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Inverse scattering problem for the nonlinear Klein–Gordon equation

1. INTRODUCTION

In this paper we shall study the inverse scattering problem for the nonlinear Klein-Gordon equation

$$u_{tt} - \Delta_x u + m^2 u = f(x,u), \quad x \in \mathbb{R}^3.$$

In [4] Morawetz and Strauss showed that the scattering operator uniquely determines any interaction function f(u) which is odd and analytic. We extend this result to

$$f(x,u) = \sum_{k=1}^{\infty} a_k(x) \, |u|^{2k} u.$$

The fundamental argument is that the solution u_m of the Klein-Gordon equation with mass $m > 0$

$$u_{tt} - \Delta_x u + m^2 u = 0 \tag{KG_m}$$

converges to the solution of the wave equation in $L^4(\mathbb{R}^4)$ as $m \to 0$. In Section 2, we treat the case $f(x,u) = q(x) \, |u|^2 u$ (see [1], [2]); in Section 3, we establish a theorem of convergence in $L^q(\mathbb{R}^{n+1})$ of solutions of the inhomogeneous Klein-Gordon equation as $m \to m_0$; in Section 4, we derive our general result of inverse scattering.

2. THE CUBIC INTERACTION

THEOREM 1. Let q be a function in $W^{1,\infty}(\mathbb{R}^3)$. Then q is determined by the scattering operator S for the nonlinear Klein-Gordon equation

$$u_{tt} - \Delta_x u + m^2 u = q(x) \, |u|^2 u, \quad x \in \mathbb{R}^3. \tag{NLKG}$$

More precisely, given $x_0 \in \mathbb{R}^3$, $\lambda > 0$, let $g \neq 0$ in $S(\mathbb{R}^3)$ be so that

$$|\xi|^{-\frac{1}{2}} \hat{g} \in L^2(\mathbb{R}^3_\xi).$$

Let ϕ_λ be the solution of (KG_m) so that $\phi_\lambda(0,x) = 0$, $\phi_{\lambda_t}(0,x) = \lambda g(\lambda(x-x_0))$ and let u be the solution of $u_{tt} - \Delta_x u = 0$, $u(0,x) = 0$, $u_t(0,x) = g(x)$; then $q(x_0)$ is given by

$$q(x_0) = [\iint |u(t,x)|^4 dt\, dx]^{-1} \lim_{\lambda\to+\infty} \lambda^4 \lim_{\varepsilon\to 0} \varepsilon^{-5}[W(S(\varepsilon\phi_\lambda), S(\varepsilon^2\phi_\lambda)) - \varepsilon^3 W(\phi_\lambda,\phi_\lambda)]$$

where $W(f,f')$ is defined for f, f' in $C^1(\mathbb{R}_t, L^2(\mathbb{R}^3_x))$ by

$$W(f,f') = \int f\, \bar{f}'_t - f_t\, \bar{f}'\, dx.$$

<u>Proof</u>: The existence of the scattering operator for small regular data is proved in [6]. We choose u_-, v_- solutions of (KG_m) so that $u_+ = Su_-$, $v_+ = Sv_-$ exist.

Let u, v be solutions of (NLKG) which verify

$$\|u(t) - u_\pm(t)\|_e \to 0, \quad \|v(t) - v_\pm(t)\|_e \to 0, \text{ as } t \to \pm\infty,$$

where $\| \;\; \|_e$ denotes the energy norm. Following [4] consider $W(u,v)$

$$W(u,v) = \int (u(t)\, \bar{v}_t(t) - u_t(t)\bar{v}(t))\, dx.$$

By differentiation and integration we obtain

$$W(u(T), v(T)) - W(u(-T),v(-T)) = \int_{-T}^{+T}\int q\, u\, \bar{v}(|u|^2 - |v|^2) dt\, dx$$

and by a passage to the limit $T \to +\infty$

$$W(Su_-, Sv_-) - W(u_-, v_-) = \iint q\, u\, \bar{v}(|u|^2 - |v|^2) dt\, dx.$$

Now, choose $u_- = \varepsilon\phi_\lambda$, $v_- = \varepsilon^2\phi_\lambda$, ε small enough; in order to estimate the integral we represent u as

$$u(t) = u_-(t) + \int_{-\infty}^{t} R(t-s) \underset{x}{*} q|u|^2 u\, ds,$$

where R is the Riemann function of (KG_m).

We solve this equation by the standard Picard method in

$$\{u \in L^4(\mathbb{R}^4) \ / \ \|u\|_{L^4(\mathbb{R}^4)} \leq 2\,\|u_-\|_{L^4(\mathbb{R}^4)}\}$$

for u_- small enough. By the L^p-L^q estimate for R (see [3]) and the standard singular integral inequality, we obtain

$$|u(t) - u_-(t)|_{L^4(\mathbb{R}^3)} \leq C \int_{-\infty}^{t} |t-s|^{-\frac{1}{2}} |u(s)|^3_{L^4(\mathbb{R}^3)}\, ds,$$

$$\|u-u_-\|_{L^4(\mathbb{R}^4)} \leq C'\,\|u\|^3_{L^4(\mathbb{R}^4)} \leq 8\,C'\,\|u_-\|^3_{L^4(\mathbb{R}^4)}.$$

Hence

$$\|u-\varepsilon\,\phi_\lambda\|_{L^4(\mathbb{R}^4)} = O(\varepsilon^3), \quad \|v-\varepsilon^2\phi_\lambda\|_{L^4(\mathbb{R}^4)} = O(\varepsilon^6).$$

Applying these estimates, we obtain

$$W(S(\varepsilon\,\phi_\lambda),\, S(\varepsilon^2\phi_\lambda)) - \varepsilon^3 W(\phi_\lambda,\phi_\lambda) = \varepsilon^5 \iint q|\phi_\lambda|^4\, dt\, dx + O(\varepsilon^7).$$

From change variable $x' = \lambda(x-x_0)$, $t' = \lambda t$ we have

$$\iint q(x_0 + \frac{x'}{\lambda})|u_\lambda(t',x')|^4\, dt'\, dx'$$

$$= \lambda^4 \lim_{\varepsilon\to 0} \varepsilon^{-5}[W(S(\varepsilon\phi_\lambda),\, S(\varepsilon^2\phi_\lambda)) - \varepsilon^3 W(\phi_\lambda,\phi_\lambda)],$$

where $u_\lambda(t', x') = \phi_\lambda(t,x)$ is a solution of

$$u_{t't'} - \Delta_{x'}u + \left(\frac{m}{\lambda}\right)^2 u = 0;\ u_\lambda(0,x') = 0;\ u_{\lambda_{t'}}(0,x') = g(x'). \qquad \text{(E)}$$

Therefore, in order to determine $q(x_0)$ it suffices to prove that solutions of equation $(KG)_m$ converge in $L^4(\mathbb{R}^4)$ to the solution of the wave equation as $m \to 0$.

3. CONVERGENCE OF SOLUTIONS OF (KG_m) AS $m \to m_0$

THEOREM 2. Let $m \geq 0$; we denote by $u_m = T_m(f,g,h)$ the solution of

$$u_{tt} - \Delta_x u + m^2 u = h(t,x), \qquad x \in \mathbb{R}^n,$$

$$u(0,x) = f(x):\ u_t(0,x) = g(x).$$

Let H, H_0 be Hilbert spaces defined by

$$H = \{(f,g)/(1+|\xi|^2)^{1/4}\hat{f} \text{ and } (1+|\xi|^2)^{-1/4}\hat{g} \in L^2(\mathbb{R}^n_\xi)\},$$

$$H_0 = \{(f,g)/(1+|\xi|^2)^{1/4}\hat{f} \text{ and } |\xi|^{-1/2}\hat{g} \in L^2(\mathbb{R}^n_\xi)\}.$$

For $n \geqslant 2$ we have the following results:

(a) if $m, m_0 > 0$, then $T_m \to T_{m_0}$ in $\mathcal{L}(H \times L^p(\mathbb{R}^{n+1}), L^q(\mathbb{R}^{n+1}))$ under the topology of simple convergence as $m \to m_0$, with p, q so that

$$\frac{1}{p} + \frac{1}{q} = 1;\ 2\,\frac{n+2}{n} \leqslant q \leqslant 2\,\frac{n+1}{n-1}.$$

(b) $T_m \to T_0$ in $\mathcal{L}(H_0 \times L^p(\mathbb{R}^{n+1}), L^q(\mathbb{R}^{n+1}))$ under the topology of simple convergence as $m \to 0$ with p, q so that

$$\frac{1}{p} + \frac{1}{q} = 1;\ q = 2\,\frac{n+1}{n-1}.$$

<u>Proof</u>: Strichartz [8] proved the equicontinuity of T_m, hence it suffices to show the convergence on a dense subset of data (f,g,h). First we study solutions of the homogeneous equation $u_m = T_m(f,g,0)$; u_m may be written:

$$u_m(t,x) = \mathcal{F}^{-1}_{\tau,\xi}\,[\tfrac{1}{2}(\sqrt{m^2+|\xi|^2}\hat{f}(\xi) + i\,\mathrm{sgn}(\tau)\,\hat{g}(\tau))\,d\mu_m],$$

where $d\mu_m$ is a measure given by

$$\langle d\mu_m, \phi(\tau,\xi)\rangle = \int (\phi(\sqrt{m^2+|\xi|^2},\xi) + \phi(-\sqrt{m^2+|\xi|^2},\xi))(m^2+|\xi|^2)^{-\frac{1}{2}}\,d\xi.$$

Suppose that: $\hat{f}, \hat{g} \in \mathcal{D}(\mathbb{R}^n)$, $0 \notin \operatorname{supp}\hat{f} \cup \operatorname{supp}\hat{g}$.

We choose $\chi \in \mathcal{D}(\mathbb{R}^{n+1}\setminus\{0\})$ so that

$$\xi \in \operatorname{supp}\hat{f} \cup \operatorname{supp}\hat{g},\ 0 \leqslant \tau^2 - |\xi|^2 \leqslant (m_0+1)^2 \Rightarrow \chi(\tau,\xi) = 1.$$

We have for $0 \leqslant m \leqslant m_0 + 1$

$$u_m(t,x) = \mathcal{F}^{-1}_{\tau,\xi}[\tfrac{1}{2}(\sqrt{m^2+|\xi|^2}\,\hat{f}(\xi) + i\,\mathrm{sgn}(\tau)\hat{g}(\xi))\chi] * \mathcal{F}^{-1}_{\tau,\xi}\,(\chi . d\mu_m).$$

Hence $u_m - u_{m_0}$ may be written

$$u_m - u_{m_0} = F_m * \mathcal{F}^{-1}_{\tau,\xi}[\chi(d\mu_m - d\mu_{m_0})] + \mathcal{F}^{-1}_{\tau,\xi}[\tfrac{1}{2}(\sqrt{m^2+|\xi|^2} - \sqrt{m_0^2+|\xi|^2})\hat{f}(\xi).d\mu_{m_0}].$$

Let

$$S_{m_0} = \{(\tau,\xi)/-\tau^2 + |\xi|^2 + m_0^2 = 0\}.$$

Then $\frac{1}{2}\hat{f}(\xi)(\sqrt{m^2 + |\xi|^2} - \sqrt{m_0 + |\xi|^2})$ converges to 0 in $L^2(S_{m_0}, d\mu_{m_0})$ as $m \to m_0$; hence it follows from [8] that for the desired q

$$\mathcal{F}^{-1}_{\tau,\xi}[\tfrac{1}{2}(\sqrt{m^2 + |\xi|^2} - \sqrt{m_0^2 + |\xi|^2})\hat{f}(\xi)\, d\mu_{m_0}] \to 0 \text{ in } L^q(\mathbb{R}^{n+1}) \text{ as } m \to m_0.$$

On the other hand, we notice that $(F_m)_{0 \leqslant m \leqslant m_0+1}$ is bounded in $L^1(\mathbb{R}^{n+1})$. Now consider the analytic family of operators T_m^z

$$T_m^z\, \phi = \phi * \mathcal{F}^{-1}_{\tau,\xi}[\gamma(z)((-\tau^2 + |\xi|^2 + m^2)_+^z - (-\tau^2 + |\xi|^2 + m_0^2)_+^z)\chi],$$

where $\gamma(z)$ is a suitable holomorphic function which cancels the poles of the distribution valued meromorphic function $(-\tau^2 + |\xi|^2 + m^2)_+^z$ and with a simple zero at $z = -1$ so that

$$T_m^{-1}\, \phi = \phi * \mathcal{F}^{-1}_{\tau,\xi}[\chi(d\mu_m - d\mu_{m_0})].$$

More precisely we choose γ

$$\gamma(z) = [\Gamma(z+1)]^{-1}(z + \tfrac{n}{2})\sin[\pi(z + \tfrac{n}{2})],\ n \text{ odd}$$

$$\gamma(z) = [\Gamma(z+1)]^{-1}(z + \tfrac{n}{2})\sin[\pi(z + \tfrac{n}{2})]\,(z+1)^{-1},\ n \text{ even.}$$

So that $(T_m^z)_{\mu \leqslant m \leqslant m_0+1}$ satisfies the hypothesis of Stein's interpolation theorem for $-(n+2)/2 \leqslant \operatorname{Re} z \leqslant 0$ if $\mu > 0$ and for $-(n+1)/2 \leqslant \operatorname{Re} z \leqslant 0$ if $\mu = 0$.

From the dominated convergence theorem it follows that

$$T_m^{iy} \to 0 \text{ in } \mathcal{L}(L^1(\mathbb{R}^{n+1}), L^2(\mathbb{R}^{n+1})) \text{ as } m \to m_0$$

and by an explicit computation we obtain that $(T_m^z)_{\mu \leqslant m \leqslant m_0+1}$ is bounded in $\mathcal{L}(L^1(\mathbb{R}^{n+1}), L^\infty(\mathbb{R}^{n+1}))$ for $-(n+2)/2 \leqslant \operatorname{Re} z \leqslant -(n+1)/2$ if $0 < \mu, m_0$ and for $\operatorname{Re} z = -(n+1)/2$ if $\mu = 0$. Stein's interpolation theorem shows that $T_m^{-1} \to 0$

in $\mathcal{L}(L^1(\mathbb{R}^{n+1}), L^q(\mathbb{R}^{n+1}))$ for suitable q as $m \to m_0$. This implies that $u_m \to u_{m_0}$ in $L^q(\mathbb{R}^{n+1})$ as $m \to m_0$.

Now, study the convergence of $u_m = T_m(0,0,h)$. Let E_m^+, (E_m^-) be the fundamental solution of (KG_m) in the forward (respectively backward) light cone. The problem is reduced to show that $(E_m^{\pm} - E_{m_0}^{\pm}) * \to 0$ simply in $\mathcal{L}(L^p(\mathbb{R}^{n+1}), L^q(\mathbb{R}^{n+1}))$ as $m \to m_0$. Following [7] we define an analytic family of operators

$$P^z_{m_\pm} h = \lim_{\sigma\to 0^\pm} F^{-1}_{\tau,\xi}[((-\tau^2 + |\xi|^2 + m^2 + \sigma^2 - 2i\sigma\tau)^z - (-\tau^2 + |\xi|^2 + m_0^2 + \sigma^2 - 2i\sigma\tau)^z)\hat{h}(\tau,\xi)].$$

We have

$$P^{-1}_{m_\pm} h = (E^{\pm}_m - E^{\pm}_{m_0}) * h.$$

On the one hand it is clear that $P^{iy}_{m_\pm} \to 0$ simply in $\mathcal{L}(L^2(\mathbb{R}^{n+1}), L^2(\mathbb{R}^{n+1}))$ as $m \to m_0$, on the other hand, it follows from [7] that $(P^z_{m_\pm})_{\mu\leqslant m\leqslant m_0+1}$ is bounded in $\mathcal{L}(L^1(\mathbb{R}^{n+1}), L^\infty(\mathbb{R}^{n+1}))$ for $-(n+2)/2 \leqslant \mathrm{Re}\, z \leqslant -(n+1)/2$ if $0 < \mu, m_0$ and for $\mathrm{Re}\, z = -(n+1)/2$ if $\mu = 0$. A slight modification of Stein's interpolation theorem proves that $P^{-1}_{m_\pm} \to 0$ simply in $\mathcal{L}(L^p(\mathbb{R}^{n+1}), L^q(\mathbb{R}^{n+1}))$. The proof of Theorem 2 is therefore complete.

4. ANALYTIC INTERACTION

THEOREM 3. Let $(a_{2n+1})_{n\geqslant 1}$ be a bounded sequence in $W^{2,\infty}(\mathbb{R}^3)$. Then this sequence is determined by the scattering operator S for the equation

$$u_{tt} - \Delta_x u + m^2 u = \sum_{n=1}^{\infty} a_{2n+1}(x)\, |u|^{2n} u;\ 0 < m,\ x \in \mathbb{R}^3.$$

Remark: Notice that this result holds whatever the sign of a_{2n+1}; indeed, the existence of S is useful only for small data (in fact only for the small regular wave packet). We sketch the proof: the scattering operator is defined for free solutions u_- so that $\|u_-\|_{scat}$ is small enough where

$$\|u_-\|^2_{scat} = \sup_{t\in\mathbb{R}} \|u(t)\|^2_e + \sup_{t\in\mathbb{R}} (1 + |t|)^3 \|u(t)\|^2_{L^\infty(\mathbb{R}^3_x)}$$

(see for instance [5]).

Notice that

$$\sup_{q \geq 4} \| \cdot \|_{L^q(\mathbb{R}^4)} \leq c \| \cdot \|_{scat}.$$

Now with the notation of Section 2 we have

$$u(t) = u_-(t) + \int_{-\infty}^{t} R(t-s) \underset{x}{*} f(x,u(s))ds.$$

This equation is solved by the Picard method for the $\| \; \|_{scat}$ norm, hence

$$u(t) = \sum_{k=0}^{\infty} \mathcal{R}^k(u_-) \qquad (*)$$

convergence taking place in the $\| \; \|_{scat}$ norm and where

$$\mathcal{R}(f) = \int_{-\infty}^{t} R(t-s) \underset{x}{*} f(x,s) \, ds.$$

Following [4] we choose $u_- = \varepsilon \, \phi_\lambda$, $v_- = 2\varepsilon\phi_-$, and we have

$$W(S(2\varepsilon \, \phi_\lambda),(\varepsilon \, \phi_\lambda)) - 2\varepsilon^2 W(\phi_\lambda,\phi_\lambda) = \iint (u\overline{f(x,v)} - f(x,u)\bar{v}) \, dt \, dx.$$

Applying (*) to u and v we see that

$$W(S(2\varepsilon \, \phi_\lambda), S(\varepsilon \, \phi_\lambda)) - 2 \, \varepsilon^2 W(\phi_\lambda,\phi_\lambda) = \sum_{n \geq 4} \varepsilon^h \, Q_n(\lambda)$$

where

$$Q_n(\lambda) = (2^{n-1}-2) \iint a_{n-1} \, |\phi_\lambda|^n \, dx \, dt + Q_n'(\lambda),$$

where Q_n' involves only $a_{n-2}, a_{n-3}, \ldots, a_3$ ($a_{2k} = 0$ by definition). We assume by induction that $Q_n'(\lambda)$ is already determined so that S determines

$$\iint a_{n-1}(x_0 + \frac{x'}{\lambda}) \, |u_\lambda(t',x')|^n \, dt' \, dx', \; n \text{ even},$$

u_λ being solution of (E). Theorem 2 ensures that u_λ converges in $L^4(\mathbb{R}^4)$ as $\lambda \to +\infty$, hence the problem is reduced showing that

$$\sup_\lambda \|u_\lambda\|_{L^\infty(\mathbb{R}^4)} < \infty \, .$$

By the Sobolev inequality

$$|u_\lambda(t)|_{L^\infty(\mathbb{R}^3)} \leq c\ |\nabla u_\lambda(t)|^{3/4}_{L^4(\mathbb{R}^3)}\ |u_\lambda(t)|^{1/4}_{L^4(\mathbb{R}^3)}.$$

We have

$$|u_\lambda(t)|_{L^4(\mathbb{R}^3)} \leq |u_\lambda(t)|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)}\ |u_\lambda(t)|^{\frac{1}{2}}_{L^6(\mathbb{R}^3)}$$

$$|u_\lambda(t)|_{L^6(\mathbb{R}^3)} \leq c\ |\nabla u_\lambda(t)|_{L^2(\mathbb{R}^3)}$$

and by the conservation of energy

$$|u_\lambda(t)|_{L^2(\mathbb{R}^3)} \leq |\ |\xi|^{-1}\hat{g}|_{L^2(\mathbb{R}^3_\xi)}\ ;\ |\nabla u_\lambda(t)|_{L^2(\mathbb{R}^3)} \leq |g|_{L^2(\mathbb{R}^3)}.$$

Hence

$$|u_\lambda(t)|_{L^4(\mathbb{R}^3)} \leq c\ |g|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)}\ |\ |\xi|^{-1}\hat{g}|^{\frac{1}{2}}_{L^2(\mathbb{R}^3_\xi)}.$$

Even

$$|\nabla u_\lambda(t)|_{L^4(\mathbb{R}^3)} \leq c\ |\ |\xi|\ \hat{g}|^{\frac{1}{2}}_{L^2(\mathbb{R}^3_\xi)}\ |g|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)}.$$

Finally

$$\sup_\lambda\ |u_\lambda|_{L^\infty(\mathbb{R}^4)} \leq c\ |g|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)}\ |\ |\xi|\ \hat{g}|^{3/8}_{L^2(\mathbb{R}^3_\xi)}\ |\ |\xi|^{-1}\ \hat{g}|^{1/8}_{L^2(\mathbb{R}^3_\xi)}.$$

We choose g so that $\hat{g} \in \mathcal{D}(\mathbb{R}^3)$ $0 \notin \operatorname{supp} \hat{g}$.

This completes the proof of Theorem 3.

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