RESONANCES OF SCHWARZSCHILD BLACK HOLES

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Abstract. *This paper is devoted to the theoretical and computational investigations of the scattering frequencies of scalar, electromagnetic, gravitationnal waves around a spherical Black Hole. We adopt a time dependent approach: construction of wave operators for the equation hyperbolic Regge–Wheeler equation; asymptotic completeness; outgoing and incoming spectral representations; meromorphic continuation of the Heisenberg matrix; approximation by dumping and cut–off of the potentials and interpretation of the semi group Z(t) in the framework of the Membrane Paradigme. We developp a new procedure for the computation of resonances by spectral analysis of the transient scattered wave, based on Prony's algorithm.*

Introduction

This paper deals with the Scattering Frequencies of the *Regge–Wheeler* equation describing the perturbation of a massless field of spin s outside a Schwarzschild Black-Hole of radius 1 :

$$\partial_t^2 \Phi - \partial_x^2 \Phi + V(x)\, \Phi = 0, \quad t\in\mathbb{R},\ x\in\mathbb{R}\ , \tag{1}$$

$$V(x) = V_{l,s}(x) = (1-\frac{1}{r})\,\{\,\frac{l(l+1)}{r^2} + \frac{1-s^2}{r^3}\,\},\ \ l\in \mathbb{N},\ s\in\mathbb{R}, \tag{2}$$

$$x = r + Log\,(\,r-1)\ ,\ 1 < r < +\infty\ . \tag{3}$$

We developp the complete Scattering Theory for (1): existence and completeness of Wave Operators (part **I**); spectral representation and meromorphic continuation of the Heisenberg Matrix (part **II**); Lax-Phillips approach by cut-off approximation (part **III**); computation of the Resonances by Prony's algorithm (part **IV**). The details of the proofs will appear in [2].

I. Time Dependent One Dimensional Scattering Theory

R. Phillips [11] has studied (1) when the potential V decays as $|x|^{-2-\varepsilon}$, $\varepsilon>0$. Since $V_{l,s}$ decays more slowly we cannot apply these results. *J. Dimock* [5] has investigated the scalar case ($s=0$); our work [1] is devoted to the Maxwell System on the Schwarzschild background ($s=1$). In [2] we have considered equation (1) as a perturbation of the free wave equation

$$\partial_t^2 \Phi_0 - \partial_x^2 \Phi_0 = 0, \quad t\in\mathbb{R},\ x\in\mathbb{R}\ , \tag{4}$$

with the general assumption on the potential:

$$\begin{cases} V(x) = V_+(x) - V_-(x),\ V_\pm \geq 0, \\ V_+(x) \leq C(1+|x|)^{-1-\varepsilon},\ 0<\varepsilon, \\ V_-(x) \leq C(1+|x|)^{-2-\varepsilon},\ 0<\varepsilon. \end{cases} \tag{5}$$

If V_- is non zero, the functional framework is rather delicate to construct due to the possible existence of bounded states; to make this section simpler, we describe our results in the case

$$V = V_{l,s}\ ,\quad 0 \leq s \leq l. \tag{6}$$

We define the energy spaces

$$\mathscr{H}_o = BL^1(\mathbb{R}) \times L^2(\mathbb{R})\ ,\ \mathscr{H} = H(\mathbb{R}) \times L^2(\mathbb{R})\ , \tag{7}$$

which are respectively the closures of $C_o^\infty(\mathbb{R}) \times C_o^\infty(\mathbb{R})$ with respect to the norms

$$E_o(f) = \int_{\mathbb{R}} |f_1'|^2 + |f_2|^2\, dx\ ,\quad E(f) = \int_{\mathbb{R}} |f_1'|^2 + |f_2|^2 + V(x)\,|f_1|^2\, dx\ . \tag{8}$$

We introduce the unitary groups $U_0(t)$ on $\mathscr{H}_0$ and $U(t)$ on $\mathscr{H}$ expressing the solutions of (6) and (1) at time t from the initial data, by putting

$$U(t)\,(\Phi(0), \partial_t\,\Phi(0)) = (\Phi(t), \partial_t\,\Phi(t)),\quad U_o(t)\,(\Phi_0\,(0), \partial_t\,\Phi_0\,(0)) = (\Phi_0\,(t), \partial_t\,\Phi_0\,(t))$$

We need some dense subspace $\mathscr{D}_0$ of $\mathscr{H}$ and $\mathscr{H}_0$, invariant by $U_0(t)$:

$$\mathscr{D}_o = \{\, f = {}^t(f_1\,, f_2) \in C_o^\infty(\mathbb{R}) \times C_o^\infty(\mathbb{R})\,,\ \int f_2(x)\, dx = 0 \}\ . \tag{9}$$

We introduce the classical wave operators

$$W_\pm\, f = s - \lim_{t\to\pm\infty}\ U(-t)\, U_o(t)\, f\ \ in\ \ \mathscr{H}\ . \tag{10}$$

THEOREM 1 *–For any f in $\mathscr{D}_o$, the limits $W_\pm f$ are well defined and satisfy*

$$E(W_\pm\, f) = E_o(f),\quad \overline{W_\pm(\mathscr{D}_0)} = \mathscr{H}. \tag{11}$$

W_+, W_- can be extended by continuity as isometries from $\mathscr{H}_o$ onto $\mathscr{H}$ and the Scattering Operator

$$S = W_+^{-1}\, W_- \tag{12}$$

is an isometry from $\mathscr{H}_o$ on $\mathscr{H}_o$.

Idea of the proof: When assumption (6) is fulfilled we use Dimock's approach: the existence of $W_\pm f$ for f in $\mathscr{D}_o$ is proved by Cook's method. The Agmon-Kato-Kuroda theorem and the invariance principle give the completeness of the wave operators associated with $(-\,\partial_x^2\,)^{1/2}$ and $(-\,\partial_x^2\, + V\,)^{1/2}$ .We get that the solutions of (1) associated with initial data f in

$$\mathcal{D}_1 = \{ f = (\varphi_1 , (-\partial_x^2 + V)\varphi_2) ; \ \varphi_i \in L^2(\mathbb{R}), (-\partial_x^2 + V)\varphi_i \in L^2(\mathbb{R})\}$$

are asymptotically free:

$$\exists f_+ \in \mathcal{H}_o , \ \lim_{t \to +\infty} E_o(U(t) f - U_o(t) f_+) = 0 .$$

We conclude by noting the density of $\mathcal{D}_1$ in $\mathcal{H}$. In the general case where the potential satisfies only assumption (5), we adapt Phillips's method [11] and we prove the existence of Scattering Operator (12) (see [2] for the details).

II. Analytic Properties of the Heisenberg Matrix

The equivalence between the stationary scattering theory and the time dependent approach is expressed by the spectral representation that connects the Scattering Operator S and the Heisenberg matrix. We define the free spectral representation of $\mathcal{H}_0$ by putting for $f \in C_o^\infty(\mathbb{R}_x) \times C_o^\infty(\mathbb{R}_x)$:

$$\mathcal{R}_o f(\sigma, \omega) = E_o(f, \varphi_o(. , \sigma, \omega)) , \tag{12}$$

$$\sigma \in \mathbb{R}^* , \ \omega \in \{-1, 1\} , \ \varphi_o(x, \sigma, \omega) = \frac{1}{\sigma\sqrt{2\pi}} \ (e^{-i\sigma x\omega} , \ i\sigma e^{-i\sigma x\omega}) . \tag{13}$$

$\mathcal{R}_o$ can be extended as an isometry of $\mathcal{H}_o$ onto $L^2(\mathbb{R}_\sigma \times \{-1, 1\}_\omega)$. Then we prove that the Scattering Operator is unitarily equivalent to the Heisenberg matrix defined by the coefficients of transmission, T, and reflection, R_ω, given by

$$T(\sigma) = 2i\sigma \, [f_-(x, \sigma) \frac{df_+}{dx}(x, \sigma) - f_+(x, \sigma) \frac{df_-}{dx}(x, \sigma)]^{-1} , \quad \sigma \in \mathbb{R}^* , \tag{14}$$

$$2i\sigma \, R_\omega(\sigma) = T(\sigma) \, [f_+ (x, -\omega\sigma) \frac{df_-}{dx} (x, \omega\sigma) - f_- (x, \omega\sigma) \frac{df_+}{dx} (x, -\omega\sigma)] . \tag{15}$$

Here, $f_\pm(x, \sigma)$ are the Jost functions, solutions of the integral equation:

$$\sigma \in \mathbb{R}^*, \ f_\pm(x, \sigma) = e^{\pm i\sigma x} + \int_{\pm\infty}^{x} \frac{\sin\sigma(x-y)}{\sigma} V(y) \, f_\pm(y, \sigma) \, dy . \tag{16}$$

THEOREM 2 - *Assume* (5) *fulfilled. Then for any* $F \in L^2(\mathbb{R}_\sigma \times \{-1, 1\}_\omega)$ *we have*:

$$[\mathcal{R}_o \, S \, \mathcal{R}_o^{-1} F] (\sigma, \omega) = T(\sigma) \, F(\sigma, \omega) + R_{-\omega}(\sigma) \, F(\sigma, -\omega) , \ \sigma \in \mathbb{R}^*, \ \omega \in \{-1, 1\}.$$

We are interested in the meromorphic continuation of $T(\sigma)$; we know it is connected with the analyticity of V with respect to x. Therefore we have investigated the analytic continuation of the inverse function of (3)

$$x \to r \, ; \ x = r + \log (r-1)$$

and we prove the following

PROPOSITION - *For any potential* $V_{l,s}$ *given by* (2), *there exists* $B>0$ *such that* :

$$\begin{cases} (i)\ V \text{ is continuous on } \mathbb{R}\cup\{x\in\mathbb{C}, |\mathcal{R}e\,x|\geq B\} \text{ analytic inside,} \\ (ii)\ \forall\theta\in]-\frac{\pi}{2},+\frac{\pi}{2}[,\ \int_{\rho>0}|V(\pm B\pm\rho e^{i\theta})|\,d\rho<+\infty \\ and:\quad \sup_{|\alpha|\leq|\theta|}|\rho V(\rho e^{i\alpha})|\rightarrow 0, \rho\in\mathbb{R}, \rho\rightarrow\pm\infty . \end{cases} \tag{17}$$

By rotating the path of integration in (16) we prove the main result of this part:

THEOREM 3 - *We assume that assumptions* (5), (17) *are fulfilled. Then for any* $x\in\mathbb{R}$, $f_{\pm}(x,\sigma)$ *is analytic by respect to* σ *in* $\mathbb{C}\setminus i\mathbb{R}^{-}$.

The poles of $T(\sigma)$ are so called *resonances*, *scattering frequencies*, or *quasi–normal modes of the Black–Hole*.

III. Cut-off Approximation

The Lax-Phillips theory provides a time dependent characterization of the resonances associated with a compactly supported perturbation of the wave equation in Minkowski space time: the solution has a spectral decomposition in terms of generalised eigenfunctions. According to a suggestion of *B. Schmidt* [12], we apply the Lax-Phillips approach to the wave propagation on the Schwarzschild background by cutting off the Schwarzschild metric near the horizon and at large radius, and taking repectively the Rindler metric near the horizon and the flat metric near infinity. Therefore we make the assumption:

$$V\in L^{\infty}(\mathbb{R}),\ V\neq 0,\ \exists\,\rho>0;\ |x|\geq\rho,\ V(x)=0. \tag{18}$$

Then the solution Φ of (1) on $\mathbb{R}_t^{+}\times\mathbb{R}_x$ satisfies

$$\partial_t\,\Phi(t,\pm\rho)\pm\partial_x\,\Phi(t,\pm\rho)=0,\quad t\in\mathbb{R}^{+}. \tag{19}$$

Hence Φ is solution of (1) on $\mathbb{R}_t^{+}\times]-\rho,\rho[$ with boundary condition (19). The propagator of this mixed problem plays the role of the semi group $Z(t)$ of Lax-Phillips; this approximation by truncated potential is called *Membrane Paradigm*: (19) at $-\rho$ is the *T. Damour* impedance condition on the streched horizon [6]. From a numerical point of view (19) is the zero order absorbing condition of *Engquist–Majda* [7].

THEOREM 4 - *If assumption (19) is fulfilled, then $T(\sigma)$ given by (14) is meromorphic on the whole complex plane; each horizontal strip contains a finite number of its poles; the poles with non-negative imaginary part are purely imaginary , of finite multiplicity, and their set is finite; the set of the poles with negative imaginary part is infinite.*

We arrange the frequencies σ_j in decreasing order of their imaginary parts :

$$Im\ \sigma_{j+1} \le Im\ \sigma_j < 0 \le Im\ \sigma_0 \le Im\ \sigma_{-1} \le \dots \le Im\ \sigma_{-N}\ , \quad 1 \le j.$$

Then for each $n \ge -N$, $\varepsilon > 0$, $\rho > 0$, there exists $C(n, \varepsilon, \rho) > 0$, such as

$$|\Phi(t,x) - \sum_{j=-N}^{n} e^{-i\sigma_j t}\ C_j\, f_+(x,\sigma_j)| \le C(n,\varepsilon,\rho).(1+|x|)^{1/2}\ |e^{(-i\sigma_{n+1}+\varepsilon)t}| \qquad (20)$$

holds for any solution Φ of (1) with compactly supported initial data f in $\mathcal{H}$, where

$$C_j = Residu\ (\frac{1}{W(\sigma)}\ \int_{\mathbb{R}} f_-(x,\sigma)\,(i\sigma\, f_1(x) - f_2(x))\, dx\ ;\ \sigma = \sigma_j\ ).$$

The spectral decomposition (20) characterizes the resonances and guaranties that the solution is exponentially vanishing if there is no bounded state.

R.G.Newton has emphasized in [10] that the singularity structure of the **S** matrix for any potential of infinite range must generally be expected to differ from the limit of the singularity structure for the corresponding cutoff potential when the cutoff tends to infinity. Nevertheless we prove that the resonances are the limits of the ones associated with cutoff and damped potentials: given $\gamma > 1$, $\varepsilon > 0$, $\rho > 0$, we put

$$V^{\rho}_{\gamma,\varepsilon}(x) = \chi_{[-\rho,\rho]}(x)\ .\ e^{-\varepsilon|x|^{\gamma}}\ .V(x)\ .$$

Let $\mathcal{R}(V,E)$ be the set of resonances in $E \subset \mathbb{C}$ for a potential V .

THEOREM 5 - *If assumptions (5), (17) are fulfilled, then for any $\eta > 0$, $\gamma > 1$, and bounded open Ω satisfying*

$$\overline{\Omega} \subset \{\sigma \in \mathbb{C}^*\ , -\frac{\pi}{2\gamma} < Arg\ \sigma < \pi + \frac{\pi}{2\gamma}\ \}\ ,$$

there exists $\varepsilon_0 > 0$ such as given $\varepsilon \in]0, \varepsilon_0]$, there exists $R > 0$ satisfying :

$$\forall \rho > R,\ Card\ \mathcal{R}(V,\Omega) = Card\ \mathcal{R}(V^{\rho}_{\gamma,\varepsilon}, \Omega)$$

$$\forall \sigma \in \mathcal{R}(V,\ \Omega)\ \exists \sigma^{\rho}_{\gamma,\varepsilon} \in \mathcal{R}(V^{\rho}_{\gamma,\varepsilon},\ \Omega)\ ,\ |\sigma - \sigma^{\rho}_{\gamma,\varepsilon}| < \eta\ ;$$

If $\overline{\Omega} \subset \{\sigma \in C, Im\,\sigma > 0\ \}$ we can take $\varepsilon_0 = \varepsilon = 0$.

IV. Computation of the Resonances by Prony's Algorithm

The numerical investigation of the resonances is very delicate because these poles are not characterized in a variational way and we know no error estimate. Hence the values obtained by computation can be some *artefact* ; therefore it is very important to compare the results given by different methods. All the methods used to find the Black Hole Resonances consist in solving the elliptic equation

$$-\partial_x^2 \Phi + V(x)\,\Phi = \sigma^2\,\Phi$$

provided with the outgoing radiation condition [4], [8], [12]. Instead, we solve the *time dependent* equation (1) and apply the *Prony procedure* already used for acoustic waves [9], [13]. More precisely, we compute solution Φ of (1) by a finite difference scheme. According to Theorem 4 the asymptotic expansion

$$\Phi(t, x_0) \approx \sum_{j=1}^{N} C_j\, e^{-i\sigma_j t}\ , \tag{20}$$

is valid for x_0 fixed and $t > t_0$ large enough. Then we choose a sample rate $\Delta T > 0$ and denote

$$f_k = \Phi(t_0 + k\,\Delta T, x_0)\ ,\quad z_j = e^{-i\sigma_j \Delta T}\ .$$

Hence we have to solve the polynomial system

$$\sum_{j=1}^{N} C_j (z_j)^k = f_k\ ,\quad k = 0, 1,... \tag{21}$$

Following the idea of *Prony* this problem can be reduced to finding the zeros z_j of the polynomial

$$\sum_{k=0}^{N} \alpha_k\, z^k = 0\ . \tag{22}$$

where the coefficients α_k are the solutions of the overdetermined linear system

$$\alpha_N = 1,\quad \sum_{j=0}^{N} \alpha_j\, f_{j+m} = 0\ ,\quad m = 0,...,M-1,\ M > N. \tag{23}$$

This system is solved using the generalised inverse and the singular value decomposition. Since (23) is very ill-conditioned, this step is unstable with respect to the slight variations of f_k . Therefore the computation of Φ has to be very accurate. For instance we choose the spatial and temporal grid sizes of the finite difference scheme $\Delta t = \Delta x = 10^{-4}$ on the domain $[-40, 120]_x \times [0, 160]_t$. To solve (22) we use Muller's algorithm. The following table gives the values of the resonances of gravitational waves ($s = 2$) for the modes $l = 2, 3, 4$, obtained by this Prony procedure and the results of S. Chandrasekar, S. Detweiler (*C.D.*) [4] and E. Leaver [8] who applied stationary approaches. The third values of *C.D.* for $l = 3, 4$, seem to be numerical artefacts.

table :gravitational waves, s=2.

Prony	*C.D.*	*Leaver*
l=2		
0.74734349, 0.17792462	0.74734, 0.17792	0.747343, 0.177925
0.69342, 0.54783	0.69687, 0.54938	0.693422, 0.547830
0.60, 0.95		0.602107, 0.956554
l=3		
1.198887042, 0.185406087	1.19889, 0.18541	1.198887, 0.185406
1.165288, 0.562596	1.16402, 0.56231	1.165288, 0.562596
1.1034 ± 2.10^{-4}, 0.9598 ± 2.10^{-4}	*0.85257, 0.74546*	1.103370, 0.958186
1.02, 1.38		1.023924, 1.380674
l=4		
1.6183578804, 0.1883279128	1.61835, 0.18832	1.61836, 0.18833
1.5932642, 0.5686687	1.59313, 0.56877	1.59326, 0.56867
1.5455, 0.9598	*1.12019, 0.84658*	1.54542, 0.95982
1.477 ± 2.10^{-3}, 1.367 ± 2.10^{-3}		1.47967, 1.36785

To conclude, we note the great accuracy of the Prony procedure for the computation of the first resonances and we constat an excellent agreement with the values obtained by E. Leaver. As regards the computation of high-overtone normal modes, the efficiency of this method is limited by the fast decay of the modes associated with a resonance with large imaginary part. The numerical experiment on super computer **CRAY2** is in progress.

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