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Scattering of electromagnetic field by De Sitter–Schwarzschild black hole

I - Maxwell Equations in the De Sitter - Schwarzschild Universe

We investigate the electromagnetic field outside a spherical Black-Hole with mass m in an asymptotic De Sitter spacetime with cosmological constant $\Lambda > 0$, described by the De Sitter - Schwarzschild metric

(1) $$ds^2 = \alpha^2 dt^2 - \alpha^{-2} dr^2 - r^2(d\theta^2 + \sin^2\theta \, d\varphi^2) \ ,$$

where lapse function α is given by

(2) $$\alpha = (1 - 2 m r^{-1} - \Lambda r^2 / 3)^{1/2} .$$

We assume m and Λ satisfy :

(3) $$9 \Lambda m^2 < 1$$

Then the equation " $\alpha = 0$ " admits two positive real roots r_+, r_{++}, $0 < r_+ < r_{++}$. The static De Sitter - Schwarzschild Universe is the four dimensional pseudoriemannian globally hyperbolic manifold

(4) $$\mathcal{S} = \mathbb{R}_t \times \{ r_+ < r < r_{++} \} \times S^2$$

with metric (1).

This metric has a fictitious singularity on the "Black Hole Horizon" $\Gamma_+ = \mathbb{R}_t \times \{ r = r_+ \} \times S^2$ and on the "Cosmological Horizon" $\Gamma_{++} = \mathbb{R}_t \times \{ r = r_{++} \} \times S^2$. No radial null geodesic reaches the horizons at finite time t and it is convenient to introduce the tortoise coordinate r_* defined by

(5) $$\frac{dr_*}{dr} = \alpha^{-2} , \ r_+ < r < r_{++} .$$

Then the equation of such geodesics is

(6) $$t = \pm r_* + C \quad ,$$

and

$$\Gamma_+ = \mathbb{R}_t \times \{ r_* = -\infty \} \times S^2 \ , \ \Gamma_{++} = \mathbb{R}_t \times \{ r_* = +\infty \} \times S^2$$

In the De Sitter - Schwarzschild vacuum Maxwell's tensor F satisfies equations :

(7) $$dF = 0 \quad , \quad d * F = 0 \ ,$$

where $*$ is the Hodge operator related to metric (1). We split F into electric and magnetic fields measured by an observer with four velocity u :

(8) $$E_\mu = F_{\mu,\nu} \ u^\nu \ , \quad B_\mu = -(* F)_{\mu,\nu} \ u^\nu \ .$$

Since we are concerned by the scattering theory, we consider the Black-Hole as a perturbation and we choose an observer at rest by respect to the Black-Hole (Fiducial observer of [9]), and then

(9) $$u = \alpha^{-1} \partial_t \ .$$

By putting

(10) $${}^tU = (E^{\hat{r}}, E^{\hat{\theta}}, E^{\hat{\varphi}}, B^{\hat{r}}, B^{\hat{\theta}}, B^{\hat{\varphi}}) = (E, B) \ ,$$

where

(11) $$X = X^{\hat{r}} \alpha \, \partial_r + X^{\hat{\theta}} r^{-1} \partial_\theta + X^{\hat{\varphi}} (r \sin \theta)^{-1} \partial_\varphi \quad , \quad X = E, B \ ,$$

Maxwell's equations (7) take a familiar form

(12) $$\partial_t U = -i H U \quad , \quad \nabla_{\mathcal{G}} . E = \nabla_{\mathcal{G}} . B = 0 \ ,$$

where

(13) $$H = i\begin{pmatrix} 0 & \nabla_{\mathcal{S}}\times\alpha \\ -\nabla_{\mathcal{S}}\times\alpha & 0 \end{pmatrix}\ ,\quad \nabla_{\mathcal{S}}\times\alpha = \begin{pmatrix} 0 & -\frac{\alpha}{r\sin\theta}\,\partial_\varphi & \frac{\alpha}{r\sin\theta}\,\partial_\theta \sin\theta \\ \frac{\alpha}{r\sin\theta}\,\partial_\varphi & 0 & -\frac{\alpha}{r}\,\partial_r r\alpha \\ -\frac{\alpha}{r}\,\partial_\theta & \frac{\alpha}{r}\,\partial_r r\alpha & 0 \end{pmatrix}$$

(14) $$\nabla_{\mathcal{S}}\,.\,X = \alpha\, r^{-2}\,\partial_r\,(r^2 X^{\hat{r}}) + (r\sin\theta)^{-1}\,[\partial_\theta\,(\sin\theta\, X^{\hat{\theta}}) + \partial_\varphi X^{\hat{\varphi}}\,]$$

If there is no Black-Hole, $\alpha = (1-\Lambda r^2/3)^{1/2}$ and we find the free dynamic in the static De Sitter space-time $\mathbb{R}_t \times \{\,0 \le r < r_{++} = (3/\Lambda)^{1/2}\} \times S^2$ with spherical coordinates.

We introduce the Hilbert space of finite redshifted energy :

(15) $$\tilde{\mathcal{H}} = [L^2(\,]r_+\,,\,r_{++}[_r \times S^2_\omega\,,\,r^2\,dr\,d\omega)]^6\ ,$$

the subspace of free divergence :

(16) $$\tilde{\mathcal{H}}^{(o)} = \{U \in \tilde{\mathcal{H}}\ \ ;\ \nabla_{\mathcal{S}}\,.\,E = \nabla_{\mathcal{S}}\,.\,B = 0\}\ ,$$

and the subspace $\mathcal{H}$ of fields without stationary part, i.e. orthogonal with the second space of cohomology :

(17) $$\mathcal{H} = \{U \in \tilde{\mathcal{H}}^{(o)}\ ;\ \int B^{\hat{r}}\,dr\,d\omega = \int E^{\hat{r}}\,dr\,d\omega = 0\}\ .$$

THEOREM I.1 - H *is a selfadjoint operator with dense domain on* $\tilde{\mathcal{H}}$, $\tilde{\mathcal{H}}^{(o)}$, *and on* $\mathcal{H}$.

Then we solve the Cauchy problem for (12) by Stone's theorem.

REMARK : We are not concerned by a mixed problem : we do not need any boundary condition on horizons Γ which are not time like.

We have a result of finite velocity dependence :

THEOREM I.2 - *Let's be* U *in* $\tilde{\mathcal{H}}$ *such that*

$$supp\ U \subset \{r_*^1 \le r_* \le r_*^2\} \times S^2\ ;$$

then we have

$$supp\ e^{-itH}\,U \subset \{r_*^1 - |t| \le r_* \le r_*^2 + |t|\} \times S^2\ .$$

An important result is that there exists no time periodic non null field:

THEOREM I.3 - *The ponctual spectrum of H on $\mathscr{H}$ is empty.*

We can deduct from this result, the decay of local energy ; but we developp here a complete scattering theory for the electromagnetic field and in particular, we find the result of Damour [4], [5], [6] on the behaviour of fields near the Black Hole Horizon. The study of case without cosmological term, i.e. $\Lambda = 0$, was treated in [1] , [2] .

II - Wave Operators at the Black-Hole Horizon

Hamiltonian H degenerates as $r \to r_+$, but $r\alpha H(r\alpha)^{-1}$ admits a formal limit H_1

(18) $$H_1 = i \begin{pmatrix} 0 & h_1 \\ -h_1 & 0 \end{pmatrix} \quad , \quad h_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\partial_{r_*} \\ 0 & \partial_{r_*} & 0 \end{pmatrix} .$$

H_1 is essentially the dynamic in Rindler metric that approximates De Sitter - Schwarzschild metric near the horizon. We introduce Hilbert spaces :

(19) $$\tilde{\mathscr{H}}_1 = \{U_1 = {}^t(E_1^{\hat{r}}, E_1^{\hat{\theta}}, E_1^{\hat{\varphi}}, B_1^{\hat{r}}, B_1^{\hat{\theta}}, B_1^{\hat{\varphi}}) \in [L^2(\mathbb{R}_{r_*} \times S^2_\omega , dr_* \, d\omega)]^6\} ,$$

(20) $$\mathscr{H}_1^{\pm} = \{U_1 \in \tilde{\mathscr{H}}_1 \, ; E_1^{\hat{r}} = B_1^{\hat{r}} = \pm E_1^{\hat{\theta}} + B_1^{\hat{\varphi}} = \pm E_1^{\hat{\varphi}} - B_1^{\hat{\theta}} = 0\} .$$

The fields in $\mathscr{H}_1^{+(-)}$ have an left (right) polarization and behave like a plane wave, coming out of the past cosmological horizon and falling into the future black-hole horizon (coming out of the past black-hole horizon and falling into the future cosmological horizon):

(21) $$U_1 \in \mathscr{H}_1^{\pm} \Rightarrow [e^{-itH_1} U_1](r_* , \omega) = U_1(\pm t + r_* , \omega) .$$

Given a cut-off function $\chi_1 \in C^\infty(\mathbb{R}_r)$ satisfying $\chi_1(r_*) = 1$ for $r_* < a$, $\chi_1(r_*) = 0$ for $r_* > b$, for some $a < b$, we construct an identification operator

$$\mathscr{J}_1 : \tilde{\mathscr{H}}_1 \to \tilde{\mathscr{H}}$$

by putting

(22) $$\mathscr{J}_1 U_1 = (r\alpha)^{-1} \chi_1 U_1 .$$

We define classical wave operators

(23) $$W_1^{\pm} U_1 = s - \lim_{t \to \pm\infty} e^{itH} \mathscr{J}_1 e^{-itH_1} U_1 \quad in \quad \tilde{\mathscr{H}} .$$

Because the gravitational potential is exponentially decreasing as $r_* \to -\infty$, we prove easily by Cook's method the :

THEOREM II.1- $W_1^{\pm} : \mathscr{H}_1^{\pm} \to \mathscr{H}$ *exist, are independent of* χ_1 *and* $\|W_1^{\pm}\| \leq 1$.

We deduct from this result, the existence of infalling fields, similar to the disappearing solutions in dissipative scattering :

THEOREM II.2 - *If* $U_1 \in \mathscr{H}_1^{\pm}$ *satisfies*

$$U_1(r_*, \omega) = 0 \quad \text{for} \quad r_* \geq c$$

then we have

$$e^{-itH} W_1^{\pm} U_1 = 0 \quad \text{for} \quad r_* \pm t \geq c .$$

To describe the field near the black-hole horizon as $t \to +\infty$ we define

(24) $$W_1 U = s - \lim_{t \to +\infty} e^{itH_1} \mathscr{I}_1^* e^{-itH} U \quad \text{in} \quad \tilde{\mathscr{H}}_1 .$$

THEOREM II.3 - $W_1 : \mathscr{H} \to \mathscr{H}_1^+$ *exists, is independent of* χ_1 *and* $\|W_1\| \leq 1$.

The physical meaning of this result of completeness is the famous "impedence condition" of Damour [4] , [5] , [6] and Znajeck [12]. More precisely the asymptotic profile of regular fields satisfies a dissipative condition or infalling left-polarization :

THEOREM II.4 - *Let's be* U *in* $\mathscr{H}$ *such that*

(25) $$U = Hf \ , \quad f \in [C_o^{\infty}(]r_+, r_{++}[_r \times S^2)]^6 .$$

We note $e^{-itH} U = {}^t(E^{\hat{r}}, ..., B^{\hat{\varphi}})$. *Then, for any* $s \in \mathbb{R}$, *there exist* $e^{\hat{r}}, ..., b^{\hat{\varphi}}$ *in* $L^2(S^2)$ *such that, as*

(26) $$r \to r_+ \quad , \quad t + r_* = s ,$$

the following limits exist in $L^2(S^2)$:

(27) $$r^2 E^{\hat{r}} \to e^{\hat{r}} , \ r^2 B^{\hat{r}} \to b^{\hat{r}} , \ r\alpha E^{\hat{\theta}} \to e^{\hat{\theta}} , \ r\alpha E^{\hat{\varphi}} \to e^{\hat{\varphi}} , \ r\alpha B^{\hat{\theta}} \to b^{\hat{\theta}} , \ r\alpha B^{\hat{\varphi}} \to b^{\hat{\varphi}} .$$

Moreover, we have

(28) $$e^{\hat{\theta}} = -b^{\hat{\varphi}} \quad , \quad e^{\hat{\varphi}} = b^{\hat{\theta}} ,$$

(29) $$\partial_s e^{\hat{r}} + (\sin\theta)^{-1} [\partial_\theta(\sin\theta e^{\hat{\theta}}) + \partial_\varphi e^{\hat{\varphi}}] = 0 .$$

Remark by Theorem I.3, the set of data satisfying (25) is dense in $\mathscr{H}$.

So, the Black Hole Horizon is rather similar to a dissipative membrane in euclidian space with surface resistivity 377 ohms (impedence of vacuum): (28) is formally the impedence condition and (29) the charge conservation law ; but we emphasize that, unlike the euclidian case for which the dissipative condition is posed at each time and is necessary to solve the mixed problem, impedence property (28) is a consequence of Maxwell equations satisfied at infinity of infalling null geodesics.

III - Wave Operators at the Cosmological Horizon

Hamiltonian H degenerates again as $r \rightarrow r_{++}$, but $r\alpha H(r\alpha)^{-1}$ admits the formal limit H_1 defined by (18) .

Given a cut-off function $\chi_0 \in C^{\infty}(\mathbb{R}_{r*})$ satisfying $\chi_0(r_*) = 1$ for $r_* > d$, $\chi_0(r_*) = 0$ for $r_* < c$, for some $c < d$, we construct an identification operator

$$\mathscr{I}_0 : \tilde{\mathscr{H}}_1 \rightarrow \tilde{\mathscr{H}}$$

by putting

(30) $$\mathscr{I}_0 U_1 = (r\alpha)^{-1} \chi_0 U_1 .$$

We define classical wave operators

(31) $$W_0^{\pm} U_1 = s - \lim_{t \rightarrow \pm\infty} e^{itH} \mathscr{I}_0 e^{-itH_1} U_1 \quad in \quad \tilde{\mathscr{H}} .$$

Because the gravitational potential is exponentially decreasing as $r_* \rightarrow \infty$, we prove easily by Cook's method the :

THEOREM III.1- *$W_0^{\pm} : \mathscr{H}_1^{\mp} \rightarrow \mathscr{H}$ exist, are independent of χ_0 and $\|W_0^{\pm}\| \leq 1$.*

We deduct from this result, the existence of outgoing fields:

THEOREM III.2 - *If $U_1 \in \mathscr{H}_1^{\mp}$ satisfies*

$$U_1(r_* , \omega) = 0 \quad for \quad r_* \leq c$$

then we have

$$e^{-itH} \, W_0^{\pm} U_1 = 0 \quad for \quad r_* \pm t \leq c .$$

To describe the field near the cosmological horizon as $t \to +\infty$ we define

(32) $$W_0\, U = s - \lim_{t\to+\infty} e^{itH_1} \mathcal{J}_0^* e^{-itH} U \quad in \quad \tilde{\mathcal{H}}_1 .$$

THEOREM III.3 - *$W_0 : \mathcal{H} \to \tilde{\mathcal{H}}_1$ exists, is independent of χ_0 and $\|W_0\| \leq 1$.*

Therefore the field satisfies at infinity a "Sommerfeld condition" like in the euclidian case. More precisely the asymptotic profile of regular fields satisfies a dissipative condition or outgoing right-polarization :

THEOREM III.4 - *Let's be U in $\mathcal{H}$ such that*

(33) $$U = Hf \quad , \quad f \in [C_0^\infty(]r_+ , r_{++}[_r \times S^2)]^6 .$$

We note $e^{-itH} U = {}^t(E^{\hat{r}} ,..., B^{\hat{\varphi}})$. Then, for any $s \in \mathbb{R}$, there exist $e^{\hat{r}} ,..., b^{\hat{\varphi}}$ in $L^2(S^2)$ such that, as

(34) $$r \to r_{++} , \quad r < r_{++} , \quad t - r_* = s ,$$

the following limits exist in $L^2(S^2)$:

(35) $$r^2 E^{\hat{r}} \to e^{\hat{r}} , r^2 B^{\hat{r}} \to b^{\hat{r}} , r\alpha E^{\hat{\theta}} \to e^{\hat{\theta}} , r\alpha E^{\hat{\varphi}} \to e^{\hat{\varphi}} , r\alpha B^{\hat{\theta}} \to b^{\hat{\theta}} , r\alpha B^{\hat{\varphi}} \to b^{\hat{\varphi}} .$$

Moreover, we have

(36) $$e^{\hat{\theta}} = b^{\hat{\varphi}} \quad , \quad e^{\hat{\varphi}} = - b^{\hat{\theta}} ,$$

(37) $$-\partial_s e^{\hat{r}} + (sin\theta)^{-1} [\partial_\theta (sin\theta\, e^{\hat{\theta}}) + \partial_\varphi e^{\hat{\varphi}}] = 0 .$$

Remark by Theorem I.3, the set of data satisfying (33) is dense in $\mathcal{H}$.

So, the cosmological horizon is again rather similar to a dissipative membrane with the impedence of vacuum. In fact the previous theorems are so true if there is no black hole, i.e. $m = 0$. Hence, far from the Black-Hole we can compare the electromagnetic fields with the solutions of Maxwell's equations in the static De Sitter space time $\mathcal{S}_0$ with some cosmological constant $\Lambda_0 > 0$, discribed by :

(38) $$\mathcal{S}_0 = \mathbb{R}_t \times \{0 \leq \rho < \rho_{++} = (3/\Lambda_0)^{1/2}\} \times S^2 = \mathbb{R}_t \times \{0 \leq \rho_* < \infty\} \times S^2 ,$$

(39) $$ds^2 = A^2 dt^2 - A^{-2} d\rho^2 - \rho^2 (d\theta^2 + sin^2\theta\, d\varphi^2) ,$$

where the lapse function A is given by

(40) $$A(\rho) = (1 - \Lambda_0 \rho^2 / 3)^{1/2},$$

and the tortoise coordinate ρ_* is defined by

(41) $$\rho_* = (3 / 4 \Lambda_0)^{1/2} \{ \ln [(3/\Lambda_0)^{1/2} + \rho] - \ln [(3/\Lambda_0)^{1/2} - \rho] \}$$

Maxwell's equations in the De Sitter space time are given by :

(42) $$\partial_t U_0 = - i H_0 U_0 \ , \quad \nabla_{\mathscr{S}_0} . E_0 = \nabla_{\mathscr{S}_0} . B_0 = 0 ,$$

where

(43) $$H_0 = i \begin{pmatrix} 0 & \nabla_{\mathscr{S}_0} \times A \\ - \nabla_{\mathscr{S}_0} \times A & 0 \end{pmatrix} , \quad \nabla_{\mathscr{S}_0} \times A = \begin{pmatrix} 0 & - \frac{A}{\rho sin\theta} \partial_\varphi & \frac{A}{\rho sin\theta} \partial_\theta \sin\theta \\ \frac{A}{\rho sin\theta} \partial_\varphi & 0 & - \frac{A}{\rho} \partial_\rho \rho A \\ - \frac{A}{\rho} \partial_\theta & \frac{A}{\rho} \partial_\rho \rho A & 0 \end{pmatrix}$$

(44) $$\nabla_{\mathscr{S}_0} . X = A \rho^{-2} \partial_\rho (\rho^2 X^{\hat\rho}) + (\rho \sin\theta)^{-1} [\partial_\theta (\sin\theta X^{\hat\theta}) + \partial_\varphi X^{\hat\varphi}] .$$

We introduce the Hilbert space of finite energy fields :

(45) $$\tilde{\mathscr{H}}_0 = [L^2([0 , \rho_{++} [_\rho \times S^2_\omega , \rho^2 d\rho \, d\omega)]^6 ,$$

and the subspace of free divergence :

(46) $$\mathscr{H}_0 = \{ U \in \tilde{\mathscr{H}}_0 \ ; \ \nabla_{\mathscr{S}_0} . E = \nabla_{\mathscr{S}_0} . B = 0 \} .$$

To avoid long range interaction between gravitational and electromagnetic fields , we identify the tortoise coordinates of De Sitter and De Sitter - Schwarzschild Universes :

(47) $$r_* = \rho_* .$$

Given a cut-off function $\chi_{00} \in C^\infty(\mathbb{R}^+_{\rho_*})$ satisfying $\chi_{00}(\rho_*) = 1$ for $\rho_* > d_0$, $\chi_{00}(\rho_*) = 0$ for $\rho_* < c_0$, for some $0 < c_0 < d_0$, we construct an identification operator

$$\mathscr{I}_{00} : \mathscr{H}_0 \rightarrow \mathscr{H}$$

by putting

(48) $$(\mathscr{I}_{00} U_0)(r, \theta, \varphi) = (r\alpha)^{-1} \rho A(\rho) \chi_{00}(\rho_*) U_0(\rho_*, \theta, \varphi) \quad for \quad r_* \geq 0 ,$$

(49) $$(\mathscr{I}_{00} U_0)(r, \theta, \varphi) = 0 \quad for \quad r_* < 0 ,$$

where r, ρ, r_*, ρ_* are related by relations (5) , (41) , (47) .

We define classical wave operators

(50) $$W_{00}^{\pm} U_0 = s - \lim_{t \to \pm\infty} e^{itH} \mathscr{I}_{00} e^{-itH_0} U_0 \quad in \quad \tilde{\mathscr{H}} .$$

(51) $$W_{00} U = s - \lim_{t \to +\infty} e^{itH_0} \mathscr{I}_{00}^{*} e^{-itH} U \quad in \quad \tilde{\mathscr{H}}_0 .$$

Given an electromagnetic field in the De Sitter space time , there exists a unique asymptotic field in the De Sitter - Schwarzschild spacetime :

THEOREM III.5- $W_{00}^{\pm} : \mathscr{H}_0 \to \mathscr{H}$ *exist, are independent of* χ_{00} *and* $\| W_{00}^{\pm} \| \leq 1$.

Moreover , operators $W_{00}^{\pm}$ are complete, i.e. the fields in the De Sitter - Schwarzschild space time are asymtotic to a free field in the De Sitter space time :

THEOREM III.6 - $W_{00} : \mathscr{H} \to \mathscr{H}_0$ *exists, is independent of* χ_{00} *and* $\| W_{00} \| \leq 1$.

Now, we can introduce scattering operator S by putting

(52) $$W^{-} : \mathscr{H}_1^{-} \times \mathscr{H}_0 \to \mathscr{H} \; , \; W^{-}(U_1, U_0) = W_1^{-} U_1 + W_0^{-} U_0 ,$$

(53) $$W : \mathscr{H} \to \mathscr{H}_1^{+} \times \mathscr{H}_0 \; , \; W U = (W_1 U , W_0 U) ,$$

(54) $$S = W W^{-} : \mathscr{H}_1^{-} \times \mathscr{H}_0 \to \mathscr{H}_1^{+} \times \mathscr{H}_0 ,$$

and we resume the whole Scattering Theory in the following

THEOREM III.7 - W^{-} *is isometric from* $\mathscr{H}_1^{-} \times \mathscr{H}_0$ *onto* $\mathscr{H}$; W *is isometric from* $\mathscr{H}$ *onto* $\mathscr{H}_1^{+} \times \mathscr{H}_0$, S *is isometric from* $\mathscr{H}_1^{-} \times \mathscr{H}_0$ *onto* $\mathscr{H}_1^{+} \times \mathscr{H}_0$.

IV - Membrane Paradigm

The Membrane Paradigm [9] states that if we are concerned only by the behaviour, far from the Black-Hole, of an initially incoming field, we may approximate the Black-Hole by a dissipative spherical membrane of radius $r_+ + \varepsilon$, $0 < \varepsilon$, called "stretched horizon". We consider the mixed problem for Maxwell equations (7) in $\mathbb{R}_t \times]r_+ + \varepsilon,\; r_{++}[_r \times S^2$ and we impose impedence condition:

(55) $$E^{\hat{\theta}} = - B^{\hat{\varphi}} \quad , \quad E^{\hat{\varphi}} = B^{\hat{\theta}} \; on \; \Gamma_{\varepsilon} = \mathbb{R}_t \times \{r = r_+ + \varepsilon\} \times S^2 ,$$

It is a classical dissipative hyperbolic problem of which the solution is given by a semigroup $V_\varepsilon(t)$ on Hilbert space $\mathscr{H}_\varepsilon = [L^2(]r_+ + \varepsilon, r_{++}[_r \times S^2_\omega, r^2\, dr\, d\omega)]^6$. For $0 < \varepsilon$ small enough we define scattering operator

(56)
$$S_\varepsilon U_0 = s - \lim_{t\to+\infty} e^{itH_0} \mathscr{I}^*_{00} V_\varepsilon(2t) \mathscr{I}_{00} e^{itH_0} U_0 \quad in \quad \tilde{\mathscr{H}}_0 .$$

THEOREM IV.1 - *$S_\varepsilon : \mathscr{H}_0 \to \mathscr{H}_0$ exists, is independent of χ_{00} and $\| S_\varepsilon \| \leq 1$.*

Now, in the De Sitter - Schwarzschild universe, the asymptotic behaviour at infinity of an initially incoming field is described by operator S_{00} defined by

(57)
$$\forall U_0 \in \mathscr{H}_0 \, , \quad S_{00}\, U_0 = \Pi_0\, S(0, U_0)$$

where Π_0 is the projector from $\mathscr{H}^+_1 \times \mathscr{H}_0$ onto $\mathscr{H}_0$. The following result is the mathematical foundation of Membrane Paradigm :

THEOREM IV.2 - *For any $U_0 \in \mathscr{H}_0$, $S_\varepsilon U_0$ tends to $S_{00} U_0$ in $\mathscr{H}_0$ as $\varepsilon \to 0$.*

Of numerical analysis view point, impedence condition (55) is an absorbing boundary condition on artificial boundary Γ_ε , so called Silver-Müller radiation condition in euclidian case [8]. So, Theorem IV.2 gives a method of numerical approximation, already used in [10] .

V Idea of Proofs and Concluding Remarks

We start by investigating vector wave equation in $\mathscr{S}$:

(58)
$$\partial_t^2 X - (\nabla_{\mathscr{S}} \times \alpha)\, (\nabla_{\mathscr{S}} \times \alpha)\, X = 0$$

with the constraint of free divergence

(59)
$$\nabla_{\mathscr{S}} \, . \, X = 0 \, .$$

We split X into radial and transverse components A^o, A^+, A^- and we expand A^o, A^+, A^- in series of generalised vector spherical functions $T^\ell_{m,n}$:

(60)
$$A^o(t, r_*, \theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{n=-\ell}^{\ell} a^o_{l,n}(t, r_*)\, T^\ell_{0,n}\left(\frac{\pi}{2} - \varphi, \theta, 0\right)$$

(61)
$$A^\pm(t, r_*, \theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{n=-\ell}^{\ell} a^\pm_{l,n}(t, r_*)\, T^\ell_{\pm1,n}\left(\frac{\pi}{2} - \varphi, \theta, 0\right) .$$

For simplicity we omit subscript ℓ, n and now a^ν are solutions of scalar one dimensional wave equation

(62)
$$\partial_t^2 a^0 - \partial_{r_*}^2 a^0 = -\ell(\ell+1)\frac{\alpha^2}{r^2} a^0,$$

(63)
$$\partial_t^2 a^+ - \partial_{r_*}^2 a^+ = -\ell(\ell+1)\frac{\alpha^2}{r^2} a^+ + \left(\frac{\ell(\ell+1)}{2}\right)^{1/2} \partial_{r_*}\left(\frac{\alpha^2}{r^2}\right) a^0,$$

(64)
$$\partial_t^2 a^- - \partial_{r_*}^2 a^- = -\ell(\ell+1)\frac{\alpha^2}{r^2} a^- + \left(\frac{\ell(\ell+1)}{2}\right)^{1/2} \partial_{r_*}\left(\frac{\alpha^2}{r^2}\right) a^0,$$

(65)
$$\partial_{r_*} a^0 + \left(\frac{\ell(\ell+1)}{2}\right)^{1/2} (a^+ + a^-) = 0.$$

Therefore a^0, $a^+ - a^-$, are solutions of scalar wave equation :

(66)
$$\partial_t^2 u - \partial_{r_*}^2 u = -\ell(\ell+1)\frac{\alpha^2}{r^2} u.$$

By noting that the potential α^2/r^2 is short range type as $r \to r_+$ and $r \to r_{++}$ we apply a Birman - Kato method to prove there exist u_0, u_1 such that

$$u(t, r_*) \cong u_0(t - r_*) + u_1(t + r_*), \quad t \to \infty,$$

where u_0, u_1 are respectively the asymptotic profiles at the cosmological horizon and at the black - hole horizon. Finaly, by using (65) we obtain the asymptotic behaviour of transverse components.

To justify the Membrane Paradigm we note $a_X^\nu = a_X^\nu(t,r_*)$ the coefficients of $T^\ell_{\nu,n}$ in (60) (61), associated to $X = E, B$, $\nu = 0,+,-$. By using Maxwell's equations and impedence condition (55), we obtain boundary conditions at $r_*^\varepsilon = r_*$ $(r = r_+ + \varepsilon)$:

(67)
$$(\partial_t - \partial_{r_*})\, a_X^o = 0, \quad t > 0, \quad r_* = r_*^\varepsilon, \quad X = E, B,$$

(68)
$$(\partial_t - \partial_{r_*})\, a_E^\pm = \mp \frac{i}{\sqrt{2}} \sqrt{\ell(\ell+1)}\frac{\alpha^2}{r^2} a_B^o, \quad t > 0, \quad r_* = r_*^\varepsilon,$$

(69)
$$(\partial_t - \partial_{r_*})\, a_B^\pm = \pm \frac{i}{\sqrt{2}} \sqrt{\ell(\ell+1)}\frac{\alpha^2}{r^2} a_E^o, \quad t > 0, \quad r_* = r_*^\varepsilon,$$

(70)
$$(\partial_t - \partial_{r_*})(a_E^+ + a_E^-) = (\partial_t - \partial_{r_*})(a_B^+ + a_B^-) = 0, \quad t > 0, \quad r_* = r_*^\varepsilon,$$

(71)
$$(\partial_t - \partial_{r_*})\left(a_E^+ - a_E^- - i\frac{\sqrt{2}}{\sqrt{\ell(\ell+1)}} \partial_t a_B^o\right) = 0, \quad t > 0, \quad r_* = r_*^\varepsilon,$$

(72)
$$(\partial_t - \partial_{r_*})\left(a_B^+ + a_B^- + i\frac{\sqrt{2}}{\sqrt{\ell(\ell+1)}} \partial_t a_E^o\right) = 0, \quad t > 0, \quad r_* = r_*^\varepsilon,$$

We conclude that

(73) $$u \in \{\ a_E^o,\ a_B^o,\ \partial_t a_E^o,\ \partial_t a_B^o,\ a_E^+ + a_E^-,\ a_B^+ + a_B^-,$$

$$a_E^+ - a_E^- + i\frac{\sqrt{2}}{\sqrt{\ell(\ell+1)}}\,\partial_t a_B^o,\ \ a_B^+ - a_B^- - i\frac{\sqrt{2}}{\sqrt{\ell(\ell+1)}}\,\partial_t a_E^o\ \}$$

is solution of

(74) $$\partial_t^2 u - \partial_{r_*}^2 u = -\,\ell(\ell+1)\,)\frac{\alpha^2}{r^2}\,u\ ,\ t > 0\ ,\ \ r_* > r_*^\varepsilon\ ,$$

(75) $$\partial_t u - \partial_{r_*} u = 0\ ,\ t > 0\ ,\ r_* = r_*^\varepsilon\ .$$

But (75) is a perfecty transparent condition, hence

(76) $$u = \tilde{u}\,\Big|_{r_* > r_*^\varepsilon}$$

where $\tilde{u}$ is solution of

(77) $$\partial_t^2\,\tilde{u} - \partial_{r_*}^2\,\tilde{u} = -\,\ell(\ell+1)\,)\frac{\tilde{\alpha}^2}{r^2}\,\tilde{u}\ ,\ t > 0,\ \ r_*\in\mathbb{R}\ ,$$

(78) $$\tilde{u}(0, r_*) = u(0, r_*)\ ,\ r_* > r_*^\varepsilon\ \ and\ \ \tilde{u}(0, r_*) = 0\ ,\ r_* \le r_*^\varepsilon\ ,$$

(79) $$\partial_t\,\tilde{u}(0, r_*) = \partial_t\,u(0, r_*)\ ,\ r_* > r_*^\varepsilon\ and\ \partial_t\,\tilde{u}(0, r_*) = 0\ ,\ r_* \le r_*^\varepsilon\ ,$$

with

(80) $$\tilde{\alpha}\,\Big|_{r_* \ge r_*^\varepsilon} = \alpha\,\Big|_{r_* \ge r_*^\varepsilon}\ ,\ \ \tilde{\alpha}\,\Big|_{r_* < r_*^\varepsilon} = 0\ .$$

Then we can apply again a Birman - Kato method and prove the existence of S_ε. Finaly , to establish the convergence of S_ε, we note that $\tilde{\alpha}$ tends exponentialy to α as $\varepsilon \to 0$.

To end we make some remarks :

We can interpret the whole Scattering Theory in terms of Characteristic Cauchy Problem thanks to the Penrose Transform (see [1] for $\Lambda = 0$) : the fictitious singularities at the Horizons which become from the choice of coordinates (t, r, θ, φ) , can be avoid by using the Kruskal type coordinates ; then the past and future black - hole and cosmological horizons are simply null submanifolds of globally hyperbolic curved spacetime ; hence the fields are there obviously well defined ; moreover the existence of wave operator W^- assures the characteristic Cauchy problem is well posed with data on the past horizons , and the existence of W means the fields can be extended up to the future horizons .

At last we note that our methods can be used to study the asymptotic behaviours of relativistic massless fields in the case of a general spherical Black - Hole with a mass $m > 0$ and a charge Q in a asymptotically De Sitter ($\Lambda > 0$) or Minkowski ($\Lambda > 0$) space described by the (De Sitter-) Reissner- Nordström metric:

(82) $$ds^2 = \alpha^2\,dt^2 - \alpha^{-2}\,dr^2 - r^2(d\theta^2 + \sin^2\theta\,d\varphi^2)\ ,$$

where lapse function α is given by

(83) $$\alpha = (1 - 2\,m\,r^{-1} + Q^2\,r^{-2} - \Lambda\,r^2/\,3\,)^{1/2}\ .$$

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