conformally invariant wave equation for which:

$$V=\frac{1}{6}R_g, \tag{III.4}$$

where R_g is the scalar curvature of $(\mathcal{M},g)$. We use $R_0>0$ be such that

$$R_0\leq\mid x\mid\Longrightarrow C(r,z)=V(x)=0. \tag{III.5}$$

We know that the D'Alembertian on a lorentzian curved space-time is strictly hyperbolic in a local sense (see e.g. [11]). The global hyperbolicity is more delicate. We denote

$$P_2(m,\xi):=g^{\mu,\nu}(m)\xi_\mu\xi_\nu,\ \ m\in\mathcal{M},\ \ \xi\in T_m^*\mathcal{M}, \tag{III.6}$$

the principal symbol of L.

Proposition III.1. (1) *Let α be in $\mathbb{R}$. Then, $P_2(m,.)$ is (strictly) hyperbolic with respect to the covector $dt+\alpha d\varphi$ iff α satisfies:*

$$-C(m)-r<\alpha<r-C(m). \tag{III.7}$$

(2) *If $\Sigma\neq\emptyset$, there does not exist $F\in C^1(\mathcal{M};\mathbb{R})$ such that L is hyperbolic with respect to the level surfaces of F.*

The previous result implies in particular that in the interesting case where $\mathbb{T}\neq\emptyset$, the initial value problem for L with data specified on $M_{t_0}=\{t_0\}\times\mathbb{R}^3$ is not well posed. (III.7) shows that the failure of the global hyperbolicity is due to the very fast rotation of the torus. Nevertheless, since ∂_t is a Killing vector field, it will be interesting to investigate the solutions of $Lu=0$ as some distributions on $\mathbb{R}_t$, valued in some spaces of distributions on $\mathbb{R}^3_x$. In order to choose the functional framework, it is useful to note that since the time translation leaves the wave equation invariant, the Noether's theorem assures the existence of a conserved current. We formally obtain the conserved energy

$$E(u;t):=\frac{1}{2}\int_{\mathbb{R}^3}\left(1-\frac{C^2}{r^2}\right)\mid\partial_t u(t,x)\mid^2+\mid\nabla u(t,x)\mid^2+V(x)\mid u(t,x)\mid^2\,dx. \tag{III.8}$$

Therefore it is natural to look for the solutions of

$$Lu=0,\quad u\in L^2_{loc}\left(\mathbb{R}_t;W^1(\mathbb{R}^3_x)\right), \tag{III.9}$$

where $W^1(\mathbb{R}^3_x)$ is the Beppo-Levi space defined as the completion of $C_0^\infty(\mathbb{R}^3_x)$ with respect to the norm:

$$\| f\|^2_{W^1}=\int_{\mathbb{R}^3}\mid\nabla f(x)\mid^2\,dx,\quad \nabla:={}^t(\partial_{x^1},\partial_{x^2},\partial_z). \tag{III.10}$$

We recall the L^2-type estimate:

$$W^1(\mathbb{R}^3_x)\subset L^2_\rho(\mathbb{R}^3_x):=L^2\left(\mathbb{R}^3_x,\frac{1}{1+\mid x\mid^2}dx\right),\quad \| f\|_{L^2_\rho}\leq K\| f\|_{W^1}. \tag{III.11}$$

The choice of the regularity of $\partial_t u$ is less clear when $\mathcal{M}$ is not globally hyperbolic since $(1-\frac{C^2}{r^2})$ is negative on $\mathbb{T}_0$ and the energy is not a positive form. We introduce the space:

$$L^2_C(\mathbb{R}^3_x):=L^2\left(\mathbb{R}^3_x,\mid 1-\frac{C^2}{r^2}\mid dx\right), \tag{III.12}$$

and we investigate the solutions u of (III.9) satisfying:

$$\partial_t u\in L^2_{loc}\left(\mathbb{R}_t;L^2_C(\mathbb{R}^3_x)\right). \tag{III.13}$$

With this functional framework, we define the local energy by

$$E_R(u;t):=\frac{1}{2}\int_{|x|\leq R}\left(1-\frac{C^2}{r^2}\right)\mid\partial_t u(t,x)\mid^2+\mid\nabla u(t,x)\mid^2+V(x)\mid u(t,x)\mid^2\,dx. \tag{III.14}$$

Lemma III.2. *Given u,v satisfying (III.9) and (III.13), we have for $R\geq R_0$, and almost all $t,s\in\mathbb{R}$:*

$$E_R(u,t)\leq E_{R+|t-s|}(u,s),\quad E_\infty(u,t)=E_\infty(u,s), \tag{III.15}$$

When $u,v\in C^0\left(\mathbb{R}_t;W^1(\mathbb{R}^3_x)\right)$, $\partial_t u,\partial_t v\in C^0\left(\mathbb{R}_t;L^2_C(\mathbb{R}^3_x)\right)$, (III.15) is satisfied for any $s,t\in\mathbb{R}$, and the conserved quantity $E(u):=E_\infty(u,t)$ is the total energy of u. If $\mathbb{T}$ is not empty, this quadratic form is not definite positive.

We could only consider solutions of (III.9) such that $\partial_t u\in L^2_{loc}(\mathbb{R}_t;L^2_C(\mathbb{R}^3_x))$, but if Σ_0 is non-confining, we prove that $\partial_t u$ is much more regular by using the results of J-M. Bony [4] :

Theorem III.3. *We assume that Σ_0 is Non-Confining. Let u be such that*

$$u \in L^2_{loc}\left(\mathbb{R}_t ; W^1(\mathbb{R}^3_x)\right), \quad Lu \in L^2_{loc}\left(\mathbb{R}_t ; L^2(\mathbb{R}^3_x)\right). \tag{III.16}$$

Then we have:

$$\partial_t u \in L^2_{loc}\left(\mathbb{R}_t ; L^2(\mathbb{R}^3_x)\right). \tag{III.17}$$

The previous result allows define the trace of u and $\partial_t u$ on M_t. We refer to [27] for the definitions and properties of the usual Sobolev spaces H^s, H^s_0.

Proposition III.4. *We assume that Σ_0 is Non-Confining. Let u be such that*

$$u \in L^2_{loc}\left(\mathbb{R}_t ; W^1(\mathbb{R}^3_x)\right), \quad Lu \in L^2_{loc}\left(\mathbb{R}_t ; L^2(\mathbb{R}^3_x)\right).$$

Then we have:

$$u \in C^0\left(\mathbb{R}_t ; H^{\frac{1}{2}}(\mathbb{R}^3_x)\right), \quad \left(1 - \frac{C}{r}\right)\partial_t u \in C^0\left(\mathbb{R}_t ; H^{-\frac{1}{2}}(\mathbb{R}^3_x)\right), \quad \partial_t u \in C^0\left(\mathbb{R}_t ; H^{-1}(\mathbb{T}_0)\right). \tag{III.18}$$

Thanks to the result of continuity stated in Proposition III.4, we may investigate the uniqueness of a possible solution of $Lu = 0$ for data specified on M_{t_0}. First we prove that $u = 0$ on $\mathcal{M}$ when $u = (C - r)\partial_t u = 0$ on M_{t_0}. This result is neither a consequence of the uniqueness theorem for the strictly hyperbolic operators ([19], Theorem 23.2.7) because the level surfaces M_t are not non-characteristic since $P_2(m, dt) = 0$ on Σ, nor a direct application of the conservation of the energy since $E(u)$ is not definite positive.

Moreover, when $\mathcal{M}$ is totally vicious, i.e. $\mathbb{T} \neq \emptyset$, and the Non-Confining Condition is fullfiled, we would like that $u = 0$ on $\mathcal{M}$ when $u = 0$ on $\mathbb{T}$. Unfortunately, although Σ is non-characteristic, we cannot use the classical results of unique continuation: on the one hand, 0 is a double real root of $P_2(m, dt + \tau N) = 0$ for $m \in \Sigma$, $N = (\partial_r C(m) - 1)dr + \partial_z C(m)dz$, hence we cannot apply the Calderon Theorem ([19], theorem 28.1.8). On the other hand, we have for $m \in \Sigma$:

$$\{P_2, \{P_2, C - r\}\}(m, dt) = -4\left(\mid \partial_r C(m) - 1 \mid^2 + \mid \partial_z C(m) \mid^2\right) < 0,$$

hence Σ is nowhere strongly pseudo-convex, and we can no more use the uniqueness theorems for second order operators of real principal type due to N. Lerner and L. Robbiano (see [19], Theorem 28.4.3) to deduce that $u = 0$ on $\mathcal{M}$, from $u = 0$ on $\mathbb{T}$. This leads to make some assumption of analyticity on C and V near Σ_0, in order to apply the Holmgren Theorem.

Theorem III.5. *We assume that Σ_0 is Non-Confining and $\mathbb{T}_0 \neq \emptyset$. Let u be satisfying (III.9) and one of the following conditions for some $t_0 \in \mathbb{R}$:*

(1) $u = \left(1 - \frac{C}{r}\right)\partial_t u = 0$ *on* M_{t_0}.

(2) $u = \partial_t u = 0$ *on* $\{t_0\} \times \mathbb{T}_0$ *and V and C are real analytic in a neighborhood of Σ_0.*

Then

$$u = 0 \ \text{ on } \ \mathcal{M}. \tag{III.19}$$

We shall see in Part 5 another uniqueness result for the incoming solutions.

A key ingredient is the following result involving the Aronszajn-Cordes theorem :

Lemma III.6. *We assume that Σ_0 is Non-Confining. Let u satisfying (III.9) and such that for some $t_0 \in \mathbb{R}$:*

$$u = \partial_t u = 0 \ \text{ on } \ \{t_0\} \times \mathbb{T}_0. \tag{III.20}$$

Then

$$u = 0 \ \text{ on } \ \mathbb{T}. \tag{III.21}$$

The sequel of this work deals with the problem of the existence of such solutions, that is not obvious when the manifold is not causal. We introduce the vector space

$$\mathcal{E} := \left\{ u \in C^0\left(\mathbb{R}_t; W^1(\mathbb{R}^3_x)\right); \;\; Lu = 0, \;\; \partial_t u \in C^0\left(\mathbb{R}_t; L^2_C(\mathbb{R}^3_x)\right) \right\}, \tag{III.22}$$

endowed with the indefinite form $E(u)$ given by (III.8) and the space of the admissible Cauchy data:

$$\mathcal{H} := \left\{ (f,g) \in W^1(\mathbb{R}^3_x) \times L^2_C(\mathbb{R}^3_x); \;\; \exists u \in \mathcal{E}, \;\; \mathbf{u}(0) = (f,g) \right\}, \tag{III.23}$$

where for $v \in C^1\left(\mathbb{R}_t; \mathcal{D}'(\mathbb{R}^3_x)\right)$, we put

$$\mathbf{v} := \begin{pmatrix} v \\ \partial_t v \end{pmatrix}. \tag{III.24}$$

A priori, when $\mathbb{T} \neq \emptyset$, $\mathcal{H}$ is not an Hilbert space for the norm of $W^1 \times L^2_C$. The previous Theorem assures that the family of maps

$$U(t): \; \mathbf{u}(0) \in \mathcal{H} \longmapsto \mathbf{u}(t) \in \mathcal{H}. \tag{III.25}$$

is a strongly continuous group of linear operators on $\mathcal{H}$. In the following parts we construct global solutions u with $E(u) = 0$ or $E(u) > 0$. We let open the problem of the existence of global solution with negative energy.

IV. The Resonant States

In this section, we investigate the global solutions $u \in H^1_{loc}(\mathcal{M})$ by separation of the variable t:

$$u(t,x) = e^{\lambda t} v(x), \tag{IV.1}$$

with $\lambda \in \mathbb{C}$ and v is a distribution on $\mathbb{R}^3_x$. Then u is solution of

$$Lu = 0 \;\; in \;\; \mathcal{M}, \tag{IV.2}$$

iff $v \in L^2_{loc}(\mathbb{R}^3_x)$ is solution of the homogeneous reduced wave equation:

$$\Delta v + \frac{2C\lambda}{r^2}\partial_\varphi v - \lambda^2\left(1 - \frac{C^2}{r^2}\right) v - Vv = 0 \;\; on \;\; \mathbb{R}^3. \tag{IV.3}$$

By the standard result of elliptic regularity, $v \in H^2_{loc}(\mathbb{R}^3)$ and $v \in C^\infty$ for $\mid x \mid$ large enough, since C and V are continuous and compactly supported. (IV.3) is similar to the acoustic wave equation in an inhomogeneous medium (see e.g. [7], [21], [31], [35]); the crucial difference is that $1 - r^{-2}C^2$ that plaies the role of the refractive index, is null on Σ_0 and negative in $\mathbb{T}_0$.

We start by proving a result of Rellich type, stating that there exists no t-periodic, non constant, solution of $Lu = 0$ satisfying some natural constraint at the space infinity.

Lemma IV.1. *Let v be a solution of (IV.3) for $\lambda \in i\mathbb{R}^*$, satisfying one of the following condition:*

$$v \in L^2(\mathbb{R}^3) \cup W^1(\mathbb{R}^3); \tag{IV.4}$$

$$\frac{x}{\mid x \mid}.\nabla v + \lambda v = O\left(\frac{1}{\mid x \mid^2}\right), \;\; \mid x \mid \to \infty; \tag{IV.5}$$

$$\frac{x}{\mid x \mid}.\nabla v - \lambda v = O\left(\frac{1}{\mid x \mid^2}\right), \;\; \mid x \mid \to \infty; \tag{IV.6}$$

Then $v = 0$.

For $\lambda = 0$ the result is well known: for non negative potential V, the conclusion of the Lemma is valid; for general potential V, since the form $v \mapsto \int V \mid v \mid^2$ is compact on $H^1_{loc}(\mathbb{R}^3)$, the space of solutions of (IV.3) with $\lambda = 0$ is of finite dimension.

Lemma IV.1 shows that we have to look for the non trivial solutions of the homogeneous reduced wave equation, for $\lambda \in \mathbb{C} \setminus i\mathbb{R}$. We adapt at our problem the concept of outgoing (resp. incoming) solution by

Lax-Phillips [25]. Given $\lambda \in \mathbb{C}$, $f \in \mathcal{E}'$, the space of the compactly supported distributions, a solution $v_\lambda^{+(-)}$ of

$$\Delta v + \frac{2C\lambda}{r^2}\partial_\varphi v - \lambda^2\left(1 - \frac{C^2}{r^2}\right)v - Vv = f \ \ on \ \ \mathbb{R}^3, \tag{IV.7}$$

is said to be λ-*outgoing* (resp. λ-*incoming)* if

$$v_\lambda^{+(-)} = \gamma_\lambda^{+(-)} * \left[f - \frac{2C\lambda}{r^2}\partial_\varphi v_\lambda^{+(-)} - \lambda^2\frac{C^2}{r^2}v_\lambda^{+(-)} + Vv_\lambda^{+(-)}\right], \tag{IV.8}$$

where

$$\gamma_\lambda^{+(-)}(x) := -\frac{e^{-(+)\lambda|x|}}{4\pi \mid x \mid}. \tag{IV.9}$$

It is well known that in the case $\lambda \in i\mathbb{R}$, the λ-outgoing (resp. λ-incoming) condition is equivalent to the Sommerfeld radiation condition (IV.5) (resp. (IV.6) A complex number λ is an *outgoing resonance (resp. incoming resonance)*, if there exists a non null λ-outgoing (resp. λ-incoming) solution $v_\lambda^{+(-)}$ of (IV.3), called *resonant state.* We remark that when a resonant state v_λ has a finite energy, i.e. $v_\lambda \in H^1(\mathbb{R}^3)$, the total energy (III.8) of the time dependant solution $u_\lambda(t,x) = e^{\lambda t}v_\lambda(x)$ is zero:

$$E(u_\lambda) = \frac{1}{2}e^{2\Re(\lambda)t}\int_{\mathbb{R}^3} \mid \lambda \mid^2 \left(1 - \frac{C^2}{r^2}\right) \mid v_\lambda \mid^2 + \mid \nabla v_\lambda \mid^2 + V \mid v_\lambda \mid^2 dx = 0. \tag{IV.10}$$

We denote $\mathcal{R}^{+(-)}$ the set of the outgoing (incoming) resonances. Because C and V are real axisymmetric, and since we may take $v_\lambda^+(x^1, -x^2, z) = v_{-\lambda}^-(x^1, x^2, z)$, it is easy to see that:

$$\lambda \in \mathcal{R}^+ \Longleftrightarrow \overline{\lambda} \in \mathcal{R}^+, \tag{IV.11}$$

$$\lambda \in \mathcal{R}^+ \Longleftrightarrow -\lambda \in \mathcal{R}^-. \tag{IV.12}$$

Hence we shall consider only the set of the outgoing resonances, simply called "resonances", and we omit the superscript $+$: $\mathcal{R} := \mathcal{R}^+$, $v_\lambda := v_\lambda^+$.

We summarize the properties of the set of the resonances:

Theorem IV.2. *$\mathcal{R}$ is a discrete subset of $\mathbb{C}$, and we have:*

$$\mathcal{R} \cap \rangle\mathbb{R}^* = \emptyset; \tag{IV.13}$$

$$\lambda \in \mathcal{R}, \ \ 0 < \Re(\lambda) \Longrightarrow v_\lambda \in H^2(\mathbb{R}^3); \tag{IV.14}$$

$$\mathbb{T}_0 = \emptyset \Longrightarrow Card\{\lambda \in \mathcal{R}; \ 0 \leq \Re(\lambda)\} < \infty; \tag{IV.15}$$

$$\mathbb{T}_0 = \emptyset, \ \ 0 \leq V \Longrightarrow \{\lambda \in \mathcal{R}; \ 0 \leq \Re(\lambda)\} = \emptyset; \tag{IV.16}$$

$$\mathcal{T}_0 \neq \emptyset, \ \ \lambda \in \mathcal{R}\cap]0,\infty[\Longrightarrow \partial_\varphi v_\lambda = 0; \tag{IV.17}$$

$$\mathbb{T}_0 \neq \emptyset \Longrightarrow Card(\mathcal{R}\cap]0,\infty[) = \infty. \tag{IV.18}$$

We know that for the scattering by obstacle there exists no real resonance, and for the scattering by non positive potential, or metric perturbation, or Schwarzschild black-hole, there exists only a finite set of real resonances with finite energy (see e.g. [2], [25]). (IV.15) and (IV.16) show that this remains true even if there is a closed null geodesic ($\Sigma_0 \neq \emptyset$) but no closed timelike curve ($\mathbb{T}_0 = \emptyset$). The main novelty, (IV.18), due to the existence of a closed timelike curve, is that this set is *infinite.* This last result can be physically interpreted as follows: in the framework of the studies of the stability of the manifolds of the General Relativity, the existence of an infinite set of resonant states with finite energy means that we cannot prove the possible stability of the metric (II.2) by a method of perturbation (see e.g. the works of Y. Choquet-Bruhat, A. Fischer, J. Marsden); hence we can suspect that the manifold is actually nonlinearly instable in a suitable set of solutions of inhomogeneous Einstein equations. This agrees with the "conjecture of chronological protection" by S. Hawking [17], that states that any universe with closed timelike curve is instable.

V. Scattering States

When $\mathbb{T}$ is not empty, the manifold is totally vicious, hence there exists no Cauchy hypersurface. Nevertheless the global Cauchy problem is well posed for regular data specified at the past null infinity, and these solutions are asymptotically free at the future null infinity (*Scattering States*). Furthermore, the Scattering Operator S is well defined for any free wave with finite energy, but, unlike the usual situations, the wave operators are *not* causal. As regards the mathematical tools, we keep the features of the scattering theory, that involve neither the positivity of the energy, nor the existence of a unitary group: we use the generalised eigenfunctions method.

We start with a uniqueness result for the solutions with some given asymptotic behaviour. We recall some basic notations for the wave equation on the Minkowski space-time:

$$L_0 u_0 := \partial_t^2 u_0 - \Delta_x u_0 = 0, \ \ (t,x) \in \mathbb{R} \times \mathbb{R}^3. \tag{V.1}$$

The Cauchy problem is solved in $\mathcal{D}'(\mathbb{R}^3_x)$ by the group $U_0(t)$:

$$U_0(t)\mathbf{u}_0(0) = \mathbf{u}_0(t). \tag{V.2}$$

We introduce: the spaces associated with the finite energy waves,

$$\mathcal{E}_0 := \left\{u_0 \in C^0\left(\mathbb{R}_t; W^1(\mathbb{R}^3_x)\right); \ L_0 u_0 = 0, \ \partial_t u_0 \in C^0\left(\mathbb{R}_t; L^2(\mathbb{R}^3_x)\right)\right\}, \ \mathcal{H}_0 := W^1(\mathbb{R}^3_x) \times L^2(\mathbb{R}^3_x), \tag{V.3}$$

which are Hilbert spaces for the energy norm

$$\| u_0 \|^2_{\mathcal{E}_0} = \| \mathbf{u}_0(t) \|^2_{\mathcal{H}_0} := \frac{1}{2}\int_{\mathbb{R}^3} | \partial_t u_0(t,x) |^2 + | \nabla u_0(t,x) |^2 \, dx, \tag{V.4}$$

and $U_0(t)$ is a strongly continuous unitary group on $\mathcal{H}_0$.

Theorem V.1. *Let u be in $\mathcal{E}$. We assume that one of the both following conditions is fullfiled:*

(1)

$$u \in L^1\left(\mathbb{R}_t; L^2_{loc}(\mathbb{R}^3_x)\right), \tag{V.5}$$

$$\| \mathbf{u}(t) \|_{W^1 \times L^2_C} \longrightarrow 0, \ \ t \to -\infty. \tag{V.6}$$

(2) $\mathbb{T} \neq \emptyset$, *and there exist $a, c, R \geq 0$, such that*

$$\| u(t) \|_{W^1} \leq c e^{a|t|}, \tag{V.7}$$

$$| x | \leq -t - R \Longrightarrow u(t,x) = 0. \tag{V.8}$$

Then

$$u = 0 \ \ on \ \ \mathcal{M}. \tag{V.9}$$

We make some remarks. 1) The global constraint (V.5) is usefull when $\mathbb{T} \neq \emptyset$: the outgoing resonant states with finite energy satisfy (V.6) but are exponentially increasing as $t \to +\infty$. 2) It is known that when $\mathbb{T} = \emptyset$ and $0 \leq V$ there exists non null solutions satisfying (V.7) and (V.8). 3) We deduce from Lemma III.6 that (V.7) is a consequence of (V.8) when Σ_0 is Non Confining.

We now return to the problem of global solutions by constructing Wave Operators. We denote $\mathcal{E}_0^\infty$ the space of the *regular wave packets* that are the smooth solutions u_0 of (V.1) such that

$$\hat{u}_0(0,\xi) := \int e^{-ix.\xi} u_0(0,x)dx, \ \ \partial_t \hat{u}_0(0,\xi) := \int e^{-ix.\xi} \partial_t u_0(0,x)dx \in C_0^\infty\left(\mathbb{R}^3_\xi \setminus \{0\}\right). \tag{V.10}$$

Theorem V.2. *Given $u_0^- \in \mathcal{E}_0^\infty$, there exists a unique $u \in \mathcal{E}$ such that $\partial_t u \in C^0\left(\mathbb{R}_t; L^2(\mathbb{R}^3_x)\right)$ and satisfying (V.5) and*

$$\| \mathbf{u}(t) - \mathbf{u}_0^-(t) \|_{\mathcal{H}_0} \to 0, \ \ t \to -\infty. \tag{V.11}$$

Moreover there exists a unique $u_0^+ \in \mathcal{E}_0$ such that:

$$\| \mathbf{u}(t) - \mathbf{u}_0^+(t) \|_{\mathcal{H}_0} \to 0, \quad t \to +\infty, \tag{V.12}$$

and we have:

$$\| u_0^- \|_{\mathcal{E}_0}^2 = E(u) = \| u_0^+ \|_{\mathcal{E}_0}^2, \tag{V.13}$$

$$u_0^+ \in \mathcal{E}_0^\infty. \tag{V.14}$$

This Theorem allows to introduce the Wave Operators

$$W^- : u_0^- \longmapsto u, \quad W^+ : u_0^+ \longmapsto u. \tag{V.15}$$

To make the link between these both operators, we use the time reverse operator

$$R: \; u(t, x^1, x^2, z) \longmapsto (Ru)(t, x^1, x^2, z) = u(-t, x^1, -x^2, z). \tag{V.16}$$

Since $R(Lu) = L(Ru)$, we have

$$W^+ = RW^-R. \tag{V.17}$$

These wave operators are defined on $\mathcal{E}_0^\infty$, but when the chronology is violated, $\mathbb{T} \neq \emptyset$, we do know to characterize neither their ranges, nor the possible continuity property. Furthermore, they are no causal in the usual sense, since Theorem V.1 shows that if $u = W^- u_0^-$ exists for some free wave $u_0^- \in \mathcal{E}_0$ satisfying the initially incoming condition (V.8), and $u = u_0^-$ for $t << 0$, then $u = u_0^- = 0$.

We now consider the Scattering Operator

$$S: \; u_0^- \longmapsto u_0^+. \tag{V.18}$$

The previous Theorem assures that S is an isometry from $\mathcal{E}_0^\infty$ onto $\mathcal{E}_0^\infty$, and by (V.17) we have

$$S^{-1} = RSR. \tag{V.19}$$

Therefore S can be extended by continuity and density, into an unitary operator on $\mathcal{E}_0$, denoted S again. To investigate this operator, we recall two important tools (see [7], [25],[30]): the translation representation for the free wave equation is the map

$$u_0 \in \mathcal{E}_0 \longmapsto f^\sharp \in L^2\left(\mathbb{R}_s \times S^2_\omega, ds d\omega\right), \;\; f^\sharp(s,\omega) = - \lim_{|t|\to\infty} t \partial_t u_0(t, x = (t+s)\omega) \;\; in \;\; L^2_{loc}\left(\mathbb{R}_s \times S^2_\omega, ds d\omega\right), \tag{V.20}$$

that is an isometry from $\mathcal{E}_0$ onto $L^2\left(\mathbb{R}_s \times S^2_\omega, ds d\omega\right)$; the spectral representation is the isometry $u_0 \mapsto \tilde{f}$ from $\mathcal{E}_0$ onto $L^2\left(\mathbb{R}_k \times S^2_\omega, dk d\omega\right)$ defined by :

$$\tilde{f}(k,\omega) = \frac{1}{\sqrt{2\pi}} \int e^{iks} f^\sharp(s,\omega) ds. \tag{V.21}$$

We put

$$\mathbf{S}: \; \mathbf{u}_0^-(0) \longmapsto \mathbf{u}_0^+(0). \tag{V.22}$$

Then $\mathbf{S}$ is an isometry from $\mathcal{H}_0$ onto $\mathcal{H}_0$, and because of the invariance of the wave equation $Lu = 0$ by the time translation, we have for any $t \in \mathbb{R}$:

$$U_0(t)\mathbf{S} = \mathbf{S}U_0(t). \tag{V.23}$$

With obvious notations, we can also represent the scattering operator by putting:

$$S^\sharp f_-^\sharp = f_+^\sharp, \quad \tilde{S}\tilde{f}_- = \tilde{f}_+. \tag{V.24}$$

Since $\mathbf{S}$ commutes with the free group $U_0(t)$, $S^\sharp$ commutes with the s-translation. Then $\tilde{S}$ is represented as a multiplicative operator-valued function $\tilde{S}(k)$ on $L^2(S^2_\omega)$. We shall state in Proposition V.4 that we can represent $\tilde{S}(k)$ by using the distorded plane waves as well as for the usual globally hyperbolic case.

We start by constructing global solutions by using the *distorded plane waves* $\Phi(t, x; k, \omega)$:

Lemma V.3. *For all $k \in \mathbb{C}$, $ik \notin \mathcal{R}$, $\omega \in S^2$, there exists a unique ik-outgoing function $\Psi(x;k,\omega)$that is a $H^2_{loc}(\mathbb{R}^3_x)$-valued analytic function on $(\mathbb{C}_k \setminus i\mathcal{R}) \times S^2_\omega$, such that*

$$\Phi(t,x;k,\omega) := e^{ik(t-x.\omega)} + e^{ikt}\Psi(x;k,\omega) \tag{V.25}$$

is solution of $L\Phi = 0$.

For any $\tilde{f}_- \in C_0^\infty\left(\mathbb{R}^*_k \times S^2_\omega\right)$, the function

$$u(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{S^2}\Phi(t,x;k,\omega)\tilde{f}_-(k,\omega)dkd\omega \tag{V.26}$$

satisfies

$$u \in C^0\left(\mathbb{R}_t; W^1(\mathbb{R}^3_x)\right),\quad \partial_t u \in C^0\left(\mathbb{R}_t; L^2(\mathbb{R}^3_x)\right),\quad Lu = 0. \tag{V.27}$$

Proposition V.4. *There exists a function $\tilde{S}(\omega',k,\omega)$ analytic on $S^2_{\omega'} \times (\mathbb{C}_k \setminus i\mathcal{R}) \times S^2_\omega$ such that*

$$\Psi(x;k,\omega) = \frac{e^{-ik|x|}}{\mid x \mid}\tilde{S}\left(\frac{x}{\mid x \mid},k,\omega\right) + O\left(\frac{e^{-ik|x|}}{\mid x \mid^2}\right),\quad \mid x \mid \to \infty, \tag{V.28}$$

$$\frac{x}{\mid x \mid}.\nabla_x\Psi(x;k,\omega) = -ik\frac{e^{-ik|x|}}{\mid x \mid}\tilde{S}\left(\frac{x}{\mid x \mid},k,\omega\right) + O\left(\frac{e^{-ik|x|}}{\mid x \mid^2}\right),\quad \mid x \mid \to \infty. \tag{V.29}$$

For any $\tilde{f}_- \in L^2\left(\mathbb{R}_k \times S^2_\omega\right)$, we have

$$\left(\tilde{S}\tilde{f}_-\right)(k,\omega) = \tilde{f}_-(k,\omega) - \frac{ik}{2\pi}\int_{S^2}\tilde{S}\left(\omega,k,\omega'\right)\tilde{f}_-(k,\omega')d\omega'. \tag{V.30}$$

When the manifold is globally hyperbolic, i.e. $\mathbb{T} = \Sigma = \emptyset$, we can apply the general results of the "black box" scattering (see e.g. [37]), that assure that $k \in \mathbb{C} \mapsto \tilde{S}(k) \in \mathcal{L}\left(L^2(S^2)\right)$ defined by (V.30) is meromorphic on $\mathbb{C}$ and the poles essentially correspond to the resonances. More precisely, the multiplicity of a pole k of $\tilde{S}$ is equal to the difference between the multiplicities of the possible resonances ik and $-ik$. We state a less precise result when the metric is not causal.

Theorem V.5. *The $\mathcal{L}\left(L^2(S^2)\right)$ valued scattering matrix $\tilde{S}(k)$ is meromorphic on $\mathbb{C}_k$. If $k_0 \in \mathbb{C}$ is a pole, then $ik_0 \in \mathcal{R}$. Conversely a complex number k_0 satisfying*

$$\Re(ik_0) > 0,\quad ik_0 \in \mathcal{R},\quad -ik_0 \notin \mathcal{R}, \tag{V.31}$$

is a pole of $\tilde{S}(k)$.

When the manifold is chronological, $\mathbb{T} = \emptyset$, but non causal, $\Sigma \neq \emptyset$, and if $0 \leq V$, then there exists no resonance with positive real part (Theorem IV.2, (IV.16)). In this case, the Fourès-Segal Theorem [10] implies that the scattering operator S is causal. When the manifold is non chronological, $\mathbb{T} \neq \emptyset$, we have stated in Theorem IV.2, (IV.18), that there exists infinitely many resonances with positive part. We conjecture that some resonance satisfies (V.31) and the scattering operator is not causal.

VI. Scattering by a Causality Violation in a Chronological Space-Time

In this part we prove the completeness of the wave operators in the case where the manifold is chronological but non globally hyperbolic:

$$\mathbb{T} = \emptyset, \tag{VI.1}$$

$$\Sigma \neq \emptyset. \tag{VI.2}$$

The case of the globally hyperbolic space-time, $\mathbb{T} = \emptyset$, $\Sigma = \emptyset$, has been treated by D. Häfner [16]. Thus we assume that:

$$\sup\frac{C}{r} = 1. \tag{VI.3}$$

In order to use some energy estimates, we impose the positivity of the total energy:

$$0 \leq V. \tag{VI.4}$$

First we consider the Cauchy problem with data on M_{t_0}. We show that this problem is well posed despite the existence of closed null geodesics. That is not entirely surprising since M_{t_0} is *weakly spacelike* according to the terminology of L. Hörmander who has studied the characteristic Cauchy problem on a globally hyperbolic manifold [20]. Nevertheless, because of the violation of the causality, we have to be carefull to define the set of the possible initial data.

We remark that Σ_0 is necessarily confining, hence we cannot invoke Theorem III.5 to assure the uniqueness. But since the conserved energy $E(u)$ is now positive, $\mathcal{E}$, $\mathcal{H}$ defined by (III.22), (III.23), are Hilbert spaces, and $u \mapsto \mathbf{u}(0)$ is an isometry from $\mathcal{E}$ onto $\mathcal{H}$, for the norms

$$\| u \|_{\mathcal{E}}^2 := E_\infty(u,t) = \| \mathbf{u}(0) \|_{\mathcal{H}}^2 := \frac{1}{2} \| \partial_t u(0) \|_{L_C^2}^2 + \frac{1}{2} \| u(0) \|_1^2 . \tag{VI.5}$$

We have used the equivalent norm on $W^1(\mathbb{R}_x^3)$:

$$\| f \|_1^2 := \int_{\mathbb{R}^3} | \nabla f(x) |^2 + V(x) \mid f(x) |^2 \, dx.$$

Since $U(t)$ given by (III.25) is a strongly continuous unitary group $U(t)$ on $\mathcal{H}$, the Stone theorem assures that there exists a self-adjoint operator A on $\mathcal{H}$, with dense domain $D(A)$, such that

$$U(t) = e^{itA}.$$

It is easy to characterize $D(A)$ in terms of more regular solutions:

$$D(A) = \left\{ \mathbf{u}(0); \;\; u \in \mathcal{E}^1 \right\}, \;\; \mathcal{E}^1 := \left\{ u \in \mathcal{E}; \;\; \partial_t u \in \mathcal{E} \right\}. \tag{VI.6}$$

To state that the space of the admissible Cauchy data is large, we introduce the set

$$\mathcal{D} := \left\{ (f,g) \in W^1(\mathbb{R}_x^3) \times H^1(\mathbb{R}_x^3); \;\; \Delta f \in L^2(\mathbb{R}_x^3), \;\; \Delta f + 2\frac{C}{r^2}\partial_\varphi g - Vf = 0 \;\; on \;\; a \;\; neighborhood \;\; \mathcal{V}_{(f,g)} \;\; of \;\; \Sigma_0 \right\}, \tag{VI.7}$$

and the Beppo-Levi space $W_0^1\left(\mathbb{R}_x^3 \setminus \Sigma_0\right)$ as completion of $C_0^\infty\left(\mathbb{R}_x^3 \setminus \Sigma_0\right)$ for the norm (III.10).

Theorem VI.1. *We assume that (VI.1) and (VI.4) are fullfiled. Then we have:*

$$C_0^\infty\left(\mathbb{R}_x^3 \setminus \Sigma_0\right) \times C_0^\infty\left(\mathbb{R}_x^3 \setminus \Sigma_0\right) \subset D(A), \tag{VI.8}$$

$$W_0^1\left(\mathbb{R}_x^3 \setminus \Sigma_0\right) \times L_C^2\left(\mathbb{R}_x^3\right) \subset \mathcal{H}, \tag{VI.9}$$

$$\mathcal{D} \subset \mathcal{H}. \tag{VI.10}$$

Moreover if the Lebesgue measure of Σ_0 is zero, then

$$\overline{\mathcal{D}} = \mathcal{H} = W^1(\mathbb{R}_x^3) \times L_C^2(\mathbb{R}_x^3). \tag{VI.11}$$

We now return to the scattering theory. We have seen that the scattering operator S is an isometry from $\mathcal{E}_0$ onto $\mathcal{E}_0$. Nevertheless, when the space time is totally vicious ($\mathbb{T} \neq \emptyset$), we can define the wave operators $W^{+(-)}$ only on the dense set of the regular wave packets, $\mathcal{E}_0^\infty$, and the range of these operators is not known. Taking advantage of the fact that the conserved energy is positive when $\mathbb{T} = \emptyset$, we could extend by continuity the wave operators (V.15) previously defined on $\mathcal{E}_0^\infty$, but in order to be more concrete, we prefer to directely construct them, by replacing $W^1 \times L^2$ by $W^1 \times L_C^2$ in the control of the asymptotic behaviour, and using a time-dependent method. Despite the violation of the causality ($\Sigma_0 \neq \emptyset$), we are able to develop a strategy *à la* Lax-Phillips [25] because the chronology is respected, and we get

$$\operatorname{Ran} W^+ = \operatorname{Ran} W^- = \mathcal{E}. \tag{VI.12}$$

We need the R-outgoing (R-incoming) subspaces:

$$D_R^{+(-)} := \left\{ F = (f,g) \in \mathcal{H}_0; \;\; | x | \leq +(-)t + R \Rightarrow U_0(t)F = 0 \right\}, \;\; 0 \leq R. \tag{VI.13}$$

Proposition VI.2. *We assume that (VI.1) and (VI.4) are fullfiled. Given* $u_0^{+(-)} \in \mathcal{E}_0$*, there exists a unique* $u^{+(-)} \in \mathcal{E}$ *such that:*

$$\| \mathbf{u}^{+(-)}(t) - \mathbf{u}_0^{+(-)}(t) \|_{W^1 \times L^2_C} \to 0, \quad t \to +(-)\infty. \tag{VI.14}$$

Moreover we have:

$$\| u^{+(-)} \|_{\mathcal{E}} = \| u_0^{+(-)} \|_{\mathcal{E}_0} . \tag{VI.15}$$

Therefore we have proved that the Wave Operators

$$W^{+(-)} : \ u_0^{+(-)} \longmapsto u^{+(-)} \tag{VI.16}$$

extend the wave operators (V.15) defined only on $\mathcal{E}_0^\infty$, and are isometries from $\mathcal{E}_0$ to $\mathcal{E}$. The main result of this part states these operators are onto.

Theorem VI.3. *We assume that (VI.1) and (VI.4) are fullfiled. Then for all* $u \in \mathcal{E}$*, there exists a unique* $u_0^{+(-)} \in \mathcal{E}_0$ *such that:*

$$\| \mathbf{u}(t) - \mathbf{u}_0^{+(-)}(t) \|_{W^1 \times L^2_C} \to 0, \quad t \to +(-)\infty. \tag{VI.17}$$

Moreover we have:

$$\| u \|_{\mathcal{E}} = \| u_0^{+(-)} \|_{\mathcal{E}_0} . \tag{VI.18}$$

The crucial point is the decay of the local energy that we establish by using the RAGE theorem.

Lemma VI.4. *Let* $u \in \mathcal{E}$*. Then for all* $R \geq R_0$ *we have:*

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T \sqrt{E_R(u,t)} dt = 0. \tag{VI.19}$$

$$\overline{\cup_{t \in \mathbb{R}} U(t) D_R^+} = \overline{\cup_{t \in \mathbb{R}} U(t) D_R^-} = \mathcal{H}. \tag{VI.20}$$

We achieve this study by some remarks on the Scattering Operator S. We have shown that even if the chronology is violated ($\mathbb{T} \neq \emptyset$), the scattering operator is a well defined isometry on $\mathcal{E}_0$, but in this case, its meaning is somewhat mysterious since we can construct the wave operators only on $\mathcal{E}_0^\infty$. When the chronology is not violated, we deduce from the previous theorem that $(W^+)^{-1}$ is well defined from $\mathcal{E}$ to $\mathcal{E}_0$, and with Proposition VI.2 we conclude that the Scattering Operator is actually defined by

$$S := \left(W^+\right)^{-1} W^-. \tag{VI.21}$$

Moreover since D_R^+ and D_R^- are orthogonal, the scattering operator S is causal in the usual sense (e.g. [25]), i.e.

$$\left(| x | \leq -t \Rightarrow u_0^-(t,x) = 0\right) \Longrightarrow \left(| x | \leq -t \Rightarrow u_0^+(t,x) = 0\right),$$

although the manifold $\mathcal{M}$ is non causal (it would be preferable to say S is chronological, since this is this property of $\mathcal{M}$ that assures the so called causality of S). This is also a consequence of the Theorem of Fourès, Segal [10], and of the spectral representation of S, Proposition V.4, since we have stated in Theorem IV.2 (IV.16) that there exists no resonance with positive real part.

It is without saying that this work is only a first incursion in the mathematically widely unexplored domain of the field equations on the non globally hyperbolic manifolds (for a rather significant bibliography of the physical litterature see e.g. [9], [11], [13], [14], [17], [23], [34], [36]). We have not dealed with many important questions such that: the asymptotic repartition of the resonances; the singularities of the scattering kernel; the existence of a "trace formula" making a link between some geometric quantities (e.g. the lenght of the closed null geodesics), and the spectral numbers; the Strichartz type estimates, etc. Last but not least, the field of the nonlinear wave equations on a non causal space-time is *terra incognita*.

References

[1] A. Bachelot. Global properties of the wave equation on non-globally hyperbolic manifolds. *J. Math. Pures Appl.*, 81: 35–65, 2002.

[2] A. Bachelot, A. Motet-Bachelot. Les résonances d'un trou noir de Schwarzschild. *Ann. Inst. Henri Poincaré - Physique théorique*, 59(1): 3–68, 1993.

[3] M. Beals, M. Reed. Microlocal regularity theorems for nonsmooth pseudodifferential operators and applications to non-linear problems. *Trans. Amer. Math. Soc.*, 285(1): 159–184, 1984.

[4] J-M. Bony. Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. *Ann. Sci. École Norm. Sup.*, 14(2): 209–246, 1981.

[5] Y. Bruhat. Sur la théorie des propagateurs. *Ann. di Mat. Pura Appl.*, 64: 191–228, 1964.

[6] Y. Choquet-Bruhat. Hyperbolic partial differential equations on a manifold. *Battelle Rencontres*, 1967, Benjamin, New York, 84–106, 1968.

[7] D. Colton, R. Kress. *Inverse Acoustic and Electromagnetic Scattering Theory.* Springer-Verlag, second edition 1998.

[8] J. Cooper, W. Strauss. Representations of the Scattering Operator for Moving Obstacles. *Indiana Univ. Math. J.*, 28(4): 643–671, 1979.

[9] S. Deser, R. Jackiw, G. 't Hooft. Three dimensional Einstein gravity: Dynamics of flat space. *Ann. Phys. (NY)*, 152: 220–235, 1984.

[10] Y. Fourès, I.E. Segal. Causality and analyticity. *Trans. Am. Math. Soc.*, 78: 385–405, 1955.

[11] F. G. Friedlander. *The wave equation on a curved space-time.* Cambridge University Press, 1975.

[12] J. L. Friedman, M.S. Morris. Existence and Uniqueness Theorems for Massless Fields on a Class of Spacetimes with Closed Timelike Curves. *Commun. Math. Phys.*, 186: 495–529, 1997.

[13] R. P. Geroch. The domain of dependence. *J. Math. Phys.*, 11: 437–449, 1970.

[14] G. W. Gibbons, C.A.R. Herdeiro. Supersymmetric rotating black holes and causality violation. *Class. Quantum Grav.*, 16: 3619–3652, 1999.

[15] K. Gödel. An example of a new type of cosmological solution of Einstein's field equations of gravitation. *Rev. Mod. Phys.*, 21: 447–450, 1949.

[16] D. Häfner. Complétude asymptotique pour l'équation des ondes dans une classe d'espaces-temps stationnaires et asymptotiquement plats. *Ann. Institut Fourier*, 51(3): 779–833, 2001.

[17] S. W. Hawking. Chronology protection conjecture. *Phys. Rev. D*, 46(2): 603–611, 1992.

[18] S. W. Hawking, G. F. R. Ellis. *The large scale structure of space-time.* Cambridge University Press, 1974.

[19] L. Hörmander. *The Analysis of Linear Partial Differential Operators, I, II, III, IV.* Springer Verlag, second printing 1994.

[20] L. Hörmander. A Remark on the Characteristic Cauchy Problem. *J. Funct. Anal.*, 93: 270–277, 1990.

[21] T. Ikebe. A Spectral Theory for the reduced Wave Equation with a Complex Refractive Index. *Publ. RIMS, Kyoto Univ.*, 8: 579–606, 1972/73.

[22] T. Kato. *Perturbation Theory for Linear Operators.* Springer Verlag, second edition, 1980.

[23] B. S.. Kay. Application of linear hyperbolic PDE to linear quantum fields incurved spacetimes: especially black holes, time machines and a new semi-local vacuum concept. *Journées Equations aux dérivées partielles*, Exposé IX, Nantes, 2000.

[24] R. P. Kerr. Gravitational field of a spinning mass as an example of algebraically special metrics. *Phys. Rev. Lett.*, 11: 237–238, 1963.

[25] P. Lax, R. Phillips. *Scattering Theory.* Academic Press, Revised Edition 1989.

[26] J. Leray. *Hyperbolic differential equations.* Princeton University, 1952.

[27] J-L. Lions, E. Magenes. *Problèmes aux limites non homogènes et applications I.* Dunod, 1968.

[28] R. B. Melrose. *Geometric Scattering Theory.* Cambridge University Press, 1995.

[29] A. Papapetrou. Champs gravitationnels à symétrie axiale. *Ann. Inst. Henri Poincaré*, A4: 83–105, 1966.

[30] V. Petkov. *Scattering Theory for Hyperbolic Operators.* North Holland, 1989.

[31] M. Reed, B. Simon. The Scattering of Classical Waves from Inhomogeneous Media. *Math. Z.*, 155: 163–180, 1977.

[32] M. Reed, B. Simon. *Methods of modern mathematical physics, Vol. II, III, IV.* Academic Press, 1975, 1978.

[33] W. J. van Stockum. Gravitational field of a distribution of particles rotating about an axis of symetry. *Proc. R. Soc. Edin.*, 57: 135–154, 1937.

[34] F. J. Tipler. Singularities and Causality Violation. *Ann. Phys.*, 108: 1–36, 1977.

[35] B. R. Vainberg. *Asymptotic Methods in Equations of Mathematical Physics.*
Gordon and Breach Science Publishers, 1989.

[36] M. Visser. *Lorentzian Wormholes.* Springer Verlag, 1995.

[37] M. Zworski. Poisson formulae for resonances. *Séminaire E.D.P.*, Ecole Polytechnique, Exposé XIII, 1996–1997.

Université Bordeaux-1, Institut de Mathématiques, UMR CNRS 5466, F-33405 Talence Cedex
E-mail address: `bachelot@math.u-bordeaux.fr`