GLOBAL SOLUTIONS TO NON LINEAR DIRAC EQUATIONS IN MINKOWSKI SPACE

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INTRODUCTION

The purpose of this paper is to expose the recent results of global existence for non linear Dirac equations or Dirac-Klein-Gordon systems in both cases where blow up can generally occur : 1) if the order of the non linearities is critic with respect to the space dimension ; 2) if the Cauchy data is large. More precisely let's consider an hyperbolic symetric system

$$\mathscr{L}(\partial_t,\partial_x)\psi = f(\psi),\ x\in\mathbb{R}^n \tag{1}$$

where

$$|f(\psi)| = O(|\psi|^d),\ |\psi| \to 0\ .$$

If we want to obtain global asymptotically free solutions of (1) for small initial data, we expect that the energy contribution of $f(\psi)$ is finite, i.e.

$$\int_{-\infty}^{+\infty} \|f(\psi(t))\|_{L^2(\mathbb{R}^n_x)} dt<+\infty\ . \tag{2}$$

We test estimate (2) with regular wave packet free solutions of $\mathscr{L}(\partial_t,\partial_x)\psi_0=0$ which satisfy

$$\|\psi_0(t)\|_{L^2(\mathbb{R}^n_x)}=cst,\ |\psi_0(t)|_{L^\infty(\mathbb{R}^n_x)}=O(|t|^{-(n-1)/2})\ ;$$

therefore,

$$\|f(\psi_0(t))\|_{L^2(\mathbb{R}^n_x)}=O(|t|^{-(d-1)(n-1)/2})$$

then the case of quadratic non linearities in Minkowski space $\mathbb{R}^{3+1}$ is critic and we know that the local solution can blow up [10]. So it is interesting to find bilinear interaction f such that

$$\|f(\psi_0(t))\|_{L^2(\mathbb{R}^3_x)} \in L^1(\mathbb{R}_t). \qquad (3)$$

We know that a sufficient condition for (3) is that f is null on the kernel of the symbol of $\mathcal{L}$: it is the algebraic condition of *compatibility* of f with $\mathcal{L}$ introduced by B. Hanouzet and J.L. Joly [9] (and so [1]) which is related to the Lorentz invariance and the *Null Condition* of S. Klainerman [13]. We study these notions in part I and apply them in part II to solve the global Cauchy problem with small initial data for the Dirac-Klein-Gordon systems

$$\begin{aligned} -i\gamma^\mu\partial_\mu\psi+M\psi &= f(\varphi,\psi), \quad M\neq 0 \\ \Box\varphi+m^2\varphi &= g(\varphi,\psi), \qquad m\geq 0 \end{aligned} \qquad (4)$$

Now, if we take large initial data, we cannot expect that (2) is verified and in fact all cases can occur : global existence, blow up or stationary solutions, even for d=3, n=3 ; for instance, M. Balabane, Th. Cazenave, A. Douady, F. Merle prove in [5] the existence of infinitely many stationary states for

$$-i\gamma^\mu\partial_\mu\psi+M\psi = (\bar{\psi}.\psi)\psi \;, \quad M\neq 0 \qquad (5)$$

and we establish in part IV the existence of global asymptotically free solutions to systems generelising (4) and (5) for arbitrarly large initial data under two algebraic assumptions on the non linearities and on the polarization of initial data : on the one hand we suppose that the systems are Lorentz invariant, and on the other hand, we consider initial data satisfying an approximated Lochak-Majorana condition. We study in part III this last condition which implies that the chiral invariant is small ; then, the Lorentz-invariance implies that the non linearities are small, and (2) is true again and we obtain the global existence and the asymptotic freedom. The complete demonstrations are published in [2], [3], [4].

I - COMPATIBILITY - LORENTZ-INVARIANCE - NULL CONDITION

We specify some notations $g^{\mu,\nu}=\mathrm{diag}(1,-1,-1,-1)$ is the Lorentz metric on Minkowski space $\mathbb{R}^4=\mathbb{R}_t\times\mathbb{R}^3_x$, $x^0=t$, $(x^1,x^2,x^3)=x$. The Dirac matrices γ^μ, $0\leq\mu\leq 3$, satisfy the relations

$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu,\nu}I \;, \quad \tilde{\gamma}^\mu=g^{\mu,\mu}\gamma^\mu$$

where $\tilde{A}$ notes the conjugate transposate of matrix A. We introduce so the matrix $\gamma^5=-i\gamma^0\gamma^1\gamma^2\gamma^3$.

We consider the generators $(\Gamma_\sigma)_{1\leq\sigma\leq 10}$ of Poincaré group

$$(\Gamma_\sigma)_{1\leq\sigma\leq 10} = (\partial_\mu=\partial/\partial x^\mu,\; \Omega_{\mu,\nu}=x_\mu\partial_\nu-x_\nu\partial_\mu)_{0\leq\mu,\nu\leq 3}$$

The Lorentz invariance of the wave equation is expressed by the commuting relations

$$[\Gamma_\sigma,\Box] = 0 .$$

To study the Dirac system we introduce

$$(\hat{\Gamma}_\sigma)_{1\leq\sigma\leq 10} = (\partial_\mu,\Omega_{\mu,\nu}+\tfrac{1}{2}\gamma_\mu\gamma_\nu)_{0\leq\mu,\nu\leq 3}$$

which satisfy

$$[\hat{\Gamma}_\sigma,-i\gamma^\mu\partial_\mu] = 0 .$$

In the case of massless fields, we can use the scaling invariance with the radiation operator Γ_0 :

$$\Gamma_0=x^\mu\partial_\mu,\ [\Box,\Gamma_0]=2\Box,\ [-i\gamma^\mu\partial_\mu,\Gamma_0]=-i\gamma^\mu\partial_\mu .$$

We recall some definitions :

i) A sesquilinear form f on $\mathbb{C}^N$ is said compatible with a first order differential system

$$A(\partial) = \sum_{j=1}^{n} A_j \frac{\partial}{\partial x^j} +iB$$

where A_j and B are P×N matrices with constant coefficients if

$$\forall\xi\in\mathbb{R}^n\setminus\{0\},\ V\in\operatorname{Ker}(\sum_{j=1}^{n}\xi_j A_j+B) \Rightarrow f(V,V)=0 .$$

ii) A sesquilinear form $N(\psi_1',\psi_2')$ on $\mathbb{C}^{16}\times\mathbb{C}^{16}$ satisfies the null condition if $\forall\psi_i\in C^1(\mathbb{R}^4,\mathbb{C}^4)$, ${}^t\psi_i=(\psi_i^1,\psi_i^2,\psi_i^3,\psi_i^4)$, $\psi_i'=(\partial_\mu\psi_i)_{0\leq\mu\leq 3}$,

$$\forall(X_\mu)\in\mathbb{R}^4, g^{\mu,\nu}X_\mu X_\nu=0 \Rightarrow \forall h,k,\ \sum_{0\leq\mu,\nu\leq 3} \frac{\partial^2 N}{\partial(\partial_\mu\tilde{\psi}_1^h)\partial(\partial_\nu\psi_2^k)} X_\mu X_\nu=0 .$$

iii) The spinorial representation of the whole Lorentz group O(3,1) is the mapping

$$L=(L^\mu_\nu)\in O(3,1) \to \Lambda\in SL(4,\mathbb{C})/\{-1,1\},$$

defined by

$$L^\mu_\nu\gamma^\nu = \Lambda^{-1}\gamma^\mu\Lambda .$$

We note DO(3,1) the orthochroneous proper Lorentz subgroup.
The compatible forms for massless Dirac system are described by the following :

THEOREM I.1 - Let f be a sesquilinear form on $\mathbb{C}^4$; the following assertions are equivalent :

1) $\forall\psi_i \in \mathbb{C}^4$, $\forall L \in DO(3,1)$, $f(\Lambda\psi_1,\Lambda\psi_2)=f(\psi_1,\psi_2)$.
2) f is compatible with the massless Dirac system $\mathscr{L}_0=-i\gamma^\mu\partial_\mu$.
3) $N(\psi_1',\psi_2')=f(\mathscr{L}_0\psi_1,\mathscr{L}_0\psi_2)$ satisfies the null condition.
4) $\exists C>0$, $\forall\psi_i \in C^1(\mathbb{R}^4,\mathbb{C}^4)$,

$$|N(\psi_1',\psi_2')|\leqslant C(1+|t|+|x|)^{-1} \sup_{0\leqslant\sigma,\tau\leqslant 10} |\Gamma_\sigma\psi_1(t,x)|\,|\Gamma_\tau\psi_2(t,x)| \tag{6}$$

5) $\exists\alpha,\beta\in\mathbb{C}/f(\psi_1,\psi_2)=\tilde{\psi}_1(\alpha\gamma^0+\beta\gamma^0\gamma^5)\psi_2$.

If the mass is non null we must avoid Γ_0 and the is only one compatible form.

THEOREM I.2 - Let f be a sesquilinear form on $\mathbb{C}^4$; the following assertions are equivalent :

1) $\forall\psi_i \in \mathbb{C}^4$, $\forall L \in O(3,1)$, $f(\Lambda\psi_1,\Lambda\psi_2)=(\det L)\ f(\psi_1,\psi_2)$.
2) f is compatible with the mass Dirac system $\mathscr{L}_M=-i\gamma^\mu\partial_\mu+M$, $M\neq 0$.
3) $N(\psi_1',\psi_2')=f(\mathscr{L}_0\psi_1,\mathscr{L}_0\psi_2)+g^{\mu,\nu}f(\partial_\mu\psi_1,\partial_\nu\psi_2)$ satisfies the null condition and

$$|N(\psi_1',\psi_2')|\leqslant C(1+|t|+|x|)^{-1} \sup_{1\leqslant\sigma,\tau\leqslant 10} |\Gamma_\sigma\psi_1(t,x)|\Gamma_\tau\psi_2(t,x)|. \tag{7}$$

4) $\exists\alpha\in\mathbb{C}/f(\psi_1,\psi_2)=\alpha\tilde{\psi}_1\gamma^0\gamma^5\psi_2$.

It is crucial that the radiation operator does not appear in estimate (7). The estimates (6)(7) play a fundamental role in our study : if ψ_j are regular wave packet free solutions (6)(7) imply

$$\|N(\psi_1',\psi_2')(t)\|_{L^2(\mathbb{R}^3_x)}=O(|t|^{-\alpha}),\alpha=2 \text{ if } M=0,\ \alpha=5/2 \text{ if } M\neq 0\ ,$$

instead of $\alpha=1$ if $M=0$, $\alpha=3/2$ if $M\neq 0$, for ordinary product.

II - GLOBAL SOLUTIONS FOR MASS AND MASSLESS FIELDS INTERACTING.

We consider system (4) with quadratic nonlinearities

$$-i\gamma^\mu\partial_\mu\psi+M\psi=\varphi V\psi, \tag{8.1}$$

$$\Box\varphi+m^2\varphi=\tilde{\psi}F\psi \tag{8.2}$$

where V and F are 4×4 matrices with constant coefficients. If both masses M and m are non null, the uniform decay is fast enough to assure the global existence [12]. If both masses M and m are null, the conformal invariance allows us to use Penrose transform and to obtain global solutions [7]. The interesting case is :

$$M\neq 0\ ,\ m=0 \tag{9}$$

We make two algebraic assumptions on the nonlinearities :

$$\tilde{V}\gamma^{0}=\gamma^{0}V,\ \ \tilde{F}=F, \tag{10}$$

$$F=ig\gamma^{0}\gamma^{5},\ \ g\in\mathbb{R} \tag{11}$$

(10) implies the conservation of the spinorial charge and (11) has two equivalent interpretations according to theorem I.2 : φ is a pseudoscalar Lorentz invariant field ; F is compatible with the mass Dirac system. The main example is the Yukawa model of nuclear forces with the interaction lagrangien $ig\varphi\tilde{\psi}\gamma^{0}\gamma^{5}\psi$.
We choose small very regular initial data :

$$\psi(0,x)=\varepsilon\psi_{0}(x),\ \varphi(0,x)=\varepsilon\varphi_{0}(x),\ \partial_{t}\varphi(0,x)=\varepsilon\varphi_{1}(x),$$
$$\psi_{0}\in\mathcal{D}(\mathbb{R}^{3}_{x},\mathbb{C}^{4}),\ \varphi_{j}\in\mathcal{D}(\mathbb{R}^{3}_{x},\mathbb{R}),\ \ 0<\varepsilon\ . \tag{12}$$

THEOREM II.1 - Under hypotheses (9)(10)(11), there exists $\varepsilon_{0}>0$ such that for $\varepsilon\in]0,\varepsilon_{0}[$, the Cauchy problem (8)(12) has a unique solution $(\psi,\varphi)\in C^{\infty}(\mathbb{R}^{4})$. Moreover (ψ,φ) is asymptoticaly free : there exist $\psi^{\pm}$ $\varphi^{\pm}$ satisfying

$$-i\gamma^{\mu}\partial_{\mu}\psi^{\pm}+M\psi^{\pm}=0,\ \ \Box\varphi^{\pm}=0$$

$$\lim_{t\to\pm\infty}\left(\|\psi(t)-\psi^{\pm}(t)\|_{L^{2}(\mathbb{R}^{3}_{x})}+\sum_{\mu=0}^{3}\|\partial_{\mu}\varphi(t)-\partial_{\mu}\varphi^{\pm}(t)\|_{L^{2}(\mathbb{R}^{3}_{x})}\right)=0\ .$$

In fact φ has even a nice behaviour in L^{2}-norm :

$$\varphi,\varphi^{\pm}\in C^{0}(\mathbb{R},L^{2}(\mathbb{R}^{3}_{x})),\ \|\varphi(t)-\varphi^{\pm}(t)\|_{L^{2}(\mathbb{R}^{3}_{x})}\longrightarrow 0,\ \ t\to\pm\infty. \tag{13}$$

The key of the proof of theorem II.1 is the compatibility of F, (11) which allows to transform the nonlinear wave equation (8.2) into a better wave equation

$$\Box(\varphi+(2M)^{-2}\tilde{\psi}F\psi)=N(\psi',\psi')+\mathcal{R}$$

where N satisfies the null condition and (7) and $\mathcal{R}$ is cubic.
An interesting question is the necessity of (11) to solve the global Cauchy problem. We show that we can replace (11) by

$$V=ig\gamma^{5},\ \ g\in\mathbb{R}\ . \tag{14}$$

We point out that, if system (8) comes from a lagrangian, then (11) and (14) are equivalent and we find the Yukawa model again.

THEOREM II.2 - Under hypotheses (9)(10)(14), the conclusions of theorem II.1 hold again.

But in this case φ has not necessarly a good behaviour in L^{2}-norm ;

we prove only $\|\varphi(t)\|_{L^2(\mathbb{R}^3_x)}=O(t^{1/2})$. This time, algebraic condition (14) is used to transform equation (8.1) :

$$(\Box+M^2)\psi=-g(\partial_\mu\varphi)\gamma^\mu\gamma^5\psi+\text{cubic}$$

and we note that in the quadratic part, φ appears only with the first derivatives which are easily estimated in L^2.

III - CHIRAL INVARIANT AND MAJORANA CONDITION

To estimate a Lorentz-invariant non linearity $F(\tilde{\psi}\gamma^0\psi, i\tilde{\psi}\gamma^0\gamma^5\psi)$, we introduce according to G. Lochak [14] the *chiral invariant* of ψ, $\rho(\psi)$ defined by

$$\rho^2=|\tilde{\psi}\gamma^0\psi|^2+|\tilde{\psi}\gamma^0\gamma^5\psi|^2.$$

We are concerned by the spinors ψ for which the chiral invariant is null. In the case of a free solution of the linear Dirac equation, a necessary and sufficient condition to $\rho=0$ is the Majorana condition generalized :

$$\exists z\in\mathbb{C}\ ,\ |z|=1,\ \psi=z\gamma^2\psi^+,$$

where ψ^+ is the complex conjugate of ψ and we have choosen the Dirac matrices in order to γ^2 is symetric :

$$\gamma^2=\begin{pmatrix}0 & \sigma^2\\ -\sigma^2 & 0\end{pmatrix},\quad \sigma^2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}.$$

We have the same result for the Dirac systems with a time dependent potential A

$$-i\gamma^\mu\partial_\mu\psi=A\psi \tag{15}$$

where A satisfies

$$A,\partial_\mu A\in L^\infty_{loc}(\mathbb{R}_t;L^\infty(\mathbb{R}^3_x;\mathbb{R}Id+i\mathbb{R}\gamma^5)). \tag{16}$$

PROPOSITION III.1 - Let ψ be a solution of (15) and $\psi\in C^0(\mathbb{R}_t,(L^2(\mathbb{R}^3_x))^4)$ $\psi(0,x)=\psi_0(x)$. Then the following assertions are equivalent :

i) $\exists z\in\mathbb{C}$, $|z|=1$, $\psi_0=z\gamma^2\psi_0^+$;

ii) $\forall x\in\mathbb{R}^3$, $\rho(\psi_0(x))=0$;

iii) $\forall (t,x)\in\mathbb{R}^4$, $\rho(\psi(t,x))=0$;

iv) $\forall (t,x)\in\mathbb{R}^4$, $\exists z\in\mathbb{C}$, $|z|=1$, $\psi(t,x)=z\gamma^2\psi^+(t,x)$.

For z=1, the implication i) $\Rightarrow$ iv) was proved by J. Chadam and R. Glassey [6].

IV - GLOBAL EXISTENCE OF LARGE AMPLITUDE SOLUTIONS

First, we consider the mass Dirac-Klein-Gordon system in Minkowski space $\mathbb{R}^{3+1}$:

$$\begin{cases} -i\gamma^\mu\partial_\mu\psi+M\psi=\varphi V\psi+F(\overline{\psi}\psi,i\overline{\psi}\gamma^5\psi)\psi, \\ \Box\varphi+m^2\varphi=G(\overline{\psi}\psi,i\overline{\psi}\gamma^5\psi)-k\varphi^3 \end{cases} \tag{17}$$

where the masses M and m are non null

$$M\neq 0,\ m\neq 0. \tag{18}$$

We define the vector space $\mathcal{M}$ of 4×4 matrices

$$\mathcal{M}=\{\alpha I+i\beta\gamma^5,\ \alpha,\beta\in\mathbb{R}\}.$$

The hypotheses on the non linearities are following :

$$V\in\mathcal{M} \tag{19}$$
$$F\in C^\infty(\mathbb{R}^2,\mathcal{M}),\ |F(u,v)|=O(|u|+|v|),\ |u|+|v|\to 0 \tag{20}$$
$$G\in C^\infty(\mathbb{R}^2,\mathbb{R}),\ |G(u,v)|=O(|u|+|v|),\ |u|+|v|\to 0 \tag{21}$$
$$k\geqslant 0 \tag{22}$$

$\overline{\psi}$ notes the usual Dirac conjugate

$$\overline{\psi}=\tilde{\psi}\gamma^0\ . \tag{23}$$

Many models of the relativistic fields theory satisfy these hypotheses ; the scalar and pseudo-scalar Yukawa models of the nuclear forces, the interactions of Heisenberg, Federbusch, the magnetic monopole of G. Lochak.
We choose arbitrarly large Cauchy data in a neighboorhood of decoupling data for which the chiral invariant is null

$$\psi\Big|_{t=0}=\Psi_0+\varepsilon X_0\ ,\ 0<\varepsilon, \tag{24}$$
$$\Psi_0,\ X_0\in\mathcal{D}(\mathbb{R}^3_x,\mathbb{C}^4), \tag{25}$$

$$\Psi_0=z\gamma^2\Psi_0^+\ ,\ z\in\mathbb{C}\ ,\ |z|=1, \tag{26}$$

$$\varphi\Big|_{t=0}=\varphi_0\ ,\ \partial_t\varphi\Big|_{t=0}=\varphi_1 \tag{27}$$
$$\varphi_0,\varphi_1\in\mathcal{D}(\mathbb{R}^3_x,\mathbb{R}). \tag{28}$$

According to proposition III.1, assumption (26) implies that Ψ_0 is a decoupling data : for $\varepsilon=0$, the scalar field φ does not depend on ψ and ψ satisfies a linear equation. So we solve the Cauchy problem in a neighbourhood of such a solution. Remark Ψ_0 and scalar field φ can be as large as we want.

THEOREM IV.1 - There exists ε_0 such that for any $0\leqslant\varepsilon\leqslant\varepsilon_0$ the Cauchy problem (17) to (28) has a unique solution (ψ,φ) in $C^\infty(\mathbb{R}^4)$. Moreover, this solution is asymptotically free : there exist $\psi^\pm$, $\varphi^\pm$ satisfying

$$\psi^\pm\in C^0(\mathbb{R}_t,(L^2(\mathbb{R}^3_x))^4),\ -i\gamma^\mu\partial_\mu\psi^\pm+M\psi^\pm=0,$$
$$\varphi^\pm\in C^0(\mathbb{R}_t,H^1(\mathbb{R}^3_x))\cap C^1(\mathbb{R}_t,L^2(\mathbb{R}^3_x)),\Box\varphi^\pm+m^2\varphi^\pm=0,$$
$$\lim_{t\to\pm\infty}\|\psi(t)-\psi^\pm(t)\|_{L^2}+\|\varphi(t)-\varphi^\pm(t)\|_{H^1}+\|\partial_t\varphi(t)-\partial_t\varphi^\pm(t)\|_{L^2}=0.$$

This result shows how complicated is the problem of large amplitude solutions : indeed, we have obtained large solutions asymptotically free but we know that there exist so stationnary solutions for V=0, $F(\overline{\psi}\psi,\overline{\psi}\gamma^5\psi)=\overline{\psi}\psi$, [5].
Now we consider the non linear massless Dirac system

$$-i\gamma^\mu\partial_\mu\psi=F(\overline{\psi}\psi,i\overline{\psi}\gamma^5\psi)\psi, \tag{29}$$

and F verifies (20).
The global Cauchy problem for small initial data was solved by J.P. Dias, M. Figueira [8]. Here we choose larga data satisfying (24) (25) (26).

THEOREM IV.2 - There exists $\varepsilon_0>0$ such that for any $0\leqslant\varepsilon\leqslant\varepsilon_0$, the Cauchy problem (29)(24)(25)(26) has a unique solution $\psi\in C^\infty(\mathbb{R}^4)$ which is asymptotically free : there exists $\psi^\pm$ verifying

$$\psi^\pm\in C^0(\mathbb{R}_t,(L^2(\mathbb{R}^3_x))^4),\ -i\gamma^\mu\partial_\mu\psi^\pm=0,$$
$$\lim_{t\to\pm\infty}\|\psi(t)-\psi^\pm(t)\|_{L^2}=0\ .$$

Moreover ψ and the free solution have the same decay inside the light cone : for $0\leqslant C<1$

$$|\psi(t)|_{L^\infty(\{|x|<C|t|\})}=O(|t|^{-2}). \tag{30}$$

(30) is a consequence of the important fact following : let's consider Φ defined by

$$\Phi=x_\mu\gamma^\mu\psi\ . \tag{31}$$

Then Φ is solution of a non linear wave equation which allows us to make convenient estimates and we have

$$|\Phi(t)|_{L^\infty(\mathbb{R}^3_x)}=O(|t|^{-1}) \tag{32}$$

that proves (30).
Now we want point out the remarkable properties of asymptotic behaviour of relativistic quantities $\overline{\psi}\psi$ and $\overline{\psi}\gamma^3\psi$. We use another characterization of the compatibility of a sesquilinear form with the massless Dirac system : the factorization formula of B. Hanouzet and J.L. Joly [9] :

PROPOSITION IV.1 - Let f be a sesquilinear form on $\mathbb{C}^4$; the following assertions are equivalent :

i) f is compatible with the massless Dirac system $-i\gamma^\mu\partial_\mu$;

ii) there exist 4×4 matrices $P(x^\mu)$,$Q(x^\mu)$, homogeneous of order -1 such that

$$f(\psi,\psi)=\tilde{\Phi}P\psi+\tilde{\psi}Q\Phi, \tag{33}$$

where Φ is defined by (31).

Theorem I.1 and (32)(33) allow to obtain for non linear system (29) a strong result of equipartition of energy which is well known in the linear case [1] [11] :

THEOREM IV.2 - Let ψ be the solution of (29) given by theorem IV.1. Then the relativistic quantities satisfy :

$$|\overline{\psi}(t)\psi(t)|_{L^1(\mathbb{R}^3_x)}+|\overline{\psi}(t)\gamma^5\psi(t)|_{L^1(\mathbb{R}^3_x)}=O(|t|^{-1}),$$
$$|\overline{\psi}(t)\psi(t)|_{L^\infty(\mathbb{R}^3_x)}+|\overline{\psi}(t)\gamma^5\psi(t)|_{L^\infty(\mathbb{R}^3_x)}=O(|t|^{-3}).$$

We can prove by the same methods the existence of global solutions for massless Dirac equation with cubic relativistic nonlinearity in two space dimension.

CONCLUSION

If we consider mass and massless fields interacting in Minkowski space, there are two difficulties : on the one hand the mass breaks the conformal invariance and we cannot transform the global Cauchy problem in $\mathbb{R}^{3+1}$ into a local Cauchy problem on $S^3\times\mathbb{R}$ by using Penrose transform. On the other hand, the uniform decay of massless field is a priori only t^{-1} and the energy of quadratic nonlinearities does not decay fast enough to assure the global existence.

Nevertheless, we have proved that the global Cauchy problem with small data is well posed for the mass Dirac system quadratically coupled with a massless scalar field if the nonlinearities satisfy the algebraic property of *compatibility* with the Dirac system, related to the Lorentz invariance and the null condition.

If the initial data are large, some blow-up can occur or certain stationnary solutions can exist. In the case of the mass Dirac-Klein-Gordon system with quadratic nonlinearities or the massless Dirac system with cubic nonlinearity, we have proved the existence of global solutions asymptotically free, for arbitrarly large Cauchy data under two assumptions : on the one hand the nonlinearities are Lorentz-invariant, on the other hand the polarization of initial spinor satisfies a generalized Majorana condition ; then the nonlinearities are small again and decay more fast than an ordinary product.

Therefore we constat that in critic cases, quadratic nonlinearities in three space dimension, or large initial data, we can obtain global asymptotically free solutions under algebraic hypotheses on the nonlinearities and the Cauchy data.

REFERENCES

[1] A. Bachelot, *Equipartition de l'énergie pour les systèmes hyperboliques et formes compatibles*. Ann. Inst. Henri Poincaré, Physique Théorique, vol.46, n°1, 1987, pp.45-76.

[2] A. Bachelot, *Problème de Cauchy global pour des systèmes de Dirac-Klein-Gordon*. Ann. Inst. Henri Poincaré, Physique Théorique, vol.48, n°4, 1988, pp.387-422.

[3] A. Bachelot, *Global existence of large amplitude solutions for Dirac-Klein-Gordon systems in Minkowski space.* Colloque international "Problèmes hyperboliques non linéaires", Bordeaux, Juin 1988, to appear in Lecture Notes in Math, Springer-Verlag.

[4] A. Bachelot, *Global existence of large amplitude solutions for nonlinear massless Dirac equation*. To appear in Portugaliae Math.

[5] M. Balabane, T. Cazenave, A. Douady, F. Merle, *Existence of excited states for a nonlinear Dirac field.* To appear.

[6] J. Chadam, R. Glassey, *On certain global solutions of the Cauchy problem for the (classical) coupled Klein-Gordon-Dirac equations in one and three space dimensions.* Arch. Rat. Mech. Anal. 54, 1974, pp.223-237.

[7] Y. Choquet-Bruhat, D. Christodoulou, *Existence of global solutions of the Yang-Mills, Higgs and spinor field equations in 3+1 dimensions.* Ann. Scient. Ec. Norm. Sup. 4ème série, t.14, 1981, pp.481-500.

[8] J.P. Dias, M. Figueira, *Sur l'existence d'une solution globale pour une équation de Dirac non linéaire avec masse nulle.* C.R. Acad. Sci. Paris, série I, t.305, 1987, pp.469-472.

[9] B. Hanouzet, J.L. Joly, *Applications bilinéaires sur certains sous-espaces de type Sobolev.* C.R. Acad. Sci. Paris, série I, t.294, 1982, pp.745-747.

[10] B. Hanouzet, J.L. Joly, *Explosion pour des problèmes hyperboliques semi-linéaires avec second membre non compatible.* C.R. Acad. Sci. Paris, t.301, n°11, 1985, pp.581-584.

[11] B. Hanouzet, J.L. Joly, *Applications bilinéaires compatibles avec un système hyperbolique.* Ann. Inst. Henri Poincaré, Analyse non linéaire, t.4, n°4, 1987, pp.357-376.

[12] S. Klainerman, *Global existence of small amplitude solutions to nonlinear Klein-Gordon equations in four space-time dimensions.* Comm. Pure and Appl. Math., t.38, 1985, pp.631-641.

[13] S. Klainerman, *The null condition and global existence to nonlinear wave equations.* Lectures in Appl. Math., t.23, 1986, pp.293-326.

[14] G. Lochak, *Wave equation for a magnetic monopole.* Int. J. Theor. Phys. 24, n°10, 1985, pp.1019-1050.