Global existence of large amplitude solutions for Dirac-Klein-Gordon systems in Minkowski space

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INTRODUCTION

The purpose of this paper is to prove the existence of some global solutions, with *large* energy, of Dirac-Klein-Gordon systems with quadratic coupling and cubic autointeractions in Minkowski space. We know that some algebraic conditions on the nonlinearities, allow to solve the global Cauchy problem for classical fields with *small* initial data [2] [9] : the notion of compatibility of a product with a differential system, introduced by B. Hanouzet and J.L. Joly [5,6,7], and the null condition of S. Klainerman [9]. These both conditions are related to the Lorentz invariance.

In this work we show that the global Cauchy problem is wellposed again for arbitrarly large initial data if the non linearities and the data satisfy some algebraic properties ; more precisely we assume the system is Lorentz-invariant and the polarization of the Cauchy data is such that the *chiral invariant* is small.

Let's consider the mass Dirac-Klein-Gordon system in Minkowski space $\mathbb{R}^{3+1}$ with Lorentz metric $g_{\mu,\nu}=\mathrm{diag}(1,-1,-1,-1)$

$$-i\gamma^{\mu}\partial_{\mu}\psi + M\psi = f(\varphi,\psi), \tag{1}$$

$$\Box\varphi + m^2\varphi = g(\varphi,\psi) . \tag{2}$$

We suppose the masses are non null

$$M \neq 0 \ , \ m \neq 0 . \tag{3}$$

Now we introduce the Lorentz invariants

$$\bar{\psi}\psi = \tilde{\psi}\gamma^{o}\psi, \quad \tilde{\psi}=\text{transposate conjugate of } \psi,$$

$$\bar{\psi}\gamma^{5}\psi, \quad \gamma^{5}=-i\gamma^{o}\gamma^{1}\gamma^{2}\gamma^{3},$$

and we define the vector space $\mathcal{M}$ of 4×4 matrices

$$\mathcal{M} = \{\alpha I+i\beta\gamma^{5}, \ (\alpha,\beta)\in\mathbb{R}^{2}\} .$$

The hypotheses on the nonlinearities are following :

$$f(\varphi,\psi)=\varphi V\psi+F(\bar{\psi}\psi, i\bar{\psi}\gamma^{5}\psi)\psi \tag{4}$$

where V is a 4×4 matrix with constant coefficients and

$$V\in\mathcal{M}, \ F\in C^{\infty}(\mathbb{R}^{2},\mathcal{M}), \ |F(u,v)|=O(|u|+|v|), |u|+|v|\to 0 , \tag{5}$$

$$g(\varphi,\psi)=G(\bar{\psi}\psi, i\bar{\psi}\gamma^{5}\psi)-k\varphi^{3} \tag{6}$$

where k is a real constant and

$$G\in C^{\infty}(\mathbb{R}^{2},\mathbb{R}), |G(u,v)|=O(|u|+|v|), |u|+|v|\to 0 . \tag{7}$$

Obviously, to obtain large solutions, we must assume

$$k \geqslant 0 . \tag{8}$$

Many models of the relativistic fields theory satisfy these hypotheses : the scalar and pseudoscalar Yukawa models of the nuclear forces, the interactions of Heisenberg, Federbusch, the magnetic monopole of G. Lochak.

Now, we recall that J. Chadam and R.T. Glassey established in [4] the existence of global solutions to the scalar Yukawa model, for which the Dirac system and the Klein-Gordon equation are decoupled and $\bar{\psi}\psi\equiv 0$.

Here, we solve the global Cauchy problem for (1)-(8) in a neighborhood of such a decoupling solution. More precisely, we choose

$$\psi\Big|_{t=0} = \Psi_{o} + \varepsilon\chi_{o} , \quad 0<\varepsilon \tag{9}$$

$$\Psi_{o}, \ \chi_{o}\in\mathcal{D}(\mathbb{R}^{3}_{x},\mathbb{C}^{4}) \tag{10}$$

$$\varphi\Big|_{t=0} = \varphi_0, \quad \partial_t\varphi\Big|_{t=0} = \varphi_1 \tag{11}$$

$$\varphi_j \in \mathcal{D}(\mathbb{R}^3_x, \mathbb{R}) \quad . \tag{12}$$

The algebraic hypothese on the polarization of Ψ_0 is

$$\Psi_0 = z\gamma^2\Psi_0^+, z\in\mathbb{C}, |z|=1, \Psi_0^+=\text{conjugate of } \Psi_0 \tag{13}$$

where

$$\gamma^2 = \begin{bmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} .$$

(13) is Majorana's condition generalized by G. Lochak [11].
In the first part we prove condition (13) is time independent for the solution of a Dirac system with scalar or pseudoscalar time dependent potential.
In the second part we make energy estimates and uniform decay estimates in Sobolev spaces associated with Lorentz metric for the nonlinear Klein-Gordon equation

$$\Box\varphi + \varphi = -\varphi^3 \quad .$$

We solve global Cauchy problem (1)-(13) in part III ; we obtain asymptotically free solutions.

I - CHIRAL INVARIANT OF DIRAC FIELD.

We consider a solution ψ of the Dirac system

$$-i\gamma^\mu\partial_\mu\psi + M\psi = A\psi, \quad M\in\mathbb{R} \tag{14}$$

where the time dependent potential A satisfies

$$A, \ \partial_\mu A\in L^\infty(\mathbb{R}_t\times\mathbb{R}^3_x; \mathcal{M}) . \tag{15}$$

Following G. Lochak [11] we introduce the *chiral invariant* of ψ, $\rho(\psi)$

$$\rho^2 = |\overline{\psi}\psi|^2 + |\overline{\psi}\gamma^5\psi|^2 \quad . \tag{16}$$

We are concerned by the solution ψ for which the Chiral invariant is null.

PROPOSITION I.1 :

Let ψ be a solution of (14) and $\psi\in C^0(\mathbb{R}_t(L^2(\mathbb{R}^3_x))^4)$, $\psi\Big|_{t=0}=\psi_0\in(L^2(\mathbb{R}^3_x))^4$.

Then the following assertions are equivalent :

(i) $\psi_0=z\gamma^2\psi_0^+$, $z\in\mathbb{C}$, $|z|=1$, ψ_0^+ conjugate of ψ_0,

(ii) $\forall x\in\mathbb{R}^3$, $\rho(\psi_0(x)) = 0$,

(iii) $\forall (t,x)\in\mathbb{R}^{1+3}$, $\rho(\psi(t,x)) = 0$.

Proof : We use the bispinorial representation of Weyl by putting

$$\psi = 2^{-1/2}(\gamma^0+\gamma^5)\begin{pmatrix}\xi\\ \eta\end{pmatrix}, \quad \xi,\eta\in\mathbb{C}^2 \quad . \tag{17}$$

We verify easily that

$$\overline{\psi}\psi = \xi^+\eta+\eta^+\xi \quad , \quad \overline{\psi}\gamma^5\psi = \xi^+\eta-\eta^+\xi \quad .$$

Therefore $\rho=0$ if and only if

$$\xi^+\eta = 0 \quad .$$

This condition is equivalent to

$$\xi = z\sigma^2\eta^+ \quad , \quad z\in\mathbb{C} \quad , \quad |z|=1 \quad .$$

By using (17) we see that this equality means

$$\psi = z\gamma^2\psi^+$$

and we conclude that

$$\psi = z\gamma^2\psi^+ \iff \rho(\psi) = 0 \quad . \tag{18}$$

Now it is obvious that it is sufficient to prove (ii) $\iff$ (iii) for $\psi_0\in(H^1(\mathbb{R}^3_x))^4$ with compact support. Equation (14) can be written

$$\partial_0\psi+\sum_{j=1}^{3}\gamma^0\gamma^j\partial_j\psi+iM\gamma^0\psi=i\gamma^0A\psi \quad . \tag{19}$$

By multiplying (19) by $\tilde{\psi}$ we find

$$\partial^{o}|\psi|^{2} + \sum_{j=1}^{3} \partial_{j}(\tilde{\psi}\gamma^{o}\gamma^{j}\psi) = 0 .$$

We integrate (20) over $\mathbb{R}_x^3$ and we obtain the charge conservation

$$\int_{\mathbb{R}^3} |\psi(t,x)|^2 dx = cst . \tag{21}$$

Now we multiply (19) by ${}^t\psi\gamma^2$, ${}^t\psi$ is the transposate of ψ and it follows

$$\partial_{o}({}^t\psi\gamma^2\psi) + \sum_{j=1}^{3} \partial_{j}({}^t\psi\gamma^2\gamma^{o}\gamma^{j}\psi) = 0 .$$

We integrate over $\mathbb{R}_x^3$ again and we obtain the conservation law

$$\int_{\mathbb{R}^3} {}^t\psi(t,x)\gamma^2\psi(t,x)\,dx = cst . \tag{22}$$

Let z be a complex number of modulus one. We have

$$|\psi - z\gamma^2\psi^+|^2 = 2|\psi|^2 + 2\mathcal{R}e(\overline{z}\ {}^t\psi\gamma^2\psi) .$$

Then we have thanks to (21) and (22)

$$\int_{\mathbb{R}^3} |\psi(t,x) - z\gamma^2\psi^+(t,x)|^2 dx = cst . \tag{23}$$

By (18) and (23) we conclude that $\rho(\psi_o)$ is equivalent to $\rho(\psi)\equiv 0$.
Q.E.D.

II - ESTIMATES FOR THE NONLINEAR KLEIN-GORDON EQUATION.

We define the Sobolev norms associated with the Lorentz metric ; for any test function $u \in \mathcal{D}(\mathbb{R}_t \times \mathbb{R}_x^3)$ and any integer N, we put

$$\|u(t)\|_N^2 = \sum_{|\lambda| \leq N} \|\Gamma^\lambda u(t)\|^2_{L^2(\mathbb{R}^3_x)} \tag{24}$$

$$|u(t)|_N = \operatorname{Sup}_{|\lambda| \leq N} |\Gamma^\lambda u(t)|_{L^\infty(\mathbb{R}^3_x)} \tag{25}$$

where

$$\Gamma^{\lambda}=\Gamma_1^{\lambda_1}\ldots\Gamma_{10}^{\lambda_{10}} \quad , \quad \lambda\in\mathbb{N}^{10}$$

$$|\lambda|=\lambda_1+\ldots\lambda_{10} \quad ,$$

and $(\Gamma_\sigma)_{1\leqslant\sigma\leqslant 10}$ are the generators of Poincaré group

$$(\Gamma_\sigma)_{1\leqslant\sigma\leqslant 10} = \left(\partial_\mu=\frac{\partial}{\partial x^\mu} \ , \ x_\mu\partial_\nu-x_\nu\partial_\mu\right)_{0\leqslant\mu,\nu\leqslant 3} . \tag{26}$$

We will writte so

$$\partial_0=\partial_t \quad , \quad x^0=t \quad , \quad x=(x^1,x^2,x^3) .$$

In this part, our purpose is to estimate with these norms the solution u of the nonlinear Klein-Gordon equation

$$\Box u + u = -gu^3, \ 0\leqslant g \ , \tag{27}$$

$$u\Big|_{t=0}\in\mathcal{D}(\mathbb{R}^3_x,\mathbb{R}) \ , \partial_t u\Big|_{t=0}\in\mathcal{D}(\mathbb{R}^3_x,\mathbb{R}) . \tag{28}$$

PROPOSITION II.1 -

The solution u of (27) (28) satisfies for any integer N :

$$\sup_{t\in\mathbb{R}} \|u(t)\|_N < +\infty \tag{29}$$

$$\sup_{t\in\mathbb{R}} (1+|t|)^{3/2}|u(t)|_N < +\infty . \tag{30}$$

Proof : First, we prove by iteration on N the following assertion P_N :

$$(P_N) \quad \begin{cases} \text{there exists } d_N>1 \text{ such that} \\ \sup_{t\in\mathbb{R}} \left(\|u(t)\|_N+\|u'(t)\|_N+(1+|t|)^{d_N}|u(t)|_{N-1}\right)<\infty \end{cases}$$

where $u'=(\partial_\mu u)_{0\leqslant\mu\leqslant 3}$.

Recall the result of C. Morawetz and W. Strauss [12] :

$$\sup_{t\in\mathbb{R}} (1+|t|)^{3/2}|u(t)|_0 < +\infty . \tag{31}$$

Now we note Ω_N an element of order $\leqslant N$, of the Lie algebra spanned by the generators of Poincaré group

$$\Omega_N = \sum_{\text{finite}} C_\lambda \Gamma^\lambda , \; C_\lambda \in \mathbb{C}, \; \lambda \in \mathbb{N}^{10}, \; |\lambda| \leqslant N . \tag{32}$$

The Lorentz invariance of the Klein-Gordon equation implies

$$\square\Omega_1 u+\Omega_1 u = -g\Omega_1(u^3) .$$

It follows

$$\|u(t)\|_1+\|u'(t)\|_1 \leqslant C(1+\int_0^t \|u(s)\|_1 |u(s)|_0^2 ds)$$

and by using Gronwall's lemma

$$\sup_{t\in\mathbb{R}} (\|u(t)\|_1+\|u'(t)\|_1) \leqslant C \exp(C\int |u(s)|_0^2 ds) . \tag{33}$$

We conclude by (31) and (33) that (P_1) is verified. Now, assume (P_N) is proved ; we have again

$$\square\Omega_{N+1} u+\Omega_{N+1} u = -g\Omega_{N+1}(u^3)$$

and thus

$$\|\Omega_{N+1} u(t)\|_0 + \|\Omega_{N+1} u'(t)\|_0 \leqslant C(1+\int_0^t \|\Omega_{N+1}(u^3)(s)\|_0 \, ds) .$$

We note

$$\Omega_{N+1}(u^3) = \sum_{\text{finite}} \{ (\Omega_{N+1} u)u^2 + (\Omega_N u)(\Omega_1 u)u + \prod_{j=1}^{3} (\Omega_{p_j} u) \}$$

where

$$p_j \leqslant N-1 \; , \; \sum_1^3 p_j \leqslant N+1$$

and then

$$\|u^3(s)\|_{N+1}\leqslant C(\|u(s)\|_{N+1}|u(s)|_0^2+\|u(s)\|_{N-1}|u(s)|_{N-1}^2$$

$$+\sum_{\text{finite}} \|\Omega_N u(s)\|_{L^4(\mathbb{R}^3_x)}\cdot\|\Omega_1 u(s)\|_{L^4(\mathbb{R}^3_x)}\cdot|u(s)|_0).$$

Thanks to the Sobolev injection $H^1(\mathbb{R}^3_x)\subset L^4(\mathbb{R}^3_x)$, (P_1) and (P_N) we find.

$$\|u(t)\|_{N+1}+\|u'(t)\|_{N+1}\leqslant C(1+|\int_0^t \|u(s)\|_{N+1}(1+|s|)^{-d_N}ds|)$$

and Gronwall's lemma gives

$$\sup_{t\in\mathbb{R}}(\|u(t)\|_{N+1}+\|u'(t)\|_{N+1})<+\infty \quad . \tag{34}$$

Now, we recall that the solution v of

$$\Box v+v = 0,\ v\Big|_{t=0} = 0\ ,\ \partial_0 v\Big|_{t=0}(x)=g(x),$$

verifies for $0\leqslant\alpha\leqslant1$

$$|v(t)|_0\leqslant C|t|^{-\alpha-(1-\alpha)(3/2)}\|g\|^{\alpha}_{W^{1,1}(\mathbb{R}^3_x)}\|g\|^{1-\alpha}_{W^{2,1}(\mathbb{R}^3_x)},$$

$$|v(t)|_0\leqslant C\|g\|_{W^{1,2}(\mathbb{R}^3_x)},$$

where

$$\|g\|_{W^{n,p}(\mathbb{R}^3_x)}=\sum_{|\alpha|\leqslant n}\|\partial_x^\alpha g\|_{L^p(\mathbb{R}^3_x)} .$$

It follows that

$$|\Omega_N u(t)|_0\leqslant C\Big[(1+|t|)^{-3/2}+|\int_0^t(1+|t-s|)^{-\alpha-(1-\alpha)(3/2)} \tag{35}$$

$$(\|\Omega_N u^3(s)\|_{W^{1,2}(\mathbb{R}^3_x)}+\|\Omega_N u^3(s)\|^{\alpha}_{W^{1,1}(\mathbb{R}^3_x)}\cdot\|\Omega_N u^3(s)\|^{1-\alpha}_{W^{2,1}(\mathbb{R}^3_x)}ds|\Big]$$

We write again

$$\Omega_N(u^3)=\sum_{\text{finite}}{}'\ \{(\Omega_N u)u^2+(\Omega_{N-1}u)(\Omega_1 u)u+\prod_{j=1}^{3}(\Omega_{p_j}u)\}$$

where

$p_j\leqslant N-2$.

Then hypothese (P_N) and (34) imply

$$\|\Omega_N(u^3)(s)\|_{W^{1,2}(\mathbb{R}^3_x)\cap W^{1,1}(\mathbb{R}^3_x)} \leqslant C(1+|s|)^{-d_N} \tag{36}$$

and thanks to (34)

$$\sup_{t\in\mathbb{R}} \|\Omega_N(u^3)(s)\|_{W^{2,1}(\mathbb{R}^3_x)} < +\infty \ . \tag{37}$$

The inequalities (35) (36) (37) yield

$$|u(t)|_N \leqslant C((1+|t|)^{-3/2} + |\int_0^t (1+|t-s|)^{-\alpha-(1-\alpha)3/2}(1+|s|)^{-d_N} ds|) .$$

We choose

$$\alpha = 3(2d_N+1)^{-1} \in]d_N^{-1}, 1[\ ,$$

$$d_{N+1} = \alpha + (1-\alpha)(3/2) = \alpha d_N > 1 \ .$$

Thus

$$|u(t)|_N \leqslant C(1+|t|)^{-d_{N+1}}$$

this ends the proof of (P_{N+1}). To obtain the uniform decay of $t^{-3/2}$ we apply the L^2-L^∞ estimate for the Klein-Gordon equation [2] :

$$|u(t)|_N \leqslant C(1+|t|)^{-3/2}(1+\int_{\mathbb{R}} \|u^3(s)\|_{N+4} ds)$$

(P_{N+5}) implies

$$\|u^3(s)\|_{N+4} \leqslant C(1+|s|)^{-2d_{N+5}} \in L^1(\mathbb{R}_s)$$

and we conclude that

$$\sup_{t\in\mathbb{R}} (1+|t|)^{3/2}|u(t)|_N < +\infty \quad \text{Q.E.D}$$

III - GLOBAL EXISTENCE OF LARGE AMPLITUDE SOLUTIONS.

MAIN THEOREM

There exists $\varepsilon_0>0$ depending only on the derivatives of initial data Ψ_0, χ_0, φ_0,φ_1 of order ≤ 10, such that for any $0\leq\varepsilon\leq\varepsilon_0$, the Cauchy problem (1) to (13) has a unique solution (ψ,φ) in $C^\infty(\mathbb{R}^4)$. Moreover, this solution is asymptotically free : there exists $\psi^\pm$, $\varphi^\pm$ satisfying :

$$\psi^\pm\in\bigcap_k C^k(\mathbb{R}_t,H^{10-k}(\mathbb{R}^3_x)),\ -i\gamma^\mu\partial_\mu\psi^\pm+M\psi^\pm=0\ ,$$

$$\varphi^\pm\in\bigcap_k C^k(\mathbb{R}_t,H^{11-k}(\mathbb{R}^3_x)),\ \Box\varphi^\pm+m^2\varphi^\pm=0\ ,$$

$$\forall k\in\mathbb{N},\ \lim_{t\to\pm\infty}\|\partial_t^k\psi(t)-\partial_t^k\psi^\pm(t)\|_{H^{10-k}}+\|\partial_t^k\varphi(t)-\partial_t^k\varphi^\pm(t)\|_{H^{11-k}}=0$$

where H^s is the Sobolev space $W^{s,2}(\mathbb{R}^3_x)$.

Proof : Let (Ψ,Φ) be the solution of

$$-i\gamma^\mu\partial_\mu\Psi + M\Psi = \Phi V\Psi\ , \tag{38}$$

$$\Box\Phi + m^2\Phi = -k\Phi^3\ , \tag{39}$$

$$\Psi\Big|_{t=0} = \Psi_0\ , \tag{40}$$

$$\Phi\Big|_{t=0} = \varphi_0,\ \partial_t\Phi\Big|_{t=0}=\varphi_1\ . \tag{41}$$

Lochak-Majorana's condition (13) and Proposition I.1 imply

$$\bar\Psi\Psi = \bar\Psi\gamma^5\Psi \equiv 0\ . \tag{42}$$

Now we put

$$\psi = \Psi+\chi\ ,\ \varphi = \Phi+u \tag{43}$$

and to solve the Cauchy problem for (ψ,φ) we study the problem

$$-i\gamma^\mu\partial_\mu\chi+M\chi = \tilde f(\chi,u;\Psi,\Phi) \tag{44}$$

$$\Box u+m^2u = \tilde g(\chi,u;\Psi,\Phi) \tag{45}$$

$$\chi\Big|_{t=0}=\varepsilon\chi_0\ ,\ u\Big|_{t=0}=0\ ,\ \partial_t u\Big|_{t=0}=0 \tag{46}$$

where $\tilde{f}$ and $\tilde{g}$ are C^∞ functions of its variables and verify

$$|h(\chi,u;\Psi,\Phi)|=O((|\chi|+|u|)(|\chi|+|u|+|\Psi|+|\Phi|)(1+|\chi|+|u|+|\Psi|+|\Phi|))$$

as

$$|\chi|+|u| \to 0, \; h = \tilde{f},\tilde{g} \; . \tag{47}$$

As usual we define the sequence $(\chi^\nu,u^\nu)_{\nu\geq 0}$ by putting

$$\chi^0 \equiv 0, \; u^0 \equiv 0 \; , \tag{48}$$

and for $\nu\geq 1$

$$-i\gamma^\mu\partial_\mu\chi^\nu+M\chi^\nu=\tilde{f}(\chi^{\nu-1},u^{\nu-1};\Psi,\Phi) , \tag{49}$$

$$\Box u^\nu+m^2u^\nu=\tilde{g}(\chi^{\nu-1},u^{\nu-1};\Psi,\Phi) , \tag{50}$$

$$\chi^\nu\Big|_{t=0}=\varepsilon\chi_0 \; , \; u^\nu\Big|_{t=0}=\partial_t u^\nu\Big|_{t=0}= 0 \; . \tag{51}$$

To estimate the norms $\|\chi^\nu(t)\|_N$ we replace in (24) (25) the operators $(\Gamma_\sigma)_{1\leq\sigma\leq 10}$ by the Fermi operators

$$(\tilde{\Gamma}_\sigma)_{1\leq\sigma\leq 10} = (\partial_\mu , x_\mu\partial_\nu-x_\nu\partial_\mu+\tfrac{1}{2}\gamma_\mu\gamma_\nu)_{0\leq\mu,\nu\leq 3}$$

which define obviously equivalent norms, and commute with the Dirac system. The commutation relations for $\tilde{\Gamma}_\sigma$ the charge conservation for the Dirac system and the usual energy equality for the Klein-Gordon equation imply

$$\begin{aligned}\|\chi^\nu(t)\|_N+\|u^\nu(t)\|_N+\|(u^\nu)'(t)\|_N \leq \\ \leq C\Big[\varepsilon+\Big|\int_0^t (\|\chi^{\nu-1}(s)\|_N+\|u^{\nu-1}(s)\|_N) \\ \times(|\chi^{\nu-1}(s)|_{[N/2]}+|u^{\nu-1}(s)|_{[N/2]}+|\Psi(s)|_N+|\Phi(s)|_N) \\ \times(1+|\chi^{\nu-1}(s)|_{[N/2]}+|u^{\nu-1}(s)|_{[N/2]}+|\Psi(s)|_N+|\Phi(s)|_N)ds\Big|\Big]\end{aligned}$$

where

$$(u^\nu)' = (\partial_\mu u^\nu)_{0\leq\mu\leq 3} \; .$$

Following Proposition II.1, we have

$$|\Psi(s)|_N+|\Phi(s)|_N \leq C_N(1+|s|)^{-3/2} \; .$$

We deduce that

$$\|\chi^\nu(t)\|_N+\|u^\nu(t)\|_N+\|(u^\nu(t))'\|_N \leqslant$$
$$\leqslant\Big[\varepsilon+\Big|\int_0^t (\|\chi^{\nu-1}(s)\|_N+\|u^{\nu-1}(s)\|_N)$$
$$\times(|\chi^{\nu-1}(s)|_{[N/2]}+|u^{\nu-1}(s)|_{[N/2]}+(1+|s|)^{-3/2})\tag{52}$$
$$\times(1+|\chi^{\nu-1}(s)|_{[N/2]}+|u^{\nu-1}(s)|_{[N/2]})ds\Big|\Big] .$$

We define for $n\in\mathbb{N}$, $t\in\mathbb{R}$

$$a_n(t)=\sup_{\substack{|s|\leqslant|t|\\ 0\leqslant\nu\leqslant n}} (\|\chi^\nu(s)\|_N+\|u^\nu(s)\|_N+\|(u^\nu(s))'\|_N)$$

$$b_n(t)=\sup_{\substack{|s|\leqslant|t|\\ 0\leqslant\nu\leqslant n}} ((1+|s|)^{3/2}(|\chi^\nu(s)|_{[N/2]}+|u^\nu(s)|_{[N/2]})) .$$

Inequality (52) can be written

$$a_n(t)\leqslant C(\varepsilon+\Big|\int_0^t a_{n-1}(s)(1+b_{n-1}(s))^2(1+|s|)^{-3/2}ds\Big|) . \tag{53}$$

At present our L^2-L^∞ estimate for Klein-Gordon equation [2] gives

$$|\chi^\nu(t)|_{[N/2]}+|u^\nu(t)|_{[N/2]}\leqslant C(1+|t|)^{-3/2}$$
$$(\varepsilon+\Big|\int_0^{2t} (\|\chi^{\nu-1}(s)\|_{[N/2]+4}+\|u^{\nu-1}(s)\|_{[N/2]+4})$$
$$\times(|\chi^{\nu-1}(s)|_{[([N/2]+4)/2]}+|u^{\nu-1}(s)|_{[([N/2]+4)/2]}$$
$$+|\Psi(s)|_{[N/2]+4} + |\Phi(s)|_{[N/2]+4})$$
$$\times(1+|\chi^{\nu-1}(s)|_{[([N/2]+4)/2]}+|u^{\nu-1}(s)|_{[([N/2]+4)/2]} +$$
$$+|\Psi(s)|_{[N/2]+4}+|\Phi(s)|_{[N/2]+4})ds\Big|) .$$

We choose N such that

$$\left[\frac{N}{2}\right]+4 \leqslant N,\quad \left[\frac{[N/2]+4}{2}\right] \leqslant \left[\frac{N}{2}\right]$$

i.e.

$$10 \leqslant N . \tag{54}$$

Then we have

$$b_n(t) \leqslant C(\varepsilon + |\int_0^{2t} a_{n-1}(s)(1+b_{n-1}(s))^2(1+|s|)^{-3/2}ds|). \tag{55}$$

Relations (53) and (55) show that if a_{n-1} and b_{n-1} are in $L^\infty_{loc}(\mathbb{R})$, then a_n and b_n are in $L^\infty_{loc}(\mathbb{R})$. Now $a_0=b_0\equiv 0$, then

$$\forall n\in\mathbb{N},\ a_n, b_n \in L^\infty_{loc}(\mathbb{R}). \tag{56}$$

We can apply the Gronwall lemma to (53) by noting $a_n(t)$ and $b_n(t)$ are creasing functions of n and t

$$a_n(t) \leqslant C\varepsilon\ \exp\{C(1+b_{n-1}(t))^2\} \tag{57}$$

where C is independing on n .
Let A_n, B_n be

$$A_n = \sup_{t\in\mathbb{R}} a_n(t)\quad ,\ B_n = \sup_{t\in\mathbb{R}} b_n(t)\quad .$$

(55) and (57) imply

$$A_n \leqslant C\varepsilon\ \exp\{C(1+B_{n-1})^2\} \tag{58}$$

$$B_n \leqslant C(\varepsilon + A_{n-1}(1+B_{n-1})^2) \tag{59}$$

and we have

$$A_0 = B_0 = 0\ . \tag{60}$$

We choose $0<\varepsilon_0$ such that

$$C\varepsilon(1+4C\ \exp\ 4C) \leqslant 1\ . \tag{61}$$

Suppose

$$0\leqslant\varepsilon\leqslant\varepsilon_0\ ,\ A_{n-1}\leqslant C\varepsilon\exp 4C\ ,\ B_{n-1}\leqslant 1\ , \tag{62}$$

(58) and (62) imply

$$A_n \leqslant C\varepsilon\ \exp\ 4C\ , \tag{63}$$

and (59) and (62) imply

$$B_n \leqslant C(\varepsilon + 4C\varepsilon\ \exp\ 4C),$$

and thanks to (61)

$$B_n \leqslant 1 \ . \tag{64}$$

We conclude by (60) (62) (63) (64) that

$$\operatorname*{Sup}_n (A_n + B_n) < +\infty \ . \tag{65}$$

Now, the existence of global solution follows from classical method (see e.g.[2]). At present we prove the asymptotic freedom. We note respectively D(t) and U(t) the propagators associated to the free equations of Dirac and Klein-Gordon

$$D(t) = \exp it\,\mathscr{A} \ , \ \mathscr{A} = \sum_{j=1}^{3} i\gamma^0\gamma^j\partial_j - M\gamma^0$$

$$U(t) = \exp it\,A \ , \ A = -i\begin{pmatrix} 0 & , & 1 \\ \Delta_x - m^2 , & & 0 \end{pmatrix}.$$

To obtain $\psi^{\pm}$, $\varphi^{\pm}$ it is sufficient to prove the convergence of $D(-t)\psi(t)$ and $U(-t)(\varphi(t),\partial_t\varphi(t))$ respectively in $(H^{10}(\mathbb{R}^3_x))^4$ and $H^{11}(\mathbb{R}^3_x)\times H^{10}(\mathbb{R}^3_x)$ as $t \to \pm\infty$.

We have

$$D(-t)\psi(t) = \psi\Big|_{t=0} + \int_0^t D(-s)f(\varphi(s),\psi(s))\,ds$$

$$U(-t)(\varphi(t),\partial_t\varphi(t)) = (\varphi,\partial_t\varphi)\Big|_{t=0} + \int_0^t U(-s)(0,g(\varphi(s),\psi(s))\,ds.$$

The propagators D(t) and U(t) being uniformly bounded on the Sobolev spaces we have to prove only

$$\|f(\varphi(s),\psi(s))\|_{(H^{10}(\mathbb{R}^3_x))^4} \in L^1(\mathbb{R}_s) \tag{66}$$

$$\|g(\varphi(s),\psi(s))\|_{H^{10}(\mathbb{R}^3_x)} \in L^1(\mathbb{R}_s) \ . \tag{67}$$

We deduce from (65) that

$$\operatorname*{Sup}_t \{\|\psi(t)\|_{10} + \|\varphi(t)\|_{10} + (1+|t|)^{3/2}(|\psi(t)|_5 + |\varphi(t)|_5)\} < +\infty \ .$$

We conclude that the norms in (66) and (67) are $O((1+|t|)^{-3/2})$ and this ends the proof.

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