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Scattering operator for Maxwell equations outside Schwarzschild black-hole

1. Maxwell Equations in Schwarzschild Universe

We investigate the electromagnetic field outside the spherical Black-Hole of radius $r_0 > 0$, described by Schwarzschild metric

$$ds^2 = \alpha^2\, dt^2 - \alpha^{-2}\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2),\ \ r_0 < r, \tag{1}$$

and the lapse function α is given by

$$\alpha = (1 - r_0 r^{-1})^{1/2}. \tag{2}$$

This metric is singular on the 'Horizon' $\Gamma = \mathbb{R}_t \times \{r = r_0\} \times S^2$ and no radial null geodesic reaches Γ at finite time t. With Wheeler coordinate r_*, the equation of such geodesics is

$$t = \pm r_* + C,\ \ r_* = r + r_0 \ln(r - r_0). \tag{3}$$

In Schwarzschild vacuum, Maxwell's tensor F verifies the equations:

$$dF = 0,\ \ d * F = 0, \tag{4}$$

where $*$ is the Hodge operator related to metric (1). We split F into electric and magnetic fields measured by an observer with four-velocity u:

$$E_\mu = F_{\mu,\nu}\, u^\nu,\ \ B_\mu = -(* F)_{\mu,\nu}\, u^\nu. \tag{5}$$

Since we are concerned with scattering theory, we consider the Black-Hole as a perturbation and we choose an observer at rest by respect to the Black-Hole (Fiducial observer of [6]), and then

$$u = \alpha^{-1}\partial_t. \tag{6}$$

By putting

$$ {}^{t}U = (E^{\hat{r}}, E^{\hat{\theta}}, E^{\hat{\varphi}}, B^{\hat{r}}, B^{\hat{\theta}}, B^{\hat{\varphi}}) = (e, b), \tag{7} $$

where

$$ X = X^{\hat{r}} \alpha \partial_r + X^{\hat{\theta}} r^{-1} \partial_\theta + X^{\hat{\varphi}} (r \sin\theta)^{-1} \partial_\varphi, \; X = E, B\,, $$

Maxwell's equations (4) take a familiar form

$$ \partial_t U = -i H U, \; \nabla_S \cdot E = \nabla_S \cdot B = 0\,, \tag{8} $$

where

$$ H = i \begin{pmatrix} 0 & \nabla_S \times \\ -\nabla_S \times & 0 \end{pmatrix}, \; \nabla_S \times \begin{pmatrix} 0 & -\frac{\alpha}{r\sin\theta}\partial_\varphi & \frac{\alpha}{r\sin\theta}\partial_\theta \sin\theta \\ \frac{\alpha}{r\sin\theta}\partial_\varphi & 0 & -\frac{\alpha}{r}\partial_r r\alpha \\ -\frac{\alpha}{r}\partial_\theta & \frac{\alpha}{r}\partial_r r\alpha & 0 \end{pmatrix}, \tag{9} $$

$$ \nabla_S \cdot X = \alpha r^{-2} \partial_r (r^2 X^{\hat{r}}) + (r \sin\theta)^{-1} [\, \partial_\theta \sin\theta \, X^{\hat{\theta}}) + \partial_\varphi X^{\hat{\varphi}} \,]\,. \tag{10} $$

If there is no Black-Hole, $\alpha = 1$ and we find the free dynamic in Minkowski space-time with spherical coordinates. We introduce the Hilbert space of finite redshifted energy

$$ \tilde{\mathcal{H}} = [\, L^2(\,]\, r_0\,, +\infty\,[_r \times S^2_\omega\,,\; r^2\, dr\, d\omega)\,]^6 $$

and the subspace of free divergence without stationary part:

$$ \mathcal{H} = \{ U \in \tilde{\mathcal{H}},\; \nabla_S . E = \nabla_S . B = 0\,, \int E^{\hat{r}}\, dr\, d\omega = \int B^{\hat{r}}\, dr\, d\omega = 0 \}. $$

Theorem 1.1. *H is a self-adjoint operator with dense domain on $\tilde{\mathcal{H}}$ and on $\mathcal{H}$.*

Then we solve the Cauchy problem for (8) by Stone's theorem.

Remark: We are not concerned with a mixed problem: we do not need any boundary condition on horizon Γ which is not time-like.

We have a result of finite velocity dependence:

Theorem 1.2. *Let U be in $\tilde{\mathcal{H}}$ such that*

$$supp\, U \subset \{ r_*^1 \leq r_* \leq r_*^2 \} \times S^2$$

then we have

$$\text{supp}\, e^{-itH}\, U \subset \{ r_*^1 - |t| \leq r_* \leq r_*^2 + |t| \} \times S^2 .$$

Schwarzschild metric is trapping: all great circles of sphere with radius $3r_0/2$, so-called 'Photons-sphere', are null geodesics; there exist also null geodesics asymptotic to the Photons-sphere. Therefore singularities of field can be trapped and do not escape at infinity. Despite this difficulty, there is no time-periodic solution in Schwarzschild universe, like the Euclidean case with an obstacle, for which, the second space of cohomology yields non-trivial stationary solutions:

Theorem 1.3. *The point spectrum of H on $\mathcal{H}$ is empty.*

We can deduce from this result the decay of local energy; but we develop here a complete scattering theory for the electromagnetic field and in particular, we find the result of Damour [3] on the behaviour of fields near the horizon. The study of the scalar case was treated by Dimock and Kay [4] [5].

2. Wave Operators at Infinity

Schwarzschild universe is asymptotically flat and far from the Black-Hole we compare the Hamiltonian H with the classical electromagnetic Hamiltonian H_0

$$H_0 = i \begin{pmatrix} 0 & \text{curl} \\ -\text{curl} & 0 \end{pmatrix} \tag{9}$$

in Minkowski space-time with metric

$$ds^2 = dt^2 - d\rho^2 - \rho^2 (d\theta^2 + \sin^2\theta\, d\varphi^2), \quad 0 \leq \rho . \tag{10}$$

For any choice of $\rho = \rho(r)$, the difference $H - H_0$ is a long-range type perturbation but because the radial null geodesics (3) are straight like their flat analogs, we avoid long-range interaction between gravitational and electromagnetic fields by choosing

$$\rho = r_* \geq 0 . \tag{11}$$

We introduce the usual finite energy Hilbert spaces

$$\tilde{\mathcal{H}}_0 = \{ U_0 = {}^t(E_0^{\hat r}, E_0^{\hat\theta}, E_0^{\hat\varphi}, B_0^{\hat r}, B_0^{\hat\theta}, B_0^{\hat\varphi}) \in [L^2(\mathbb{R}^+_{r_*} \times S^2_\omega, r_*^2\, dr_*\, d\omega)]^6 \},$$

$$\mathcal{H}_0 = \{ U_0 = {}^t(E_0, B_0) \in \tilde{\mathcal{H}}_0;\ \text{div}\, E_0 = \text{div}\, B_0 = 0 \}.$$

Given a cut-off function $\chi_0 \in C^\infty(\mathbb{R}^+_{r_*})$ satisfying $\chi_0(r_*) = 0$ for $0 \le r_* < a$, and $\chi_0(r_*) = 1$ for $r_* > b$, for some $0 < a < b$, we construct an identification operator $I_0 : \tilde{\mathcal{H}}_0 \to \tilde{\mathcal{H}}$ by putting

$$I_0 U_0 = \chi_0 U_0 \quad for \quad r_* \ge 0, \quad I_0 U_0 = 0 \quad for \quad r_* \le 0.$$

We define classical wave operators without Dollard's modification

$$W_0^{\pm} U_0 = s - \lim_{t \to \pm\infty} e^{itH} I_0 e^{-itH_0} U_0 \text{ in } \tilde{\mathcal{H}}.$$

The spherical invariance of Maxwell equations - that implies a t^{-2} decay of radial components - and our choice (11), cancel the long-range effects and by Cook's method we prove the

Theorem 2.1. *$W_0^{\pm} : \mathcal{H}_0 \to \mathcal{H}$ exist, are independent of χ_0 and $\| W_0^{\pm} \| \le 1$.*

We deduce from this result the existence of outgoing fields:

Theorem 2.2. *If $U_0 \in \mathcal{H}_0$ satisfies*

$$e^{-itH_0} U_0 = 0 \quad for \quad 0 \le r_* \le \pm t + C$$

then we have

$$e^{-itH} W_0^{\pm} U_0 = 0 \quad for \quad r_* \le \pm t + C.$$

3. Wave Operators near the Black-Hole

The Hamiltonian H degenerates as $r \to r_0$, but $r\alpha H (r\alpha)^{-1}$ admits a formal limit H_1:

$$H_1 = i \begin{pmatrix} 0 & h_1 \\ -h_1 & 0 \end{pmatrix}, \quad h_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\partial_{r_*} \\ 0 & \partial_{r_*} & 0 \end{pmatrix}. \tag{13}$$

H_1 is essentially the dynamic in Rindler metric that approximates the Schwarzschild metric near the horizon. We introduce Hilbert spaces :

$$\tilde{\mathcal{H}}_1 = \{ U_1 = {}^t(E_1^{\hat{r}}, E_1^{\hat{\theta}}, E_1^{\hat{\varphi}}, B_1^{\hat{r}}, B_1^{\hat{\theta}}, B_1^{\hat{\varphi}}) \in [L^2(\mathbb{R}_{r_*} \times S^2_\omega, dr_*\, d\omega)]^6 \},$$

$$\mathcal{H}_1^{\pm} = \{ U_1 \in \tilde{\mathcal{H}}_1 ;\ E_1^{\hat{r}} = B_1^{\hat{r}} = \pm {}_1^{\hat{\theta}} + B_1^{\hat{\varphi}} = \pm E_1^{\hat{\varphi}} - B_1^{\hat{\theta}} = 0 \}.$$

The fields in $\mathcal{H}_1^{+(-)}$ have a left (right) polarization and behave like a plane wave, falling into the future (coming out of the past) horizon

$$U_1 \in \mathcal{H}_1^{\pm} \Rightarrow [e^{-itH_1} U_1](r_*, \omega) = U_1(\pm t + r_*, \omega).$$

Given a cut-off function $\chi_1 \in C^\infty(\mathbb{R}_{r_*})$ satisfying $\chi_1(r_*) = 1$ for $r_* < c$, $\chi_1(r_*) = 0$ for $r_* > d$, for some $c < d < 0$, we construct an identification operator $I_1 : \tilde{\mathcal{H}}_1 \to \tilde{\mathcal{H}}$ by putting

$$I_1 U_1 = (r\alpha)^{-1} \chi_1 U_1 .$$

We define classical wave operators

$$W_1^{\pm} U_1 = s - \lim_{t \to \pm\infty} e^{itH} I_1 e^{-itH_1} U_1 \text{ in } \tilde{\mathcal{H}}. \tag{14}$$

Because the Schwarzschild potential is exponentially decreasing as $r_* \to -\infty$, we prove easily by Cook's method the :

Theorem 3.1. *$W_1^{\pm} : \mathcal{H}_1^{\pm} \to \mathcal{H}$ exist, are independent of χ_1 and $\| W_1^{\pm} \| \leq 1$.*

We deduce from this result the existence of infalling fields, similar to the disappearing solutions in dissipative scattering.

Theorem 3.2. *If $U_1 \in \mathcal{H}_1^{\pm}$ satisfies*

$$U_1(r_*, \omega) = 0 \textit{ for } r_* \geq c$$

then we have

$$e^{-itH}\, W_1^{\pm} U_1 = 0 \textit{ for } r_* \geq \pm t + c\,.$$

4. Asymptotic Completeness

To study the asymptotic behaviour far from the Black-Hole we introduce

$$W_0 U = s - \lim_{t \to \pm\infty} \; e^{itH_0}\, I_0^* e^{-itH}\, U \text{ in } \tilde{\mathcal{H}}_0\,. \tag{15}$$

At infinity of Schwarzschild universe, the electromagnetic field is asymptotic to a free field in Minkowski space-time :

Theorem 4.1. $W_0 : \mathcal{H} \to \mathcal{H}_0$ *exists, is independent of* χ_0 *and* $\| W_0 \| \leq 1$.

To describe the field near the horizon as $t \to +\infty$ we define

$$W_1 U = s - \lim_{t \to \pm\infty} \; e^{itH_1}\, I_1^* e^{-itH}\, U \text{ in } \tilde{\mathcal{H}}_1\,. \tag{16}$$

Theorem 4.2. $W_1 : \mathcal{H} \to \mathcal{H}_1^+$ *exists, is independent of* χ_1 *and* $\| W \| \leq 1$.

The physical meaning of this result of completeness is the famous 'impedence condition' of Damour and Znajeck [3]. More precisely the asymptotic profile of regular fields satisfies a dissipative condition or infalling left-polarization :

Theorem 4.3. *Let there exist* U *in* $\mathcal{H}$ *such that*

$$U = Hf,\ f \in [\, C_0^\infty(\,]r_0, +\infty[\times S^2)\,]^6. \tag{17}$$

We note $e^{-itH}\, U = {}^t(E^{\hat{r}}, ..., B^{\hat{\varphi}})$. *Then, for any* $s \in \mathbb{R}$, *there exist* $e^{\hat{r}}, ..., b^{\hat{\varphi}}$ *in* $L^2(S^2)$ *such that, as*

$$r \to r_0,\ t + r_* = s\,, \tag{18}$$

the following limits exist in $L^2(S^2)$:

$$E^{\hat r} \to e^{\hat r}, B^{\hat r},\ \alpha E^{\hat\theta} \to e^{\hat\theta},\ \alpha E^{\hat\varphi} \to e^{\hat\varphi},\ \alpha B^{\hat\theta} \to b^{\hat\theta},\ \alpha B^{\hat\varphi} \to b^{\hat\varphi}. \tag{19}$$

Moreover, we have

$$e^{\hat\theta} = - b^{\hat\varphi},\ e^{\varphi} = b^{\hat\theta}, \tag{20}$$

$$\partial_s e^{\hat r} + (\sin\theta)^{-1}[\partial_\theta \sin\theta\, e^{\hat\theta}) + \partial_\varphi e^{\hat\varphi}] = 0. \tag{21}$$

Remark by Theorem 1.3, the set of data satisfying (17) is dense in $\mathcal{H}$.

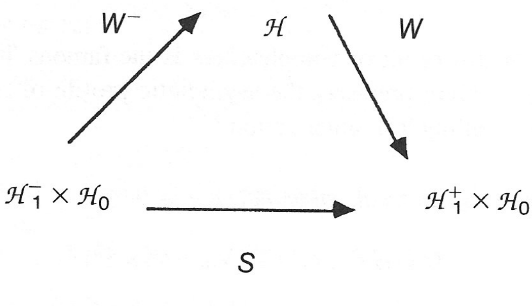
So, the horizon is rather similar to a dissipative membrane in Euclidean space with surface resistance 377 ohms : (20) is formally the impedance condition and (21) the charge conservation law; but we emphasize that, unlike the Euclidean case for which the dissipative condition is posed at each time and is necessary to solve the mixed problem, impedance property (20) is a consequence of Maxwell equations satisfied at infinity of infalling null geodesics.

Now, we can introduce scattering operator S by putting

$$W^- : \mathcal{H}_1^- \times \mathcal{H}_0 \to \mathcal{H},\ W^-(U_1, U_0) = W_1^- U_1 + W_0^- U_0,$$

$$W : \mathcal{H} \to \mathcal{H}_1^+ \times \mathcal{H}_0,\ WU = (W_1 U, W_0 U),$$

$$S = WW^- : \mathcal{H}_1^- \times \mathcal{H}_0 \to \mathcal{H}_1^+ \times \mathcal{H}_0.$$



Theorem 4.4. *W^- is isometric from $\mathcal{H}_1^- \times \mathcal{H}_0$ onto $\mathcal{H}$; W is isometric from $\mathcal{H}$ onto $\mathcal{H}_1^+ \times \mathcal{H}_0$, S is isometric from $\mathcal{H}_1^- \times \mathcal{H}_0$ onto $\mathcal{H}_1^+ \times \mathcal{H}_0$.*

We give only the idea of proof for asymptotic completeness. Since the kernel of H on $\mathcal{H}$ is null, we may use vector potential and so we have to study the vector wave equation

$$\partial_t^2 U - H^2 U = 0. \tag{22}$$

Thanks to the spherical invariance we can obtain a complete variables separation by using the generalized vector spherical harmonics of Gel'Fand and Šapiro. Roughly speaking the problem is now reduced to the study of the scalar one-dimensional wave equation

$$\partial_t^2 v - \partial_{r_*}^2 v = \alpha^2 r^{-2} v . \tag{23}$$

The crucial point is that, thanks to our choice of Wheeler coordinate, the potential $\alpha^2 r^{-2}$ in (23) is a short-range type and we apply classical results of Birman and Kato. In a suitable sense, solution v of (23) satisfies

$$v(t, r_*) \sim v_0(r_* - t) + v_1(r_* + t), \quad t \to +\infty . \tag{24}$$

v_0 and v_1 are respectively the asymptotic profiles of parts of the field, respectively, outgoing at infinity, infalling into the Black-Hole. To get the asymptotic fields and to prove the existence of W_0, W_1, S, we apply the two Hilbert spaces scattering theory.

5. Membrane Paradigm

The Membrane Paradigm [6] states that if we are concerned only with the behaviour, far from the Black-Hole, of an initially incoming field, we may approximate the Black-Hole by a dissipative spherical membrane of radius $r_0 + \varepsilon$, $0 < \varepsilon$, called 'stretched horizon'.

We consider the mixed problem for Maxwell equations (8) in $]\, r_0 + \varepsilon, +\infty\, [\times S^2$ and on stretched horizon $\Gamma_\varepsilon = \mathbb{R}_t \times \{ r = r_0 + \varepsilon \} \times S^2$ which is time-like, we impose impedence condition

$$E^{\hat{\theta}} = - B^{\hat{\varphi}}, \quad E^{\hat{\varphi}} = B^{\hat{\theta}} . \tag{25}$$

It is a classical dissipative hyperbolic problem of which the solution is given by a semigroup $V_\varepsilon(t)$ on Hilbert space $\tilde{\mathcal{H}}_\varepsilon = \{ L^2(]\, r_0 + \varepsilon, +\infty\, [_r \times S^2_\omega, r^2\, dr\, d\omega) \}^6$. For $0 < \varepsilon < a$ we define the scattering operator

$$S_\varepsilon U_0 = s - \lim_{t \to +\infty} e^{itH_0} I_0^* V_\varepsilon(2t) I_0 e^{itH_0} U_0 \text{ in } \tilde{\mathcal{H}}_0 .$$

Theorem 5.1. $S_\varepsilon : \mathcal{H}_0 \to \mathcal{H}_0$ *exists, is independent of* χ_0 *and* $\| S \| \leq 1$.

Now, in Schwarzschild universe, the asymptotic behaviour at infinity of an initially incoming field is described by operator S_{00} defined by

$$\forall U_0 \in \mathcal{H}_0, \; S_{00} U_0 = \Pi_0 S(0, U_0) , \tag{26}$$

where Π_0 is the projector from $\mathcal{H}_1^+ \times \mathcal{H}_0$ onto $\mathcal{H}_0$. The following result is the mathematical foundation of Membrane Paradigm :

Theorem 5.2. *For any* $U_0 \in \mathcal{H}_0$, $S_\varepsilon U_0$ *tends to* $S_{00} U_0$ *in* $\mathcal{H}_0$ *as* $\varepsilon \to 0$.

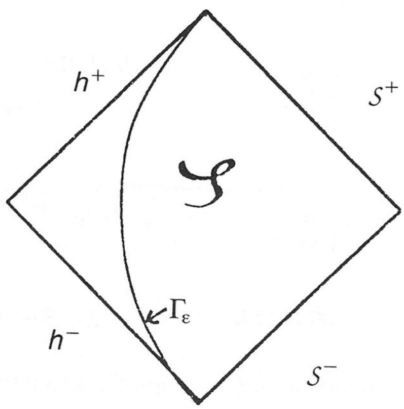
Of numerical analysis viewpoint, impedance condtiion (25) is an absorbing boundary condition on artificial boundary Γ_ε, so-called Silver–Müller radiation condition in Euclidean case [2]. So, Theorem 5.2 gives a method of numerical approximation, already used in [6].

6. Interpreation of the Kruskal Manifold

We reinterpret the whole scattering theory of electromagnetic field by Schwarzschild Black-Hole, in terms of characteristic Cauchy problem on the Kruskal manifold. We know that horizon $\mathbb{R}_t \times \{r = r_0\} \times S^2$ is a fictitious singularity arising from the choice of Schwarzschild coordinates (t, r, θ, φ). We can avoid it by using Kruskal coordinates (u, v, θ, φ) :

$$u = 2\tan^{-1}(-2r_0\, e^{-(t - r_*)/2r_0}),$$
$$v = 2\tan^{-1}(2r_0\, e^{(t + r_*)/2r_0}). \qquad (27)$$

This conformal transformation allows us to describe Schwarzschild universe $\mathcal{S}$ by the famous Penrose diagram :



Each point of this diagram is a two-sphere S^2 and $\mathcal{S} =]-\pi, 0[_u \times]0, \pi[_v \times S^2$.

$\mathscr{I}^+ =]-\pi, 0[_u \times \{v = \pi\} \times S^2$ is the future Minkowskian horizon $(t \to +\infty,\ r \to +\infty)$;
$\mathscr{I}^- = \{u = -\pi\} \times]0, \pi[_v \times S^2$ is the past Minkowskian horizon $(t \to -\infty,\ r \to +\infty)$;
$h^+ = \{u = 0\} \times]0, \pi[_v \times S^2$ is the future horizon of Black-Hole $(t \to +\infty,\ r \to r_0)$;
$h^- =]-\pi, 0[_u \times \{v = 0] \times S^2$ is the past horizon of Black-Hole $(t \to -\infty,\ r \to r_0)$.

These horizons are null submanifolds on which the asymptotic profiles of fields live. Time-like submanifold Γ_ε is the stretched horizon $\mathbb{R}_t \times \{r = r_0 + \varepsilon\} \times S^2$. Now we define

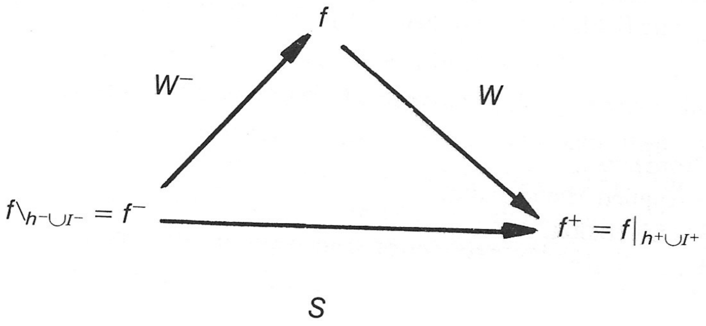
$$f(u, v, \theta, \varphi) = r\alpha U(t, r, \theta, \varphi)\,, \tag{28}$$

where U is given by (7). Maxwell equations (8) become

$$\mathcal{L}f = 0,\quad \mathcal{D}f = 0 \tag{29}$$

where $\mathcal{L}$ is an hyperbolic system for which horizons $h^\pm$, $\mathscr{I}^\pm$ are characteristic, and the operator $\mathcal{D}$ expresses the constraint of free divergence.

We can interpret our results in terms of characteristic Cauchy problem for (29) in $\mathcal{S}$: the existence of wave operator W^- implies that, given data f^- on past horizons $h^- \cup \mathscr{I}^-$ - i.e., given past asymptotic profiles - there exists a unique solution f of (29) in $\mathcal{S}$ such that $f \mid h^- \cup \mathscr{I}^- = f^-$. The existence of wave operator W implies that this solution f can be extended up to the upper boudnary $h^+ \cup \mathscr{I}^+$: its trace is the future asymptotic profile. We can resume by diagram



At last, Theorem 5.1 assures that the characteristic mixed problem for (29) in $\mathbb{R}_t \times [r_0 + \varepsilon, +\infty[_r \times S^2$ with data given on $\mathscr{I}^-$ and impedence condition on Γ_ε is well posed, and solution f_ε admits a trace on $\mathscr{I}^+$ (future asymptotic profile); we have

$$f_\varepsilon \mid \mathscr{I}^- \xrightarrow{S_\varepsilon} f_\varepsilon \mid \mathscr{I}^+ \,.$$

Now, the Membrane Paradigm states that if

$$f^- |_{h^-} = 0, \ f_\varepsilon |_{\mathcal{I}^-} = f^- |_{\mathcal{I}^-} ,$$

then

$$f_\varepsilon |_{\mathcal{I}^+} \to f |_{\mathcal{I}^+} , \ \varepsilon \to 0 ,$$

i.e.

$$S_\varepsilon (f^- |_{\mathcal{I}^-} \to S_{00} (f^- |_{\mathcal{I}^-}), \ \varepsilon \to 0 .$$

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