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Superradiance and scattering of the charged Klein–Gordon field by a step-like electrostatic potential

Alain Bachelot

Université Bordeaux-1, institut de mathématiques, UMR CNRS 5466, 33405 Talence Cedex, France

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Abstract

We develop the scattering theory for the charged Klein–Gordon equation on $\mathbb{R}_t \times \mathbb{R}_x$, when the electrostatic potential $A(x)$ has different asymptotics $a^{\pm}$ as $x \to \pm\infty$. In this case, the conserved energy is not positive definite (Klein paradox). We construct the spectral representation for the harmonic equation. Since $a^+ \neq a^-$, the distorted Fourier transform has to be defined on weighted L^2-spaces, and spectral quantities of a new type can appear, that are neither eigenvalues, nor resonances. These so called "hyperradiant modes" are real singularities of the Green function, and lead to solutions polynomially increasing in time. We investigate the asymptotic behaviours of the solutions as $t \to \pm\infty$, and we establish the existence of a Scattering operator the symbol of which has a norm strictly larger than 1, for the frequencies in (a^-, a^+). We apply these results to the DeSitter–Reissner–Nordstrøm metric, to justify rigorously the notion of superradiance of charged black-holes.

Résumé

Nous développons la théorie de la diffusion pour l'équation de Klein–Gordon chargée sur $\mathbb{R}_t \times \mathbb{R}_x$ en présence d'un potentiel électrostatique $A(x)$ admettant des limites distinctes $a^{\pm}$ quand $x \to \pm\infty$. Dans ce cas, l'énergie conservée n'est pas définie positive (paradoxe de Klein). Nous construisons la représentation spectrale associée à l'équation harmonique. Comme $a^+ \neq a^-$, la transformée de Fourier distordue doit être définie sur des espaces L^2 à poids, et il peut apparaître des quantités spectrales d'un type nouveau dans l'intervalle (a^-, a^+), qui ne sont ni des valeurs propres, ni des résonances. Ces modes "hyperradiants" sont des singularités réelles de la fonction de Green et produisent des champs polynomialement croissants en temps. Nous étudions les comportements asymptotiques des solutions quand $t \to \pm\infty$, et établissons l'existence d'un opérateur de diffusion

E-mail address: bachelot@math.u-bordeaux.fr (A. Bachelot).

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dont la norme du symbole est strictement supérieure à 1 pour les fréquences dans (a^-, a^+). Nous appliquons ces résultats à la métrique de DeSitter–Reissner–Nordstrøm, pour justifier rigoureusement la notion de superradiance des trous noirs chargés.

Keywords: Superradiance; Scattering operator; Klein–Gordon equation; Klein paradox

1. Introduction

An abstract Klein–Gordon equation with gyroscopic term has the form:

$$(\partial_t - \mathrm{i}A)^2 u + H^2 u = 0, \quad t \in \mathbb{R}, \tag{1.1}$$

where A and H are selfadjoint operators on a Hilbert space $(X, \|\cdot\|)$, A bounded, $H \geqslant \delta > 0$ with dense domain $D(H)$. The standard spectral theory assures that the Cauchy problem is well posed: given initial data $u_0 \in D(H)$, $u_1 \in X$, Eq. (1.1) has a unique solution $u \in C^1(\mathbb{R}_t; X) \cap C^0(\mathbb{R}_t; D(H))$ with initial data $u(0) = u_0$, $\partial_t u(0) = u_1$, and there exists a conserved energy:

$$E(u,t) := \left\|\partial_t u(t)\right\|^2 + \left\|Hu(t)\right\|^2 - \left\|Au(t)\right\|^2 = Cst. \tag{1.2}$$

When A is small with respect to H, i.e., $\|A\| < \delta$, this functional is positive, hence we have the uniform bound

$$\sup_{t\in\mathbb{R}} \left(\left\|\partial_t u(t)\right\| + \left\|Hu(t)\right\|\right) < \infty, \tag{1.3}$$

and numerous authors have developed the scattering theory when A and H are short range perturbations of some operators A_0, H_0 (see, e.g., K.J. Eckardt [9], L.E. Lundberg [21], B. Najman [27], M. Schechter [33], Veseliĉ [35], R. Weder [36]).

When A is not small, this energy can be indefinite, and we have only:

$$\left\|\partial_t u(t)\right\| + \left\|Hu(t)\right\| \leqslant c\mathrm{e}^{\alpha|t|}\left(\left\|\partial_t u(0)\right\| + \left\|Hu(0)\right\|\right), \tag{1.4}$$

where $c > 0$ and $\alpha > 0$ do not depend on u. In this case, the possibility of the existence of modes occurs. These modes are solutions of (1.1) of type $u(t) = \mathrm{e}^{\mathrm{i}kt}v$, $v \in D(H) \setminus \{0\}$, $k \in \mathbb{C} \setminus \mathbb{R}$, that obviously satisfy:

$$\left\|\partial_t u(t)\right\| + \left\|Hu(t)\right\| = C\mathrm{e}^{\kappa t}, \quad 0 < C,\ \kappa \in \mathbb{R}^*. \tag{1.5}$$

This difficulty can be overcome in the situations where A is a compact operator from $D(H)$ to X, because in this case, for any $\varepsilon > 0$, there exists a subspace $X_\varepsilon \subset D(H)$, with finite codimension, such that $\|Au\| \leqslant \varepsilon\|Hu\|$ for any $u \in X_\varepsilon$. As consequence, the set of modes is finite-dimensional. To our knowledge the unique paper solving completely the scattering problem is due to T. Kako [18] who investigated the Klein–Gordon equation in $\mathbb{R}_x^n$ with a short range electrostatic field $A(x)$: he established the existence and completeness of wave operators, isometric on the energy space of the free Klein–Gordon equation. The general case is open, in particular the case of the step-like potential A, and we know only that these operators are locally definitizable for a large class of perturbations (P. Jonas [17]).

In this paper, we deal with the very simple equation:

$$\big(\partial_t - \mathrm{i}A(x)\big)^2 u - \partial_x^2 u + V(x)u = 0, \quad t \in \mathbb{R},\ x \in \mathbb{R}, \tag{1.6}$$

with the main hypotheses:

$$A(x) \to a_\pm, \quad x \to \pm\infty,\ a_- \neq a_+, \tag{1.7}$$

$$V(x) \to 0, \quad |x| \to \infty. \tag{1.8}$$

Fundamental examples of such an equation arise in General Relativity for the propagation of waves on space-times of Black-Hole type with an electrostatic charge. Other one-dimensional field equations with step-like perturbations have been studied: the existence of a Scattering Operator that is unitary, was established for the Dirac system by S.N.M. Ruijsenaars and P.J.M. Bongaarts [32], and for the Schrödinger equation by E.B. Davies and B. Simon [7]. The key point for both these equations is the conservation of the L^2 norm. The situation drastically differs for the Klein–Gordon equation (1.6) since the conserved energy

$$E(u,t) := \int \big|\partial_t u(t,x)\big|^2 + \big|\partial_x u(t,x)\big|^2 + \big[V(x) - A^2(x)\big]\big|u(t,x)\big|^2 \,\mathrm{d}x \tag{1.9}$$

is not always positive. In particular, when A satisfies the step-like hypothesis (1.7), the set of modes is finite-dimensional, but there exists no finite codimensional subspace of Cauchy data, on which this energy is positive. This is the root of the so called *Klein paradox*. Nevertheless we shall be able to describe the asymptotic behaviours of the solutions of (1.6), and to prove the existence of a Scattering Operator the norm of which is always *strictly larger* than one: this is the *superradiance*. Furthermore, in some situations, there exist solutions polynomially increasing in time (*hyperradiant* modes):

$$\big|u(t,x)\big| \sim t^n, \quad t \to +\infty,\ n \geqslant 1. \tag{1.10}$$

Recently, the γ-bursts have been attributed to the superradiance of the charged black-holes (R. Ruffini [31]).

We sketch the plan of this paper. In part two we develop the spectral analysis of the time-harmonic Klein–Gordon equation:

$$u'' + \big[k - A(x)\big]^2 u - V(x)u = 0, \quad x \in \mathbb{R}, \tag{1.11}$$

where $k \in \mathbb{C}$ is the spectral parameter. The case of the Schrödinger equation with step-like potentials has been studied by A. Cohen and T. Kappeler [6] (see also F. Gesztesy et al. [10]), and for the quadratic pencil with short range potentials by A. Krall, E. Bairamov, Ö. Çakar [19], and F.G. Maksudov, G.Sh. Guseinov [22] (see also [5,15,16]). We construct the spectral representation for (1.11) with assumptions (1.7), (1.8). In Proposition 2.9, the distorted Fourier transform is defined on the weighted space $L^2(\mathbb{R}_k, (1+k^2)^s\,\mathrm{d}k)$, and because of the critical frequencies $a^{\pm}$, it is necessary to take $s > 1/2$. Moreover, due to the hyperradiant modes, the usual spectral measure on $\mathbb{R}_k$ is replaced by a distribution, that is singular at the hyperradiant modes that belong to (a_-, a_+). The main result of this harmonic analysis is the resolution of the identity stated in Theorem 2.12. We investigate the asymptotic behaviours in time of the solutions of the hyperbolic equation (1.6) in the third section. We construct in Theorem 3.7 the solutions polynomially increasing in time (1.10), associated with the hyperradiant modes. Taking advantage of the previous spectral representation, we establish in Theorem 3.11 the existence of the wave operators, without using the conservation of the non positive energy (1.9). When there exists neither usual mode, nor hyperradiant mode, we develop a complete scattering theory in Theorem 3.15. We establish the existence of the scattering operator, which is superradiant for the frequencies $\kappa \in (a_-, a_+)$. We apply these results to the field theory in General Relativity in the fourth section. We consider the propagation of charged scalar fields in spherically symmetric space-times of black-hole type, in particular the De Sitter–Reissner–Nordstrøm universe. Our analysis explains the phenomenon of superradiance of charged black-holes [11,32].

2. Spectral decomposition

In this section we investigate the spectral properties of the harmonic Klein–Gordon equation on the whole line:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}y + \left[k - A(x)\right]^2 y - V(x)y = 0, \quad x \in \mathbb{R}. \tag{2.1}$$

We assume that the potentials are real and bounded, V is short range and A is step-like, rapidly tending to a as $x \to -\infty$, and to 0 as $x \to +\infty$. More precisely:

$$A \in L^\infty(\mathbb{R};\mathbb{R}), \qquad V \in L^\infty(\mathbb{R};\mathbb{R}), \tag{2.2}$$

and there exist $\alpha > 0$, $a \in \mathbb{R} \setminus \{0\}$ such that:

$$[[A, V]] := \int_{-\infty}^{\infty} \left(\left|A(x) - a\mathbf{1}_{]-\infty,0]}(x)\right| + \left|V(x)\right|\right)\mathrm{e}^{\alpha|x|}\,\mathrm{d}x < \infty. \tag{2.3}$$

Moreover we want to be able to consider non smooth potentials, hence we only suppose that the distribution derivative of A is a short range measure in the following sense:

$$[[A']] := \sup_{0<|h|<1} \int_{-\infty}^{\infty} \left|\frac{A(x+h) - A(x)}{h}\right|\mathrm{e}^{\alpha|x|}\,\mathrm{d}x < \infty. \tag{2.4}$$

We start in the usual way by constructing suitable Jost functions, taking the different asymptotics as $x \to \pm\infty$, into account.

Proposition 2.1. *For any $k \in \mathbb{C}$, $\Im k > -\alpha/2$ (respectively $\Im k < \alpha/2$), there exists unique functions $f_{\text{in}}^{\pm}(k;x)$ (respectively $f_{\text{out}}^{\pm}(k;x)) \in C^1(\mathbb{R}_x)$, solutions of (2.1) and satisfying*:

$$f_{\text{in(out)}}^{+}(k;x) = \mathrm{e}^{+(-)\mathrm{i}kx+(-)\mathrm{i}\int_x^{\infty} A(y)\,\mathrm{d}y}\big(1+\varepsilon_{\text{in(out)}}^{+}(k;x)\big), \tag{2.5}$$

$$f_{\text{in(out)}}^{-}(k;x) = \mathrm{e}^{-(+)\mathrm{i}(k-a)x+(-)\mathrm{i}\int_{-\infty}^{x}[A(y)-a]\,\mathrm{d}y}\big(1+\varepsilon_{\text{in(out)}}^{-}(k;x)\big), \tag{2.6}$$

where $\varepsilon_{\text{in(out)}}^{\pm} \in C^0(\mathbb{R}_x)$ and $\partial_x\varepsilon_{\text{in(out)}}^{\pm} \in L_{\text{loc}}^{\infty}(\mathbb{R}_x)$ satisfy:

$$\big|\varepsilon_{\text{in(out)}}^{\pm}(k;x)\big| + \big\|\partial_x\varepsilon_{\text{in(out)}}^{\pm}(k;y)\big\|_{L^\infty(\pm y \geqslant \pm x)} \to 0, \quad x \to \pm\infty, \tag{2.7}$$

and we have:

$$f_{\text{out}}^{\pm}(k;x) = \overline{f_{\text{in}}^{\pm}(\bar{k};x)}, \tag{2.8}$$

$$f_{\text{in}}^{+}(0;x) = f_{\text{out}}^{+}(0;x), \qquad f_{\text{in}}^{-}(a;x) = f_{\text{out}}^{-}(a;x). \tag{2.9}$$

Moreover, for each $x \in \mathbb{R}$, $\varepsilon_{\text{in}}^{\pm}$ and $\partial_x\varepsilon_{\text{in}}^{\pm}$ (respectively $\varepsilon_{\text{out}}^{\pm}$ and $\partial_x\varepsilon_{\text{out}}^{\pm}$), are analytic functions of $k \in \mathbb{C}$, $\Im k > -\alpha/2$ (respectively $\Im k < \alpha/2$) that satisfy for any $R \in \mathbb{R}$, $\beta < \alpha$, $n \in \mathbb{N}$, $p = 0, 1$:

$$R \leqslant x, \quad 0 \leqslant +(-)\Im k \quad \Rightarrow \quad \big|\partial_k^n\partial_x^p\varepsilon_{\text{in(out)}}^{+}(k;x)\big| \leqslant C_{R,n,\beta}\big(1+|k|\big)^{p-1}\mathrm{e}^{-\beta x}, \tag{2.10}$$

$$x \leqslant R, \quad 0 \leqslant +(-)\Im k \quad \Rightarrow \quad \big|\partial_k^n\partial_x^p\varepsilon_{\text{in(out)}}^{-}(k;x)\big| \leqslant C_{R,n,\beta}\big(1+|k|\big)^{p-1}\mathrm{e}^{\beta x}, \tag{2.11}$$

where $C_{R,n,\beta} > 0$ depends only of R, n, β, $[[A,V]]$ and $[[A']]$.

Proof. To prove the uniqueness of the C^1 solution of (2.1) satisfying (2.5), we use the fact that the Wronskian of two C^1 solutions:

$$[f_1, f_2](k) := f_1'(k;x)f_2(k;x) - f_1(k;x)f_2'(k;x) \in C^0(\mathbb{R}_x), \tag{2.12}$$

does not depend on x since its derivative in the sense of the distributions is zero, and by (2.5), this Wronskian tends to zero as $x \to \infty$. Here f' denotes the derivative of f with respect to x.

At first we assume that $A \in C^1(\mathbb{R})$. To get f_{in}^{+}, we introduce:

$$m(k;x) := f_{\text{in}}^{+}(k;x)\mathrm{e}^{-\mathrm{i}kx-\mathrm{i}\int_x^{\infty} A(y)\,\mathrm{d}y},$$

that is solution of

$$\partial_x^2 m + 2\mathrm{i}k\partial_x m = (V - \mathrm{i}A')m + 2\mathrm{i}A\partial_x m.$$

Therefore it is sufficient to solve the integral equation:

$$m(k;x)=1+\int_x^\infty D_k(y-x)\big[\big(V(y)-\mathrm{i}A'(y)\big)m(k;y)+2\mathrm{i}A(y)\partial_x m(k;y)\big]\,\mathrm{d}y, \quad (2.13)$$

where $D_k(y):=\mathrm{e}^{\mathrm{i}ky}\frac{\sin(ky)}{k}$ for $k\in\mathbb{C}^*$, and $D_0(y)=y$. We also consider the equation satisfied by $\partial_x m$:

$$\partial_x m(k;x)=-\int_x^\infty \partial_y D_k(y-x)\big[\big(V(y)-\mathrm{i}A'(y)\big)m(k;y)+2\mathrm{i}A(y)\partial_x m(k;y)\big]\,\mathrm{d}y. \quad (2.14)$$

To estimate the kernel $D_k(y)$ for small k, we write:

$$\partial_k^n D_k(y)=(2\mathrm{i})^n y^{n+1}\int_0^1 t^n\mathrm{e}^{-2\mathrm{i}kyt}\,\mathrm{d}t,$$

hence we get:

$$\big|\partial_k^n D_k(y)\big|\leqslant 2^n|y|^{n+1}\big(1+\mathrm{e}^{-2y\Im k}\big). \quad (2.15)$$

On the other hand we have:

$$\partial_k^n D_k(y)=\frac{P_n(ky)\mathrm{e}^{2\mathrm{i}ky}-(-1)^n n!}{2\mathrm{i}k^{n+1}},$$

where P_n is a polynomial of degree n. Hence we get:

$$\big|\partial_k^n D_k(y)\big|\leqslant \frac{c_n}{|k|^{n+1}}\big(1+\mathrm{e}^{-2y\Im k}\big)\big(1+|ky|\big)^n. \quad (2.16)$$

We deduce from (2.15) and (2.16) that for any $k\in\mathbb{C}$, $y\in\mathbb{R}$:

$$\big|\partial_k^n D_k(y)\big|\leqslant \frac{C_n}{1+|k|}\big(1+|y|\big)^{n+1}\big(1+\mathrm{e}^{-2y\Im k}\big). \quad (2.17)$$

We have also the obvious estimate:

$$\big|\partial_k^n\partial_y D_k(y)\big|\leqslant C_n|y|^n\mathrm{e}^{-2y\Im k}. \quad (2.18)$$

Therefore, thanks to hypothesis (2.3), we deduce from (2.17), (2.18), that the right-hand side of (2.13), (2.14) is a contracting map of $(m(k;\cdot),\partial_x m(k;\cdot))\in C^0\cap L^\infty([R,\infty[)$ for R large enough and $\Im k>-\alpha/2$, and we get for $\beta<\alpha$, $\Im k\geqslant 0$:

$$R \leqslant x \Rightarrow \left(1+|k|\right)\left|m(k;x)-1\right| + \left|\partial_x m(k;x)\right| \leqslant C_{R,\beta} M(A,V) \mathrm{e}^{-\beta x}, \tag{2.19}$$

with

$$M(A,V) := [[A,V]] + \int_{-\infty}^{\infty} \left|A'(y)\right| \mathrm{e}^{\alpha|y|}\,\mathrm{d}y.$$

Since $D_k(y)$ is an analytic function of $k \in \mathbb{C}$ that satisfies (2.17), (2.18), we obtain (2.10) by iteration on n, and $C_{R,n,\beta}$ depends only of R, n, β, $M(A,V)$.

When A is not C^1 but satisfies (2.4), we choose $\theta \in \mathcal{D}(\mathbb{R})$ such that $0 \leqslant \theta$, $\theta(x) = \theta(-x)$, $0 \leqslant x\theta'(x)$, $\int \theta(x)\,\mathrm{d}x = 1$, and we put for $j \in \mathbb{N}$, $A_j(x) = j \int A(x-y)\theta(jy)\,\mathrm{d}y$. We easily check that

$$\limsup_{j\to\infty} M(A_j, V) \leqslant [[A,V]] + [[A']]. \tag{2.20}$$

Therefore to get (2.10) it is sufficient to prove that the Jost function $f^+_{\mathrm{in},j}$ associated with A_j tends to f^+_{in} in the sense of distributions in (k,x). We note that for $\Im k \geqslant 0$, the function $g_{(j)}(k;x) := \mathrm{e}^{-\mathrm{i}kx} f^+_{\mathrm{in}(,j)}(k;x)$ is solution of

$$g_{(j)}(k;x) = 1 + \int_x^{\infty} D_k(y-x)\left[V(y) - A^2_{(j)}(y) + 2kA_{(j)}(y)\right] g_{(j)}(k;y)\,\mathrm{d}y.$$

As previously this integral equation can be solved by a fixed point argument and we have for $x \geqslant R$:

$$\left|g(k;x)\right|, \left|g_j(k;x)\right| \leqslant C_{R,\beta} \mathrm{e}^{-\beta x} \left([[A,V]] + [[A']]\right)\left(1 + \|A\|_{L^\infty}\right).$$

Since $\|A_j\|_{L^\infty} \leqslant \|A\|_{L^\infty}$ and A_j tends to A in $L^1([R,\infty[)$ for any R, we deduce that g_j tends to g in $L^\infty_{\mathrm{loc}}(\{(k;x) \in \mathbb{C}\times\mathbb{R};\ \Im k \geqslant 0\}$. At last, the existence and the properties of f^-_{in} are obtained in the same way by replacing A by $A-a$ and x by $-x$. The construction of $f^\pm_{\mathrm{out}}$ by (2.8) is a direct consequence of the fact that A and V are real valued. Finally we get (2.9) by noting that $[f^+_{\mathrm{in}}, f^+_{\mathrm{out}}](0) = [f^-_{\mathrm{in}}, f^-_{\mathrm{out}}](a) = 0$ and $\lim_{x\to+(-)\infty} f^{+(-)}_{\mathrm{in}}(0(a);x) = \lim_{x\to+(-)\infty} f^{+(-)}_{\mathrm{out}}(0(a);x)$. □

Since the Jost functions are solutions of (2.1), the following Wronskians do not depend of x:

$$W_{\mathrm{in}}(k) := [f^+_{\mathrm{in}}, f^-_{\mathrm{in}}](k), \qquad W_{\mathrm{out}}(k) := [f^+_{\mathrm{out}}, f^-_{\mathrm{out}}](k). \tag{2.21}$$

Using (2.5)–(2.7), we evaluate:

$$[f^+_{\mathrm{in}}, f^+_{\mathrm{out}}](k) = 2\mathrm{i}k, \qquad [f^-_{\mathrm{in}}, f^-_{\mathrm{out}}](k) = -2\mathrm{i}(k-a),$$

hence for $k \neq a$, we have:

$$\begin{cases} f_{\text{in}}^{+}(k;x) = \rho_{\text{in}}^{+}(k) f_{\text{in}}^{-}(k;x) + \tau_{\text{in}}^{+}(k) f_{\text{out}}^{-}(k;x), \\ \rho_{\text{in}}^{+}(k) := -\frac{1}{2\mathrm{i}(k-a)}[f_{\text{in}}^{+}, f_{\text{out}}^{-}](k), \quad \tau_{\text{in}}^{+}(k) := \frac{1}{2\mathrm{i}(k-a)} W_{\text{in}}(k), \end{cases} \tag{2.22}$$

$$\begin{cases} f_{\text{out}}^{+}(k;x) = \tau_{\text{out}}^{+}(k) f_{\text{in}}^{-}(k;x) + \rho_{\text{out}}^{+}(k) f_{\text{out}}^{-}(k;x), \\ \rho_{\text{out}}^{+}(k) := \frac{1}{2\mathrm{i}(k-a)}[f_{\text{out}}^{+}, f_{\text{in}}^{-}](k), \quad \tau_{\text{out}}^{+}(k) := -\frac{1}{2\mathrm{i}(k-a)} W_{\text{out}}(k), \end{cases} \tag{2.23}$$

and for $k \neq 0$, we have:

$$\begin{cases} f_{\text{in}}^{-}(k;x) = \rho_{\text{in}}^{-}(k) f_{\text{in}}^{+}(k;x) + \tau_{\text{in}}^{-}(k) f_{\text{out}}^{+}(k;x), \\ \rho_{\text{in}}^{-}(k) := \frac{1}{2\mathrm{i}k}[f_{\text{in}}^{-}, f_{\text{out}}^{+}](k), \quad \tau_{\text{in}}^{-}(k) := \frac{1}{2\mathrm{i}k} W_{\text{in}}(k), \end{cases} \tag{2.24}$$

$$\begin{cases} f_{\text{out}}^{-}(k;x) = \tau_{\text{out}}^{-}(k) f_{\text{in}}^{+}(k;x) + \rho_{\text{out}}^{-}(k) f_{\text{out}}^{+}(k;x), \\ \rho_{\text{out}}^{-}(k) := -\frac{1}{2\mathrm{i}k}[f_{\text{out}}^{-}, f_{\text{in}}^{+}](k), \quad \tau_{\text{out}}^{-}(k) := -\frac{1}{2\mathrm{i}k} W_{\text{out}}(k). \end{cases} \tag{2.25}$$

From the analyticity of $f_{\text{in(out)}}^{\pm}$, and with these definitions of functions $\tau_{\text{in(out)}}^{\pm}$, $\rho_{\text{in(out)}}^{\pm}$, we directly get the following properties:

Lemma 2.2. *$W_{\text{in}}(k)$, $k\tau_{\text{in}}^{-}(k)$, $(k-a)\tau_{\text{in}}^{+}(k)$ are analytic functions of $k \in \mathbb{C}$, $\Im k > -\alpha/2$. $W_{\text{out}}(k)$, $k\tau_{\text{out}}^{-}(k)$, $(k-a)\tau_{\text{out}}^{+}(k)$ are analytic functions of $k \in \mathbb{C}$, $\Im k < \alpha/2$. $(k-a)\rho_{\text{in/out}}^{+}(k)$, $k\rho_{\text{in/out}}^{-}(k)$ are analytic functions of $k \in \mathbb{C}$, $-\alpha/2 < \Im k < \alpha/2$. Moreover we have*:

$$W_{\text{out}}(k) = \overline{W_{\text{in}}(\bar{k})}, \tag{2.26}$$

$$\tau_{\text{out/in}}^{\pm}(k) = \overline{\tau_{\text{in/out}}^{\pm}(\bar{k})}, \tag{2.27}$$

$$\rho_{\text{out/in}}^{\pm}(k) = \overline{\rho_{\text{in/out}}^{\pm}(\bar{k})}, \tag{2.28}$$

$$k\tau_{\text{out/in}}^{-}(k) = (k-a)\tau_{\text{out/in}}^{+}(k), \tag{2.29}$$

$$k\rho_{\text{out/in}}^{-}(k) = -(k-a)\rho_{\text{in/out}}^{+}(k), \tag{2.30}$$

$$\tau_{\text{in}}^{-}(k)\rho_{\text{out}}^{+}(k) + \tau_{\text{in}}^{+}(k)\rho_{\text{in}}^{-}(k) = 0, \tag{2.31}$$

$$\tau_{\text{in}}^{-}(k)\tau_{\text{out}}^{+}(k) + \rho_{\text{in}}^{-}(k)\rho_{\text{in}}^{+}(k) = 1, \tag{2.32}$$

$$k \in \mathbb{R} \setminus \{0\} \quad \Rightarrow \quad \left|\tau_{\text{in/out}}^{-}(k)\right|^2 - \left|\rho_{\text{in/out}}^{-}(k)\right|^2 = \frac{k-a}{k}, \tag{2.33}$$

$$k \in \mathbb{R} \setminus \{a\} \quad \Rightarrow \quad \left|\tau_{\text{in/out}}^{+}(k)\right|^2 - \left|\rho_{\text{in/out}}^{+}(k)\right|^2 = \frac{k}{k-a}. \tag{2.34}$$

Since W_{in} is an analytic function of $k \in \mathbb{C}$, $\Im k > -\alpha/2$, the set of its zeros is locally finite, and each of them is of finite multiplicity. We introduce:

$$\sigma_p := \{k \in \mathbb{C};\ \Im k > 0,\ W_{\text{in}}(k) = 0\}, \tag{2.35}$$

$$\sigma_{ss} := \{k \in \mathbb{R};\ W_{\text{in}}(k) = 0\}, \tag{2.36}$$

$$\mathcal{R} := \left\{k \in \mathbb{C};\ -\frac{\alpha}{2} < \Im k < 0,\ W_{\text{in}}(k) = 0\right\}. \tag{2.37}$$

We adopt the terminology used in the analogous contexts of the Dirac or Schrödinger equations, for the spectral quantities associated with (2.1): the elements of σ_p are the *eigenvalues* or *normal modes*, and the elements of $\mathcal{R}$ are the *resonances* or *quasinormal modes*. The *Klein zone* is the open interval I_K of ends 0 and a. We shall see that the asymptotic behaviour of the solutions of the Klein–Gordon equation with step-like potential A is very peculiar, and justifies to call *superradiant modes* the real frequencies in $I_K \setminus \sigma_{ss}$, and *hyperradiant modes* the elements of σ_{ss}, that play the role of the *spectral singularities* of the quadratic pencils with short range complex potential. In the simple example of the step potential $A_0(x) = a\mathbf{1}_{]-\infty,0]}(x)$, $a \in \mathbb{R}^*$, $V = 0$, we easily find $W_{\text{in}}(k) = \mathrm{i}(2k - a)$, hence $\sigma_p = \mathcal{R} = \emptyset$, $\sigma_{ss} = \{a/2\}$. There exists cases where there is no hyperradiant mode. For instance if we choose $A_1(x) = 1 - \tanh(x)$ or $A_2(x) = 1$ when $x < 0$, $A_2(x) = 0$ when $x > 1$, $A_2(x) = 1 - x$ when $0 \leqslant x \leqslant 1$, and $V(x) = 0$, we can compute $W_{\text{in}}(\kappa)$ by using some formal calculus system. We get frightful combinations of hypergeometric functions for A_1, and Bessel functions for A_2 and the investigation of the possible roots of the equation $W_{\text{in}}(\kappa) = 0$ seems to be rather delicate. A numerical evaluation of $|W_{\text{in}}(\kappa)|$ using the Maple system, clearly shows that $\sigma_{ss} = \emptyset$ for both these potentials. More generally we easily localize the spectrum:

Lemma 2.3. *σ_p and σ_{ss} are finite sets and if $m(A) \leqslant A(x) \leqslant M(A)$ a.e., we have*:

$$\sigma_p \subset \left\{k \in \mathbb{C};\ m(A) \leqslant \Re k \leqslant M(A),\ 0 < \Im k \leqslant M(A) - m(A) + \|V_-\|_{L^\infty}^{1/2}\right\},$$
$$V_- := \max(0, -V), \tag{2.38}$$

$$\sigma_{ss} \subset I_K :=]0, a[\quad \textit{if } a > 0, \qquad \textit{or }]a, 0[\quad \textit{if } a < 0. \tag{2.39}$$

Proof. (2.10) and (2.11) assure that $|W_{\text{in}}(k)| \sim 2|k|$ as $|k| \to \infty$, $\Im(k) \geqslant 0$. Hence σ_p and σ_{ss} are finite sets since W_{in} is an analytic function. Moreover (2.33) shows that

$$4k(k - a) \leqslant |W_{\text{in}}(k)|^2. \tag{2.40}$$

Thus $\sigma_{ss} \subset I_K \cup \{0, a\}$. Suppose that $W_{\text{in}}(0) = 0$. Then $\tau_{\text{in}}^+(0) = 0$ and $W_{\text{out}}(0) = 0$. Moreover, since $f_{\text{in}}^+(0; x) = f_{\text{out}}^+(0; x)$, we have $2\mathrm{i}a\rho_{\text{in}}^+(0) = W_{\text{out}}(0) = 0$, hence $f_{\text{in}}^+(0; x) = 0$, this is a contradiction. In the same way, if $W_{\text{in}}(a) = 0$, then $\tau_{\text{in}}^-(a) = 0$ and $W_{\text{out}}(a) = 0$. Moreover, since $f_{\text{in}}^-(a; x) = f_{\text{out}}^-(a; x)$, we have $2\mathrm{i}a\rho_{\text{in}}^-(a) = W_{\text{out}}(a) = 0$, hence $f_{\text{in}}^-(a; x) = 0$, this is a contradiction again. We have proved (2.39).

Now for $k = \kappa + i\lambda \in \sigma_p$, $\kappa \in \mathbb{R}$, $0 < \lambda$, we have $\partial_x^p f_{\text{in}}^+(k; x) \in L^2(\mathbb{R}_x)$ for $p = 0, 1, 2$, hence multiplying (2.1) by $\bar{f}_{\text{in}}^+$, then integrating, we get:

$$0 = 2i\lambda \int_{\mathbb{R}} \left[\kappa - A(x)\right]\left|f_{\text{in}}^+(k; x)\right|^2 \mathrm{d}x,$$

therefore $m(A) \leqslant \kappa \leqslant M(A)$. Moreover,

$$0 = \int_{\mathbb{R}} -\left|\partial_x f_{\text{in}}^+(k; x)\right|^2 + \left[\left(\kappa - A(x)\right)^2 - \lambda^2 - V(x)\right]\left|f_{\text{in}}^+(k; x)\right|^2 \mathrm{d}x,$$

thus $\lambda^2 < (M(A) - m(A))^2 + \|V_-\|_{L^\infty}$. □

In the case of the Schrödinger equation, i.e., $A = 0$, we know that the multiplicity of the zeros of W_{in} is simple. This is proved in [8] for the short range potentials V, and in [6] for the step-like case. Unlike this usual situation, when $A \neq 0$, the multiplicity $m(k) \in \mathbb{N}^*$ of $k \in \mathbb{C}$, defined by:

$$\frac{\mathrm{d}^l}{\mathrm{d}k^l} W_{\text{in}}(k) = 0, \quad 0 \leqslant l \leqslant m(k) - 1, \tag{2.41}$$

can be strictly larger that 1. As an example, we choose $A_3(x) = \mathbf{1}_{]-\infty,0]\cup[\pi/3,(2\pi)/3]}(x)$, $V = 0$. By tedious but elementary calculations, we check that $W_{\text{in}}(1/2) = W_{\text{in}}'(1/2) = 0$. Let $\lambda_j \in \sigma_p$ denote the N_p eigenvalues with multiplicity $m(\lambda_j) = m_j$, and let $\kappa_j \in \sigma_{ss}$ denote the N_{ss} hyperradiant modes with multiplicity $m(\kappa_j) = n_j$. The *principal functions* are given by:

$$\frac{\partial^h}{\partial k^h} f_{\text{in}}^{\pm}(\lambda_j; x), \quad 0 \leqslant h \leqslant m_j - 1,\ j \leqslant N_p, \tag{2.42}$$

$$\frac{\partial^h}{\partial k^h} f_{\text{in}}^{\pm}(\kappa_j; x), \quad 0 \leqslant h \leqslant n_j - 1,\ j \leqslant N_{ss}. \tag{2.43}$$

The principal functions associated with an eigenvalue are rapidly decreasing:

Lemma 2.4. *Given $k_* \in \sigma_p \cup \sigma_{ss}$, with multiplicity $m(k_*) \geqslant 1$, for $0 \leqslant l_* \leqslant m(k_*) - 1$, the principal functions $\partial_k^{l_*} f_{\text{in}}^{\pm}(k_*; x)$ satisfy for all $x \in \mathbb{R}$:*

$$\left|\partial_k^{l_*} f_{\text{in}}^{\pm}(k_*; x)\right| + \left|\partial_x \partial_k^{l_*} f_{\text{in}(k_*;x)}^{\pm}\right| \leqslant C\left(1 + |x|\right)^l \mathrm{e}^{-\Im(k_*)|x|}. \tag{2.44}$$

Proof. From (2.5), (2.6), (2.10), and (2.11), it is clear that (2.44) is satisfied for $\partial_k^{l_*} f_{\text{in}}^{+(-)}(k_*; x)$ on $\mathbb{R}_x^{+(-)}$. Now by (2.22); (2.24), we have for $0 \leqslant l_* \leqslant m(k_*) - 1$:

$$\partial_k^{l_*} f_{\text{in}}^{\pm}(k_*; x) = \sum_{p=0}^{l_*} C_{l_*}^p \frac{\mathrm{d}^{l_*-p}}{\mathrm{d}k^{l_*-p}} (\rho_{\text{in}}^{\pm})(k_*) \partial_k^p f_{\text{in}}^{\mp}(k_*; x). \tag{2.45}$$

Therefore we get (2.44) on the whole axis. □

We now investigate the asymptotic behaviours as k tends to 0 or a, and at high frequency.

Lemma 2.5. *We have* 0 *is a simple pole of* $\rho^-_{\text{in/out}}(k)$ *and* $\tau^-_{\text{in/out}}(k)$, a *is a simple pole of* $\rho^+_{\text{in/out}}(k)$ *and* $\tau^+_{\text{in/out}}(k)$, *then*:

$$\frac{\rho^-_{\text{in/out}}(k)}{\tau^-_{\text{in/out}}(k)} \to -1, \quad k \to 0, \tag{2.46}$$

$$\frac{\rho^+_{\text{in/out}}(k)}{\tau^+_{\text{in/out}}(k)} \to -1, \quad k \to a, \tag{2.47}$$

$$W_{\text{in}}(k) = 2\mathrm{i}k\mathrm{e}^{\mathrm{i}(\int_{-\infty}^0 [A(y)-a]\,\mathrm{d}y + \int_0^\infty A(y)\,\mathrm{d}y)} + \mathrm{O}(1), \quad |k| \to \infty,\ 0 \leqslant \Im k, \tag{2.48}$$

$$\left|\frac{\mathrm{d}^n}{\mathrm{d}k^n} W_{\text{in}}(k)\right| = \mathrm{O}(1), \quad |k| \to \infty,\ 0 \leqslant \Im k,\ 1 \leqslant n, \tag{2.49}$$

$$\tau^\pm_{\text{in}}(k) = \mathrm{e}^{\mathrm{i}(\int_{-\infty}^0 [A(y)-a]\,\mathrm{d}y + \int_0^\infty A(y)\,\mathrm{d}y)} + \mathrm{O}\left(\frac{1}{k}\right), \quad |k| \to \infty,\ 0 \leqslant \Im k, \tag{2.50}$$

$$\left|\frac{\mathrm{d}^n}{\mathrm{d}\kappa^n} \rho^\pm_{\text{in/out}}(\kappa)\right| = \mathrm{O}\left(\frac{1}{\kappa}\right), \quad |\kappa| \to \infty,\ \kappa \in \mathbb{R},\ 0 \leqslant n. \tag{2.51}$$

Proof. Since $a \notin \sigma_{ss}$, a is a simple pole of $\tau^+_{\text{in/out}}(k)$. (2.34) shows that $|\rho^+_{\text{in/out}}(k)| \to \infty$ as $k \to a$, $k \in I_K$. Hence a is a simple pole of $\rho^+_{\text{in/out}}(k)$. Moreover, by (2.9), $\rho^+_{\text{in/out}}(k)/\tau^+_{\text{in/out}}(k) \to -[f^+_{\text{in/out}}, f^-_{\text{out/in}}](a)/W_{\text{in/out}}(a) = -1$. We treat $\tau^-_{\text{in/out}}$, $\rho^-_{\text{in/out}}$ at $k = 0$ by a similar argument.

As regards the behaviour at high frequency, we calculate with (2.5), (2.6):

$$W_{\text{in}}(k) = \mathrm{i}\mathrm{e}^{\mathrm{i}(\int_{-\infty}^0 [A(y)-a]\,\mathrm{d}y + \int_0^\infty A(y)\,\mathrm{d}y)}\big(2k - a + \big[1 + \varepsilon^+_{\text{in}}(k; x); 1 + \varepsilon^-_{\text{in}}(k; x)\big](x = 0)\big),$$

$$\begin{aligned}[f^+_{\text{in}}, f^-_{\text{out}}](\kappa) = \mathrm{e}^{\mathrm{i}(-\int_{-\infty}^0 [A(y)-a]\,\mathrm{d}y + \int_0^\infty A(y)\,\mathrm{d}y)}&\big(1 + \varepsilon^+_{\text{in}}(k; 0)\big)\big(1 + \varepsilon^-_{\text{in}}(k; 0)\big)\\ &\times \big[1 + \varepsilon^+_{\text{in}}(k; x); 1 + \varepsilon^-_{\text{in}}(k; x)\big](x = 0),\end{aligned}$$

hence estimates (2.10), (2.11), immediately give the geometric optics approximations (2.48)–(2.51). □

We introduce the transmission coefficients $T^{+(-)}(\kappa)$ and the reflection coefficients $R^{-(+)}(\kappa)$, defined for $\kappa \in \mathbb{R} \setminus \sigma_{ss}$, $\kappa \neq a$ ($\kappa \neq 0$), by:

$$T^\pm(\kappa) := \frac{1}{\tau^\pm_{\text{out}}(\kappa)}, \qquad R^\pm(\kappa) := \frac{\rho^\mp_{\text{out}}(\kappa)}{\tau^\mp_{\text{out}}(\kappa)}. \tag{2.52}$$

Taking advantage of (2.46), (2.47), and of the fact that $0, a \notin \sigma_{ss}$, we put:

$$R^+(0) = R^-(a) = -1, \qquad T^+(a) = T^-(0) = 0. \tag{2.53}$$

These quantities describe the propagation of the field as $x \to \pm\infty$. We summarize their main properties that are direct consequences of the previous results:

Lemma 2.6. *$R^{\pm}(\kappa)$ and $T^{\pm}(\kappa)$ are analytic functions on $\mathbb{R}_\kappa \setminus \sigma_{ss}$ and satisfy*:

$$\frac{\kappa}{\kappa - a}\left|T^{+}(\kappa)\right|^2 + \left|R^{+}(\kappa)\right|^2 = 1, \tag{2.54}$$

$$\frac{\kappa - a}{\kappa}\left|T^{-}(\kappa)\right|^2 + \left|R^{-}(\kappa)\right|^2 = 1, \tag{2.55}$$

$$\left|T^{+}(\kappa)T^{-}(\kappa) - R^{+}(\kappa)R^{-}(\kappa)\right| = 1, \tag{2.56}$$

$$\kappa \in \mathbb{R} \setminus I_K \quad \Rightarrow \quad \left|R^{\pm}(\kappa)\right| \leqslant 1, \tag{2.57}$$

$$\kappa \in I_K \setminus \sigma_{ss} \quad \Rightarrow \quad \left|R^{\pm}(\kappa)\right| > 1, \tag{2.58}$$

$$\kappa \to \kappa_j \in \sigma_{ss} \quad \Rightarrow \quad \left|R^{\pm}(\kappa)\right|, \left|T^{\pm}(\kappa)\right| \to \infty, \tag{2.59}$$

$$T^{\pm}(\kappa) = \mathrm{e}^{\mathrm{i}(\int_{-\infty}^{0}[A(y)-a]\,\mathrm{d}y + \int_0^\infty A(y)\,\mathrm{d}y)} + \mathrm{O}\left(\frac{1}{\kappa}\right), \quad \left|R^{\pm}(\kappa)\right| = \mathrm{O}\left(\frac{1}{\kappa}\right), \quad |\kappa| \to \infty, \tag{2.60}$$

$$1 \leqslant n \quad \Rightarrow \quad \left|\frac{\mathrm{d}^n}{\mathrm{d}\kappa^n}T^{\pm}(\kappa)\right|, \left|\frac{\mathrm{d}^n}{\mathrm{d}\kappa^n}R^{\pm}(\kappa)\right| = \mathrm{O}\left(\frac{1}{\kappa}\right), \quad |\kappa| \to \infty. \tag{2.61}$$

We emphasize that when κ is outside the Klein zone, the reflection coefficient is not greater than one as in the usual case of the decaying potential (i.e., $a = 0$). But when κ is a superradiant mode, $|R^{\pm}(\kappa)|$ is strictly larger than one, but finite: this is the phenomenon of *superradiance* of the Klein–Gordon fields (2.58). At last $T^{\pm}$ and $R^{\pm}$ diverge at the hyperradiant modes. The situation differs for the Dirac or Schrödinger equations, for which the reflection is total in the Klein zone (i.e., $T = 0$, $R = 1$, see [7,32]).

Now we introduce the incoming (outgoing) Green functions. For $k \notin \sigma_p \cup \sigma_{ss}$, $0 \leqslant \Im k$, (respectively $\bar{k} \notin \sigma_p \cup \sigma_{ss}$, $\Im k \leqslant 0$), we define:

$$G_{\mathrm{in(out)}}(k; x, y) := \frac{1}{W_{\mathrm{in(out)}}(k)} f^{+}_{\mathrm{in(out)}}\big(k; \max(x, y)\big) f^{-}_{\mathrm{in(out)}}\big(k; \min(x, y)\big), \tag{2.62}$$

that are distribution solutions of

$$\frac{\partial^2}{\partial x^2}G + \left[k - A(x)\right]^2 G - V(x)G = \delta_0(x - y). \tag{2.63}$$

We want to separate the variables x, and y, *modulo* an entire function of k. Let $f_0(k; x)$ and $f_1(k; x)$ be the solutions of (2.1) satisfying:

$$f_0(k; 0) = \frac{\mathrm{d}}{\mathrm{d}x} f_1(k; 0) = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}x} f_0(k; 0) = f_1(k; 0) = 1.$$

For any $x \in \mathbb{R}$, the functions $f_0(k; x)$ and $f_1(k; x)$ are entire functions of $k \in \mathbb{C}$.

Lemma 2.7. *For $k \in \mathbb{C} \setminus (\sigma_p \cup \sigma_{ss} \cup \mathcal{R})$, $\Im k > -\alpha/2$ (respectively $\bar{k} \in \mathbb{C} \setminus (\sigma_p \cup \sigma_{ss} \cup \mathcal{R})$, $\Im k < \alpha/2$), We have*:

$$\begin{aligned} G_{\text{in(out)}}(k; x, y) &= \frac{1}{W_{\text{in(out)}}(k)} f^{+}_{\text{in(out)}}(k; x) f^{-}_{\text{in(out)}}(k; y) \\ &\quad + \mathbf{1}_{[0,\infty[}(y - x)\big[f_0(k; y) f_1(k; x) - f_0(k; x) f_1(k; y)\big] \\ &= \frac{1}{W_{\text{in(out)}}(k)} f^{-}_{\text{in(out)}}(k; x) f^{+}_{\text{in(out)}}(k; y) \\ &\quad + \mathbf{1}_{[0,\infty[}(x - y)\big[f_0(k; x) f_1(k; y) - f_0(k; y) f_1(k; x)\big]. \end{aligned} \tag{2.64}$$

Proof. We put:

$$f^{\pm}_{\text{in}} = \alpha^{\pm}_0 f_0 + \alpha^{\pm}_1 f_1.$$

We calculate:

$$W_{\text{in}}(k) = \alpha^{+}_0 \alpha^{-}_1 - \alpha^{-}_0 \alpha^{+}_1,$$

$$\begin{aligned} &f^{+}_{\text{in}}\big(k; \max(x, y)\big) f^{-}_{\text{in}}\big(k; \min(x, y)\big) - f^{+}_{\text{in}}(k; x) f^{-}_{\text{in}}(k; y) \\ &\quad = \mathbf{1}_{[0,\infty[}(y - x)\big(\alpha^{+}_0 \alpha^{-}_1 - \alpha^{-}_0 \alpha^{+}_1\big)\big(f_1(k; x) f_0(k; y) - f_0(k; x) f_1(k; y)\big). \end{aligned}$$

The other cases are treated in the same manner. □

We construct the distorted Fourier transforms. Given $f \in C^{\infty}_0(\mathbb{R}_x)$, $\varphi \in C^{\infty}_0(\mathbb{R}_\kappa)$ we put:

$$F^{\pm}_{\text{in(out)}}(f)(k) := \int_{-\infty}^{\infty} f^{\pm}_{\text{in(out)}}(k; x) f(x)\, \mathrm{d}x, \quad k \in \mathbb{C},\ +(-)\Im k \geqslant 0. \tag{2.65}$$

To study the continuity of these operators, we need the weighted L^2-spaces on an open interval $I \subset \mathbb{R}$, defined for $s \in \mathbb{R}$, by:

$$L^2_s(I_y) := \big\{f \in L^2_{\text{loc}}(I_y, \mathrm{d}y);\ \|f\|_{L^2_s(I)} < \infty\big\}, \quad \|f\|^2_{L^2_s(I)} := \int_I \langle y\rangle^{2s} \big|f(y)\big|^2\, \mathrm{d}y, \tag{2.66}$$

with the notation:

$$\langle y\rangle := \big(1 + y^2\big)^{1/2}, \tag{2.67}$$

and the usual Sobolev spaces on $\mathbb{R}$:

$$H^s(\mathbb{R}) := \big\{f \in \mathcal{S}'(\mathbb{R});\ \hat{f} \in L^2_s(\mathbb{R})\big\}, \quad \|f\|_{H^s(\mathbb{R})} := \big\|\hat{f}\big\|_{L^2_s(\mathbb{R})},$$

where $\hat{f} = \mathcal{F}(f)$ is the Fourier transform of a tempered distribution $f \in \mathcal{S}'(\mathbb{R})$, given when $f \in L^1(\mathbb{R}_x)$ by $\hat{f}(\xi) = \mathcal{F}(f)(\xi) := \int e^{-ix\xi} f(x)\,dx$. We need also the Hardy spaces $\mathcal{H}^2_\pm$ that are the Hilbert spaces of analytic functions of k, $\pm\Im k > 0$, satisfying:

$$\|\varphi\|^2_{\mathcal{H}^2_\pm} := \sup_{\pm\eta>0} \int_{\Im k=\eta} \left|\varphi(k)\right|^2 dk < \infty. \tag{2.68}$$

It is well known that for $f \in \mathcal{H}^2_\pm$, the limit $\lim_{\eta\to 0^+} f(\kappa \pm i\eta)$ exists a.e. and is in $L^2(\mathbb{R}_\kappa)$, and we have:

$$\|f\|_{\mathcal{H}^2_\pm} = \|f\|_{L^2(\mathbb{R}_\kappa)}. \tag{2.69}$$

For $R \in \mathbb{R}$, we put:

$$I_R^+ :=]R-1, \infty[_x, \qquad I_R^- :=]-\infty, R+1[_x. \tag{2.70}$$

Lemma 2.8. *For any $f \in C_0^\infty(\mathbb{R}_x)$, and for all $n \in \mathbb{N}$, $\partial_k^n F^\pm_{\text{in(out)}}(f)$ belongs to $\mathcal{H}^2_{+(-)}$. Moreover there exists positive functions $C(R)$, $C(s)$, $C(s,R)$, such that for all $R \in \mathbb{R}$, $f \in C_0^\infty(I_R^\pm)$, $0 \leqslant s$, $0 \leqslant p \leqslant 1$, we have*:

$$F^\pm_{\text{in(out)}}(f)(\kappa + \lambda i) \in C^0\big(\mathbb{R}_\lambda^{+(-)}; H^s(\mathbb{R}_\kappa)\big),$$
$$\left\|F^\pm_{\text{in(out)}}(f)(\kappa + \lambda i)\right\|_{H^s(\mathbb{R}_\kappa)} \leqslant C(s,R)\|f\|_{L^2_s(\mathbb{R}_x)}, \tag{2.71}$$

$$F^\pm_{\text{in(out)}}(f)(\kappa + \lambda i) \in C^0\big(\mathbb{R}_\lambda^{+(-)}; L^2_p(\mathbb{R}_\kappa)\big),$$
$$\left\|F^\pm_{\text{in(out)}}(f)(\kappa + \lambda i)\right\|_{L^2_p(\mathbb{R}_\kappa)} \leqslant C(R)\|f\|_{H^p(\mathbb{R}_x)}, \tag{2.72}$$

$$\left\|\langle\kappa\rangle^{-1}(\kappa - a)F^+_{\text{in(out)}}(f)\right\|_{L^2_p(\mathbb{R}_\kappa)} + \left\|\langle\kappa\rangle^{-1}\kappa F^-_{\text{in(out)}}(f)\right\|_{L^2_p(\mathbb{R}_\kappa)} \leqslant C\|f\|_{H^p(\mathbb{R}_x)}, \tag{2.73}$$

$$\left\|\langle\kappa\rangle^{-1}(\kappa - a)F^+_{\text{in(out)}}(f)\right\|_{H^s(\mathbb{R}_\kappa)} + \left\|\langle\kappa\rangle^{-1}\kappa F^-_{\text{in(out)}}(f)\right\|_{H^s(\mathbb{R}_\kappa)} \leqslant C(s)\|f\|_{L^2_s(\mathbb{R}_x)}, \tag{2.74}$$

$$F^-_{\text{in=out}}(f)(a) = 0 \quad\Rightarrow\quad \left\|F^+_{\text{in(out)}}(f)\right\|_{L^2_p(\mathbb{R}_\kappa)} \leqslant C\|f\|_{L^2_1\cap H^p(\mathbb{R}_x)}, \tag{2.75}$$

$$F^+_{\text{in=out}}(f)(0) = 0 \quad\Rightarrow\quad \left\|F^-_{\text{in(out)}}(f)\right\|_{L^2_p(\mathbb{R}_\kappa)} \leqslant C\|f\|_{L^2_1\cap H^p(\mathbb{R}_x)}. \tag{2.76}$$

Remark. For $f \in L^2_1(\mathbb{R}_x)$, $F^-_{\text{in(out)}}(f)(a)$ and $F^+_{\text{in(out)}}(f)(0)$ are well defined since the embedding $H^1 \subset C^0$, and (2.74) assure that $F^-_{\text{in(out)}}(f) \in C^0(\mathbb{R}\setminus\{0\})$ and $F^+_{\text{in(out)}}(f) \in C^0(\mathbb{R}\setminus\{a\})$.

Proof of Lemma 2.8. It is sufficient to establish these results for $s \in \mathbb{N}$, $p = 0, 1$, since the general case follows by interpolation. Proposition 2.1 assures that $F_{\text{in}}^{+}(f)(k)$ is an analytic function of $k \in \mathbb{C}$, $\Im k > 0$ and we have:

$$\frac{\mathrm{d}^n}{\mathrm{d}k^n} F_{\text{in}}^{+}(f)(k) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\big(\mathrm{e}^{\mathrm{i}\int_x^\infty A(y)\,\mathrm{d}y}(\mathrm{i}x)^n f(x)\big)\,\mathrm{d}x \\ + \sum_{p=0}^{n} C_n^p \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\big(\mathrm{e}^{\mathrm{i}\int_x^\infty A(y)\,\mathrm{d}y}\partial_k^p \varepsilon_{\text{in}}^{+}(k;x)(\mathrm{i}x)^{n-p} f(x)\big)\,\mathrm{d}x.$$

We deduce from (2.10) that

$$\left|\frac{\mathrm{d}^n}{\mathrm{d}k^n} F_{\text{in}}^{+}(f)(k)\right| \leqslant \left|\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\big(\mathrm{e}^{\mathrm{i}\int_x^\infty A(y)\,\mathrm{d}y}(\mathrm{i}x)^n f(x)\big)\,\mathrm{d}x\right| + \frac{1}{1+|k|} C(n,R)\|f\|_{L^2(\mathbb{R}_x)},$$

hence the inequality in (2.71) follows from the Parseval equality. We conclude that $\partial_k^n F_{\text{in}}^{+}(f) \in \mathcal{H}_{+}^{2}$, therefore we get the continuity with respect to λ.

Now by using Eq. (2.1) of which f_{in}^{+} is solution, and integrating by parts, we have:

$$k^2 F_{\text{in}}^{+}(f)(k) = k F_{\text{in}}^{+}(\mathrm{i}f' + 2Af)(k) + F_{\text{in}}^{+}\big(\big(V - A^2\big)f - \mathrm{i}Af'\big)(k) \\ + \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\partial_x \varepsilon_{\text{in}}^{+}(k;x) f'(x)\,\mathrm{d}x \tag{2.77}$$

and we deduce (2.72) for $p = 1$, thanks to (2.71) with $s = 0$, and (2.10); F_{out}^{+}, $F_{\text{in(out)}}^{\pm}$ are treated in the same way.

In what follows, given $\rho \in \mathbb{R}$, it is useful to choose some functions $\chi_\rho^{\pm} \in C^\infty(\mathbb{R}_x)$, satisfying:

$$x \leqslant \rho - \frac{1}{2} \Rightarrow \chi_\rho^{+}(x) = 0, \quad x \geqslant \rho + \frac{1}{2} \Rightarrow \chi_\rho^{+}(x) = 1, \quad \chi_\rho^{-} = 1 - \chi_\rho^{+}. \tag{2.78}$$

We get from (2.22) that

$$(\kappa - a)F_{\text{in}}^{+}(f) = (\kappa - a)F_{\text{in}}^{+}(\chi_\rho^{+} f) + (\kappa - a)\rho_{\text{in}}^{+}(\kappa)F_{\text{in}}^{-}(\chi_\rho^{-} f) \\ + (\kappa - a)\tau_{\text{in}}^{+}(\kappa)F_{\text{out}}^{-}(\chi_\rho^{-} f).$$

Since $\chi_\rho^{\pm} f \in C_0^\infty(I_\rho^{\pm})$, we can take advantage of Lemma 2.2, the asymptotic behaviours (2.50), (2.51), and apply (2.71), (2.72) to obtain:

$$\big\|(\kappa - a)F_{\text{in}}^{+}(f)\big\|_{L^2_{p-1}(\mathbb{R}_\kappa)} \leqslant C\|f\|_{H^p(\mathbb{R}_x)}, \quad p = 0, 1.$$

We also use the analyticity of $(\kappa - a)\rho_{\mathrm{in}}^{+}(\kappa)$ and $(\kappa - a)\tau_{\mathrm{in}}^{+}(\kappa)$, and (2.71) to get:

$$\left\| \langle\kappa\rangle^{-1}(\kappa - a)F_{\mathrm{in}}^{+}(f)\right\|_{H^n(\mathbb{R}_\kappa)} \leqslant C(n)\|f\|_{L_n^2(\mathbb{R}_x)}, \quad n \in \mathbb{N}.$$

To prove (2.75) we remark that (2.73) assures that for any $\eta > 0$,

$$\left\| F_{\mathrm{in(out)}}^{+}(f)\right\|_{L_p^2(\mathbb{R}_\kappa \setminus [a-\eta, a+\eta])} \leqslant C_\eta \|f\|_{H^p(\mathbb{R}_x)}, \quad p = 0, 1,$$

thus it is sufficient to show that

$$\left\| F_{\mathrm{in(out)}}^{+}(f)\right\|_{L^2([a-\eta, a+\eta]_\kappa)} \leqslant C'_\eta \|f\|_{L_1^2(\mathbb{R}_x)} \tag{2.79}$$

when $F_{\mathrm{in(out)}}^{-}(f)(a) = 0$. From (2.22), and with this assumption, we have:

$$F_{\mathrm{in(out)}}^{+}(f)(\kappa) = \int_0^1 (\kappa - a)\varrho_{\mathrm{in(out)}}^{+}(\kappa)\frac{\mathrm{d}}{\mathrm{d}k}F_{\mathrm{in(out)}}^{-}(f)\big(a + (\kappa - a)s\big)$$
$$+ (\kappa - a)\tau_{\mathrm{in(out)}}^{+}(\kappa)\frac{\mathrm{d}}{\mathrm{d}k}F_{\mathrm{out(in)}}^{-}(f)\big(a + (\kappa - a)s\big)\,\mathrm{d}s.$$

We deduce by Lemma 2.5 that

$$\left\| F_{\mathrm{in(out)}}^{+}(f)\right\|_{L^2([a-\eta, a+\eta]_\kappa)} \leqslant C\int_0^1 \left\| \frac{\mathrm{d}}{\mathrm{d}k}F_{\mathrm{in}}^{-}(f)\big(a + (\kappa - a)s\big)\right\|_{L^2([a-\eta, a+\eta]_\kappa)}$$
$$+ \left\| \frac{\mathrm{d}}{\mathrm{d}k}F_{\mathrm{out}}^{-}(f)\big(a + (\kappa - a)s\big)\right\|_{L^2([a-\eta, a+\eta]_\kappa)} \mathrm{d}s.$$

We have:

$$\left\| \frac{\mathrm{d}}{\mathrm{d}k}F_{\mathrm{in(out)}}^{-}(f)\big(a + (\kappa - a)s\big)\right\|_{L^2([a-\eta, a+\eta]_\kappa)} = \frac{1}{\sqrt{s}}\left\| \frac{\mathrm{d}}{\mathrm{d}k}F_{\mathrm{in(out)}}^{-}(f)(\kappa)\right\|_{L^2([a-s\eta, a+s\eta]_\kappa)},$$

hence

$$\left\| F_{\mathrm{in(out)}}^{+}(f)\right\|_{L^2([a-\eta, a+\eta]_\kappa)} \leqslant C\left\| \frac{\mathrm{d}}{\mathrm{d}k}F_{\mathrm{in}}^{-}(f)(\kappa)\right\|_{L^2([a-\eta, a+\eta]_\kappa)}$$
$$+ C\left\| \frac{\mathrm{d}}{\mathrm{d}k}F_{\mathrm{out}}^{-}(f)(\kappa)\right\|_{L^2([a-\eta, a+\eta]_\kappa)}.$$

We choose $0 < \eta < |a|$, then $0 \notin [a-\eta, a+\eta]$, and (2.74) implies the existence of $C''_\eta > 0$ such that

$$\left\| \frac{\mathrm{d}}{\mathrm{d}k} F^-_{\mathrm{in/out}}(f)(\kappa) \right\|_{L^2([a-\eta,a+\eta]_\kappa)} \leqslant C''_\eta \|f\|_{L^2_1(\mathbb{R}_x)}.$$

Therefore we get (2.79). The proofs for $F^-_{\mathrm{in(out)}}(f)$ are analogous. □

As we can see by (2.71) and (2.74), $F^\pm_{\mathrm{in(out)}}$ is well defined from $\bigcup_R L^2(I^\pm_R)$ to $L^2(\mathbb{R}_\kappa)$, but, unlike the short range case $a = 0$, a problem arises for the low frequencies $\kappa = 0, a$, with loss of regularity, when we want to define $F^\pm_{\mathrm{in(out)}}$ on $L^2(\mathbb{R}_x)$. It is necessary to use the weighted L^2-spaces to overcome this difficulty. The main properties of the distorted Fourier transforms are summarized in the following:

Proposition 2.9. *We have, $F^\pm_{\mathrm{in(out)}}$ that is defined from $C^\infty_0(\mathbb{R}_x)$ to $\mathcal{H}^2_\pm \cap_n [H^n \cap L^2_n](\mathbb{R}_\kappa)$, has a continuous extension*:

(1) *from $L^2_{1/2+\delta}(\mathbb{R}_x)$ to $H^{-1/2-\delta}(\mathbb{R}_\kappa) \cap \mathcal{E}'(\mathbb{R}_\kappa) + H^{1/2+\delta}(\mathbb{R}_\kappa)$, for any $0 < \delta$;*
(2) *from $L^2_1(\mathbb{R}_x)$ to $\mathcal{H}^2_\pm$;*
(3) *from $H^1 \cap L^2_1(\mathbb{R}_x)$ to $L^2_1(\mathbb{R}_\kappa)$.*

There exists no continuous extension from $L^2_{1/2}(\mathbb{R}_x)$ to $\mathcal{D}'(\mathbb{R}_\kappa)$. For any $\delta > 0$ there exists no continuous extension from $L^2_{1-\delta}(\mathbb{R}_x)$ to $L^2(\mathbb{R}_\kappa)$.

Proof. We choose some function $\chi_K \in C^\infty_0(\mathbb{R}_\kappa)$ satisfying:

$$\kappa \in I_K \quad \Rightarrow \quad \chi_K(\kappa) = 1. \tag{2.80}$$

Now we write:

$$F^+_{\mathrm{in(out)}}(f) = \mathrm{P.V.}\left(\frac{1}{\kappa-a}\right)\chi_K(\kappa)(\kappa-a)F^+_{\mathrm{in(out)}}(f) + \left(\frac{1-\chi_K(\kappa)}{\kappa-a}\right)(\kappa-a)F^+_{\mathrm{in(out)}}(f), \tag{2.81}$$

where the principal value $\mathrm{P.V.}(1/(\kappa-a))$ belongs to $H^{-1/2-\delta}(\mathbb{R}_\kappa)$, for any $\delta > 0$. Since $H^{1/2+\delta}(\mathbb{R})$ is an algebra, (2.74) assures that $f \mapsto \mathrm{P.V.}(1/(\kappa-a))\chi_K(\kappa)(\kappa-a)F^+_{\mathrm{in(out)}}(f)$ is continuous from $H^{1/2+\delta}(\mathbb{R}_x)$ to $H^{-1/2-\delta}(\mathbb{R}_\kappa)$. On the other hand (2.73) assures that $f \mapsto ((1-\chi_K(\kappa))/(\kappa-a))(\kappa-a)F^+_{\mathrm{in(out)}}(f)$ is continuous from $L^2_{1/2+\delta}(\mathbb{R}_x)$ to $H^{1/2+\delta}(\mathbb{R}_\kappa)$. Thus the first assertion is proved. To establish the second one, we choose $g_a \in C^\infty_0(\mathbb{R}_x)$ such that $F^-_{\mathrm{in=out}}(g_a)(\kappa = a) = 1$, and we write $f = f_a + F^-_{\mathrm{in=out}}(f)(\kappa = a)g_a$. Since $F^-_{\mathrm{in=out}}(f_a)(\kappa = a) = 0$, we evaluate $\|F^+_{\mathrm{in/out}}(f_a)\|_{L^2(\mathbb{R}_\kappa)}$ with (2.75) and (2.76). We deduce from this last result and (2.73) that $F^\pm_{\mathrm{in(out)}}$ has a continuous extension from $H^1 \cap L^2_1(\mathbb{R}_x)$ to $L^2_1(\mathbb{R}_\kappa)$.

To prove that F_{in}^{+} is not continuous from $C_0^\infty(\mathbb{R}_x)$ endowed with the $L^2_{1/2}$-norm, into $\mathcal{D}'(\mathbb{R}_\kappa)$, we choose $\varphi \in C_0^\infty(\mathbb{R}_\kappa)$ such that for any $N \in \mathbb{N}$ there exists $g_N \in C_0^\infty(\mathbb{R}_x)$ satisfying:

$$\|g_N\|_{L^2_{1/2}(\mathbb{R}_x)} \leqslant 1, \quad \left|\left\langle F_{\mathrm{in}}^{+}(g_N), \varphi\right\rangle_{\mathcal{D}'(\mathbb{R}_\kappa),\mathcal{D}(\mathbb{R}_\kappa)}\right| \geqslant N. \tag{2.82}$$

Using (2.22) we have:

$$\begin{aligned}
&\left\langle F_{\mathrm{in}}^{+}(f), \varphi\right\rangle_{\mathcal{D}'(\mathbb{R}_\kappa),\mathcal{D}(\mathbb{R}_\kappa)} = \iint \varphi(\kappa) f_{\mathrm{in}}^{+}(\kappa; x) f(x)\,\mathrm{d}x\,\mathrm{d}\kappa \\
&\quad = \lim_{\varepsilon\to 0} \int f(x)\left(\int_{|\kappa-a|\geqslant\varepsilon} \varphi(\kappa) f_{\mathrm{in}}^{+}(\kappa; x)\,\mathrm{d}\kappa\right)\mathrm{d}x \\
&\quad = \frac{1}{2\mathrm{i}} \int f(x)\left\langle \mathrm{P.V.}\left(\frac{1}{\kappa-a}\right), \varphi(\kappa)\left\{W_{\mathrm{in}}(\kappa) f_{\mathrm{out}}^{-}(\kappa; x)\right.\right. \\
&\qquad \left.\left. - [f_{\mathrm{in}}^{+}, f_{\mathrm{out}}^{-}](\kappa) f_{\mathrm{in}}^{-}(\kappa; x)\right\}\right\rangle_{\mathcal{D}'(\mathbb{R}_\kappa),\mathcal{D}(\mathbb{R}_\kappa)} \mathrm{d}x \\
&\quad = \frac{1}{2\mathrm{i}} \left\langle \mathrm{P.V.}\left(\frac{1}{\kappa-a}\right), \varphi(\kappa)\left\{W_{\mathrm{in}}(\kappa) F_{\mathrm{out}}^{-}(f) - [f_{\mathrm{in}}^{+}, f_{\mathrm{out}}^{-}](\kappa) F_{\mathrm{in}}^{-}(f)\right\}\right\rangle_{\mathcal{D}'(\mathbb{R}_\kappa),\mathcal{D}(\mathbb{R}_\kappa)}.
\end{aligned}$$

For $n \in \mathbb{N}^*$, $f \in C_0^\infty(]-\infty, 0[_x)$, we put $f_n(x) := (1/n) f(x/n)$. We have:

$$\|f_n\|_{L^2_{1/2}(\mathbb{R}_x)} \leqslant \|f\|_{L^2_{1/2}(\mathbb{R}_x)}. \tag{2.83}$$

Thanks to (2.11) we have:

$$\begin{aligned}
&F_{\mathrm{in(out)}}^{-}(f_n)(\kappa) = \hat{f}\big(+(-)n(\kappa-a)\big) + \psi_n(\kappa), \\
&\|\psi_{\mathrm{in(out)},n}\|_{L^\infty(\mathbb{R}_\kappa)} + \|\psi'_{\mathrm{in(out)},n}\|_{L^\infty(\mathbb{R}_\kappa)} \to 0,\ n \to \infty.
\end{aligned} \tag{2.84}$$

We deduce that as $n \to \infty$ we have:

$$\begin{aligned}
&\left\langle F_{\mathrm{in}}^{+}(f_n), \varphi\right\rangle_{\mathcal{D}'(\mathbb{R}_\kappa),\mathcal{D}(\mathbb{R}_\kappa)} \\
&\quad = \frac{1}{2\mathrm{i}} \left\langle \mathrm{P.V.}\left(\frac{1}{\kappa-a}\right), \varphi\left(a+\frac{\kappa}{n}\right)\left\{W_{\mathrm{in}}\left(a+\frac{\kappa}{n}\right)\hat{f}(-\kappa)\right.\right. \\
&\qquad \left.\left. - [f_{\mathrm{in}}^{+}, f_{\mathrm{out}}^{-}]\left(a+\frac{\kappa}{n}\right)\hat{f}(\kappa)\right\}\right\rangle_{\mathcal{D}'(\mathbb{R}_\kappa),\mathcal{D}(\mathbb{R}_\kappa)} + \mathrm{o}(1) \\
&\quad = \frac{1}{2\mathrm{i}} \varphi(a) W_{\mathrm{in}}(a) \left\langle \mathrm{P.V.}\left(\frac{1}{\kappa}\right), \hat{f}(-\kappa) - \hat{f}(\kappa)\right\rangle_{\mathcal{D}'(\mathbb{R}_\kappa),\mathcal{D}(\mathbb{R}_\kappa)} + \mathrm{o}(1) \\
&\quad = -\pi\varphi(a) W_{\mathrm{in}}(a) \int f(x)\,\mathrm{d}x + \mathrm{o}(1).
\end{aligned}$$

Since $a \notin \sigma_{ss}$, we choose $\varphi \in C_0^\infty(\mathbb{R}_\kappa)$ such that $\pi\varphi(a)W_{\mathrm{in}}(a) = 1$. We take $\chi_N \in C_0^\infty(]-\infty, -\mathrm{e}[_x)$, $0 \leqslant \chi_N \leqslant 1$, $\chi_N(x) = 1$ for $[-c_N, -2\mathrm{e}]$. We put $f(x) = \frac{1}{2|x|\ln|x|}\chi_N(x)$. We have:

$$\|f\|_{L^2_{1/2}(\mathbb{R}_x)} \leqslant 1, \qquad \int f(x)\,\mathrm{d}x \geqslant \ln\bigl(\ln(c_N)\bigr).$$

To get (2.82) it is sufficient to choose $c_N > 0$ large enough and to put $g_N = f_n$ with n large enough.

Now given $\delta > 0$, we choose $f \in C_0^\infty(]-\infty, 0[_x)$, such that $\int f(x)\,\mathrm{d}x \neq 0$, and we introduce $g_n(x) := n^{\delta-1/2} f_n(x) = n^{\delta-3/2} f(x/n)$. We have:

$$\sup_n \|g_n\|_{L^2_{1-\delta}(\mathbb{R}_x)} < \infty. \tag{2.85}$$

For any $\varphi \in C_0^\infty(\mathbb{R}_\kappa)$, we put $\varphi_n(\kappa) = n^{1/2-\delta}\varphi(n^{1-2\delta}(\kappa - a))$. We also have:

$$\|\varphi_n\|_{L^2(\mathbb{R}_\kappa)} = \|\varphi\|_{L^2(\mathbb{R}_\kappa)}. \tag{2.86}$$

By using (2.84), we obtain as before the following asymptotics as $n \to \infty$:

$$\begin{aligned}
&\bigl\langle F^+_{\mathrm{in}}(g_n), \varphi_n\bigr\rangle_{\mathcal{D}'(\mathbb{R}_\kappa),\mathcal{D}(\mathbb{R}_\kappa)} \\
&\quad = \frac{1}{2\mathrm{i}}\Bigl\langle \mathrm{P.V.}\Bigl(\frac{1}{\kappa - a}\Bigr), \varphi\bigl(n^{1-2\delta}(\kappa - a)\bigr)\bigl\{W_{\mathrm{in}}(\kappa)\hat{f}\bigl(-n(\kappa - a)\bigr) \\
&\qquad - [f^+_{\mathrm{in}}, f^-_{\mathrm{out}}](\kappa)\hat{f}\bigl(n(\kappa - a)\bigr)\bigr\}\Bigr\rangle_{\mathcal{D}'(\mathbb{R}_\kappa),\mathcal{D}(\mathbb{R}_\kappa)} + \mathrm{o}(1) \\
&\quad = \frac{1}{2\mathrm{i}}\Bigl\langle \mathrm{P.V.}\Bigl(\frac{1}{\kappa}\Bigr), \varphi\bigl(n^{-2\delta}\kappa\bigr)\Bigl\{W_{\mathrm{in}}\Bigl(a + \frac{\kappa}{n}\Bigr)\hat{f}(-\kappa) \\
&\qquad - [f^+_{\mathrm{in}}, f^-_{\mathrm{out}}]\Bigl(a + \frac{\kappa}{n}\Bigr)\hat{f}(\kappa)\Bigr\}\Bigr\rangle_{\mathcal{D}'(\mathbb{R}_\kappa),\mathcal{D}(\mathbb{R}_\kappa)} + \mathrm{o}(1) \\
&\quad = -\pi\varphi(0)W_{\mathrm{in}}(a)\int f(x)\,\mathrm{d}x + \mathrm{o}(1).
\end{aligned}$$

Since we can choose $\varphi(0)$ as large as we want with $\|\varphi\|_{L^2(\mathbb{R}_\kappa)} = 1$, we conclude that F^+_{in} is not continuous from $C_0^\infty(\mathbb{R}_x)$ endowed with the $L^2_{1-\delta}$-norm, to $L^2(\mathbb{R}_\kappa)$. The proofs for $F^\pm_{\mathrm{in(out)}}$ are similar. □

For any $f \in C_0^\infty(\mathbb{R}_x)$, $F^\pm_{\mathrm{in(out)}}(f)$ is analytic on $\Im k > 0$ ($\Im k < 0$) and

$$\frac{\mathrm{d}^n}{\mathrm{d}k^n}F^\pm_{\mathrm{in(out)}}(f)(k) = \int_{-\infty}^{\infty} \frac{\partial^n}{\partial k^n} f^\pm_{\mathrm{in(out)}}(k;x) f(x)\,\mathrm{d}x. \tag{2.87}$$

When $f\in L^2(\mathbb{R}_x)$ is not compactly supported, $F^{\pm}_{\mathrm{in(out)}}(f)$ is no more defined for non real k. Nevertheless, the right-hand side of (2.87) is well defined for $k\in\sigma_p$ ($\bar{k}\in\sigma_p$) thanks to Lemma 2.4, and it is convenient to keep the same notation of "derivative". Hence we put:

$$\forall f\in L^2(\mathbb{R}_x),\ \lambda_j\in\sigma_p,\quad \frac{\mathrm{d}^n}{\mathrm{d}k^n}F^{\pm}_{\mathrm{in(out)}}(f)\big(k=\lambda_j\big(=\bar{\lambda}_j\big)\big)$$
$$:=\int_{-\infty}^{\infty}\frac{\partial^n}{\partial k^n}f^{\pm}_{\mathrm{in(out)}}\big(k=\lambda_j\big(=\bar{\lambda}_j\big);x\big)f(x)\,\mathrm{d}x. \tag{2.88}$$

We now introduce the inverse distorted Fourier transforms, defined for $\varphi\in C_0^\infty(\mathbb{R}_\kappa)$ by:

$$\Phi^{\pm}_{\mathrm{in(out)}}(\varphi)(x):=\frac{1}{2\pi}\int_{-\infty}^{\infty}f^{\pm}_{\mathrm{in(out)}}(\kappa;x)\varphi(\kappa)\,\mathrm{d}\kappa,\quad x\in\mathbb{R}. \tag{2.89}$$

We denote $\mathcal{E}'(\mathbb{R})$ the space of compactly supported distributions on $\mathbb{R}$, and for $s\in\mathbb{R}$ and any open interval $I\subset\mathbb{R}$, we define the Sobolev spaces on I: $H^s(I):=\{g=f_{|I};\ f\in H^s(\mathbb{R})\}$, $\|g\|_{H^s(I)}:=\inf_{g=f_{|I}}\|f\|_{H^s(\mathbb{R})}$, and as usual, $H^s_{\mathrm{loc}}(\mathbb{R})$ denotes the space of distributions on $\mathbb{R}$, whose restriction to each bounded open interval I, belongs to $H^s(I)$.

Lemma 2.10. *There exist a constant $C>0$, functions $C(R)$, $C(s)$, $C(s,R)$, such that for all $R\in\mathbb{R}$, $s\in\mathbb{R}$, $-1\leqslant p\leqslant 1$, $\varphi\in C_0^\infty(\mathbb{R}_\kappa)$, we have*:

$$\big\|\Phi^{\pm}_{\mathrm{in(out)}}(\varphi)\big\|_{H^p(I_R^{\pm})}\leqslant C(R)\|\varphi\|_{L^2_p(\mathbb{R}_\kappa)}, \tag{2.90}$$

$$\big\|\Phi^{\pm}_{\mathrm{in(out)}}(\varphi)\big\|_{L^2_s(I_R^{\pm})}\leqslant C(s,R)\|\varphi\|_{H^s(\mathbb{R}_\kappa)}, \tag{2.91}$$

$$\big\|\Phi^{+}_{\mathrm{in(out)}}\big(\langle\kappa\rangle^{-1}(\kappa-a)\varphi\big)\big\|_{H^p(\mathbb{R}_x)}+\big\|\Phi^{-}_{\mathrm{in(out)}}\big(\langle\kappa\rangle^{-1}\kappa\varphi\big)\big\|_{H^p(\mathbb{R}_x)}\leqslant C\|\varphi\|_{L^2_p(\mathbb{R}_\kappa)}, \tag{2.92}$$

$$\big\|\Phi^{+}_{\mathrm{in(out)}}\big(\langle\kappa\rangle^{-1}(\kappa-a)\varphi\big)\big\|_{L^2_s(\mathbb{R}_x)}+\big\|\Phi^{-}_{\mathrm{in(out)}}\big(\langle\kappa\rangle^{-1}\kappa\varphi\big)\big\|_{L^2_s(\mathbb{R}_x)}\leqslant C(s)\|\varphi\|_{H^s(\mathbb{R}_\kappa)}. \tag{2.93}$$

Moreover $\Phi^{\pm}_{\mathrm{in(out)}}$ is a bounded operator from $\mathcal{E}'(\mathbb{R}_\kappa)$ to $H^2_{\mathrm{loc}}(\mathbb{R}_x)$.

Proof. We prove these results for $s=n\in\mathbb{Z}$ and $p=-1,0,1$. The general case follows by interpolation. Since we have:

$$\Phi^{+}_{\mathrm{in}}(\varphi)(x)=\frac{1}{2\pi}\mathrm{e}^{\mathrm{i}\int_x^\infty A(y)\,\mathrm{d}y}\big[\widehat{\varphi}(-x)+\big\langle\mathrm{e}^{\mathrm{i}\kappa x}\varepsilon^{+}_{\mathrm{in}}(\kappa;x),\varphi(\kappa)\big\rangle_{H^{-n}(\mathbb{R}_\kappa),H^n(\mathbb{R}_\kappa)}\big],$$

we deduce from (2.10) that for $x\geqslant R$,

$$\big|\Phi^{+}_{\mathrm{in}}(\varphi)(x)\big|\leqslant\big|\widehat{\varphi}(-x)\big|+C(n,R,\beta)\big(1+|x|\big)^{\max(0,-n)}\mathrm{e}^{-\beta x}\|\varphi\|_{H^n(\mathbb{R}_\kappa)},$$

hence we obtain (2.90) for $p = 0$, and (2.91). We also write,

$$\partial_x \Phi_{\text{in}}^{+}(\varphi)(x) = -\mathrm{i}A(x)\Phi_{\text{in}}^{+}(\varphi)(x) + \Phi_{\text{in}}^{+}(\mathrm{i}\kappa\varphi)(x) + \frac{1}{2\pi}\mathrm{e}^{\mathrm{i}\int_x^\infty A(y)\,\mathrm{d}y}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\kappa x}\partial_x\varepsilon_{\text{in}}^{+}(\kappa;x)\varphi(\kappa)\,\mathrm{d}\kappa,$$

and by (2.10) again:

$$\left|\partial_x \Phi_{\text{in}}^{+}(\varphi)(x)\right| \leqslant \|A\|_{L^\infty(\mathbb{R}_x)}\left|\Phi_{\text{in}}^{+}(\varphi)(x)\right| + \left|\Phi_{\text{in}}^{+}(\mathrm{i}\kappa\varphi)(x)\right| + C(R,\beta)\mathrm{e}^{-\beta x}\|\varphi\|_{L_1^2(\mathbb{R}_\kappa)}, \tag{2.94}$$

therefore we get (2.90) for $p = 1$. We treat the case $p = -1$ by duality. Given $f \in C_0^\infty(I_R^+)$, we estimate by (2.72):

$$\left|\int_{-\infty}^{\infty}\Phi_{\text{in}}^{+}(\varphi)(x)f(x)\,\mathrm{d}x\right| = \left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(\kappa)F_{\text{in}}^{+}(f)(\kappa)\,\mathrm{d}\kappa\right| \leqslant C(R)\|f\|_{H^1(I_R^+)}\|\varphi\|_{L_{-1}^2(\mathbb{R}_\kappa)}.$$

Now we write by (2.22):

$$\Phi_{\text{in}}^{+}\big((\kappa-a)\cdot\big) = \chi_\rho^{+}\Phi_{\text{in}}^{+}\big((\kappa-a)\cdot\big) + \chi_\rho^{-}\Phi_{\text{in}}^{-}\big((\kappa-a)\rho_{\text{in}}^{+}(\kappa)\cdot\big) + \chi_\rho^{-}\Phi_{\text{out}}^{-}\big((\kappa-a)\tau_{\text{in}}^{+}(\kappa)\cdot\big), \tag{2.95}$$

and we conclude with Lemma 2.2, (2.50), (2.51), and (2.90), that

$$\left\|\Phi_{\text{in}}^{+}\big((\kappa-a)\varphi\big)\right\|_{H^p(\mathbb{R}_x)} \leqslant C\|\varphi\|_{L_{p+1}^2(\mathbb{R}_\kappa)}, \quad p = 0, 1.$$

Finally, given $\psi \in C_0^\infty(\mathbb{R}_x)$, we have:

$$\int_{-\infty}^{\infty}\Phi_{\text{in}}^{+}\big((\kappa-a)\varphi\big)(x)\psi(x)\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(\kappa)(\kappa-a)F_{\text{in}}^{+}(\psi)(\kappa)\,\mathrm{d}\kappa.$$

Thanks to (2.73) we get:

$$\left|\int_{-\infty}^{\infty}\Phi_{\text{in}}^{+}\big((\kappa-a)\varphi\big)(x)\psi(x)\,\mathrm{d}x\right| \leqslant C\|\varphi\|_{L^2(\mathbb{R}_\kappa)}\|\psi\|_{H^1(\mathbb{R}_x)},$$

hence we deduce that

$$\left\|\Phi_{\mathrm{in}}^{+}\big((\kappa-a)\varphi\big)\right\|_{H^{-1}(\mathbb{R}_x)}\leqslant C\|\varphi\|_{L^2(\mathbb{R}_\kappa)},$$

and the proof of (2.92) for $\Phi_{\mathrm{in}}^{+}((\kappa-a)\varphi)$ is complete.

We also get from (2.95) and (2.90):

$$\left\|\Phi_{\mathrm{in}}^{+}\big(\langle\kappa\rangle^{-1}(\kappa-a)\varphi\big)\right\|_{L^2_n(\mathbb{R}_x)}\leqslant C\|\varphi\|_{H^n(\mathbb{R}_\kappa)}.$$

To prove the last assertion, we note that for $\varphi\in C_0^\infty(\mathbb{R}_\kappa)$, we have:

$$\Phi_{\mathrm{in}}^{+}(\varphi)(x)=\frac{1}{2\pi}\mathrm{e}^{\mathrm{i}\int_x^\infty A(y)\,\mathrm{d}y}\Big[\widehat{\varphi}(-x)+\big\langle\varphi(\kappa),\mathrm{e}^{\mathrm{i}\kappa x}\varepsilon_{\mathrm{in}}^{+}(\kappa;x)\big\rangle_{\mathcal{E}'(\mathbb{R}_\kappa),\mathcal{E}(\mathbb{R}_\kappa)}\Big],$$

hence the same formula holds for $\varphi\in\mathcal{E}'(\mathbb{R}_\kappa)$. Since such a distribution can be written as $\varphi=\sum_{j=0}^{N}\psi_j^{[j]}$, $\psi_j\in C^0(\mathbb{R}_\kappa)$, we get:

$$\Phi_{\mathrm{in}}^{+}(\varphi)(x)=\frac{1}{2\pi}\mathrm{e}^{\mathrm{i}\int_x^\infty A(y)\,\mathrm{d}y}\left[\widehat{\varphi}(-x)+\sum_{j=0}^{N}(-1)^j\int\psi_j(\kappa)\partial_\kappa^j\big(\mathrm{e}^{\mathrm{i}\kappa x}\varepsilon_{\mathrm{in}}^{+}(\kappa;x)\big)\,\mathrm{d}\kappa\right].$$

On the one hand $\widehat{\varphi}\in C^\infty(\mathbb{R}_x)$, and on the other hand, (2.10) assures that the last term belongs to $C^0(\mathbb{R}_x)$. Since:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Phi_{\mathrm{in(out)}}^{\pm}(\varphi)=-\Phi_{\mathrm{in(out)}}^{\pm}\big(\kappa^2\varphi\big)+2A\Phi_{\mathrm{in(out)}}^{\pm}(\kappa\varphi)+\big(V-A^2\big)\Phi_{\mathrm{in(out)}}^{\pm}(\varphi),\tag{2.96}$$

we conclude that $\Phi_{\mathrm{in}}^{+}(\varphi)\in H^2_{\mathrm{loc}}(\mathbb{R}_x)$. The proofs for Φ_{in}^{-}, $\Phi_{\mathrm{out}}^{\pm}$ are similar. □

Because of the hyperradiant modes, the spectral expansion involves singular distributions instead of a usual integral on $\mathbb{R}_\kappa$. We introduce:

$$\nu:=\max_{1\leqslant j\leqslant N_{ss}}(n_j)\quad\text{if }\sigma_{ss}\neq\emptyset,\qquad\nu:=0\quad\text{if }\sigma_{ss}=\emptyset.\tag{2.97}$$

Lemma 2.11. *We obtain that* $[W_{\mathrm{in}}(\kappa+\mathrm{i}\varepsilon)]^{-1}$ *and* $[W_{\mathrm{out}}(\kappa-\mathrm{i}\varepsilon)]^{-1}$ *converge in* $H^{-\nu}(\mathbb{R}_\kappa)$ *as* $\varepsilon\to0^+$ *to limits respectively denoted* $[W_{\mathrm{in}}(\kappa+\mathrm{i}0)]^{-1}$ *and* $[W_{\mathrm{out}}(\kappa-\mathrm{i}0)]^{-1}$ *that belong to* $H^s(\mathbb{R}_\kappa)$*,* $s<-\nu+\frac{1}{2}$*. The singular support of these distributions is* σ_{ss}*.*

Proof. Since $W_{\mathrm{in}}(k)$ is analytic for $\Im k>-\alpha/2$, we can use a partition of the unity to express:

$$\frac{1}{W_{\mathrm{in}}(\kappa+\mathrm{i}\varepsilon)}=\frac{\theta_0(\kappa)}{W_{\mathrm{in}}(\kappa+\mathrm{i}\varepsilon)}+\sum_{j=1}^{N_{ss}}\frac{\theta_j(\kappa)}{W_{\mathrm{in},j}(\kappa+\mathrm{i}\varepsilon)}\frac{1}{(\kappa-\kappa_j+\mathrm{i}\varepsilon)^{n_j}},$$

where $(\theta_j)_{1\leqslant j\leqslant N_{ss}} \in C_0^\infty(\mathbb{R}_\kappa)$, $\theta_j(\kappa)=1$ near κ_j, $\sum_{j=0}^{N_{ss}}\theta_j=1$, $W_{\mathrm{in},j}$ are analytic functions on some neighborhood of the support of θ_j, without zero inside. By Lemma 2.3 and (2.48), $\theta_0(\kappa)[W_{\mathrm{in}}(\kappa+\mathrm{i}\varepsilon)]^{-1}$ tends to $\theta_0(\kappa)[W_{\mathrm{in}}(\kappa)]^{-1}$ in $L^2(\mathbb{R}_\kappa)$. Now it is well known that $(\kappa+\mathrm{i}\varepsilon)^{-1}$ tends to $(\kappa+\mathrm{i}0)^{-1}:=\mathrm{P.V.}(1/\kappa)-\mathrm{i}\pi\delta_0$ in $H^{-1}(\mathbb{R}_\kappa)$ as $\varepsilon\to 0^+$, and $(\kappa+\mathrm{i}0)^{-1}\in H^s(\mathbb{R}_\kappa)$, $s<-1/2$. Hence $(\kappa+\mathrm{i}\varepsilon)^{-n-1}=\frac{(-1)^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\kappa^n}(\kappa+\mathrm{i}\varepsilon)^{-1}$ tends to $(\kappa+\mathrm{i}0)^{-n-1}:=\frac{(-1)^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\kappa^n}(\mathrm{P.V.}(1/\kappa)-\mathrm{i}\pi\delta_0)$ in $H^{-n-1}(\mathbb{R}_\kappa)$ as $\varepsilon\to 0^+$, and $(\kappa+\mathrm{i}0)^{-n}\in H^s(\mathbb{R}_\kappa)$, $s<-n+1/2$. Since $\theta_j(\kappa)[W_{\mathrm{in},j}(\kappa+\mathrm{i}\varepsilon)]^{-n_j}$ tends to $\theta_j(\kappa)[W_{\mathrm{in},j}(\kappa)]^{-n_j}$ in $H^{n_j}(\mathbb{R}_\kappa)$, we conclude that

$$\frac{1}{W_{\mathrm{in}}(\kappa+\mathrm{i}\varepsilon)}\to\frac{1}{W_{\mathrm{in}}(\kappa+\mathrm{i}0)}:=\frac{\theta_0(\kappa)}{W_{\mathrm{in}}(\kappa)}+\sum_{j=1}^{N_{ss}}\frac{\theta_j(\kappa)}{W_{\mathrm{in},j}(\kappa)}\frac{1}{(\kappa-\kappa_j+0\mathrm{i})^{n_j}},$$
$$\varepsilon\to 0^+,\ \text{in } H^{-\nu}(\mathbb{R}_\kappa). \tag{2.98}$$

From this formula, we see that $1/W_{\mathrm{in}}(\kappa+\mathrm{i}0)$ belongs to $H^s(\mathbb{R}_\kappa)$, $s<-\nu+1/2$, and the singularities of this distribution are exactly located at the hyperradiant modes. □

We are now ready to state the main result of this part.

Theorem 2.12. *There exist complex numbers $c_{j,l}$, for $1\leqslant j\leqslant N_p$, $0\leqslant l\leqslant m_j-1$, with $c_{j,m_j-1}\neq 0$, such that for all $f\in L^2_s(\mathbb{R}_x)$, $s>\max(1/2,\nu-1/2)$, where ν is defined by* (2.97)*, we have for $p=0,1$:*

$$\begin{aligned}pf&=\Phi_{\mathrm{in}}^{\pm}\left(\frac{\mathrm{i}\kappa^p}{W_{\mathrm{in}}(\kappa+\mathrm{i}0)}F_{\mathrm{in}}^{\mp}(f)\right)-\Phi_{\mathrm{out}}^{\pm}\left(\frac{\mathrm{i}\kappa^p}{W_{\mathrm{out}}(\kappa-\mathrm{i}0)}F_{\mathrm{out}}^{\mp}(f)\right)\\&\quad+\sum_{\lambda_j\in\sigma_p}\sum_{l=0}^{m_j-1}c_{j,l}\partial_k^l\big(k^p f_{\mathrm{in}}^{\pm}(k;x)F_{\mathrm{in}}^{\mp}(f)\big)(k=\lambda_j)\\&\quad+\sum_{\lambda_j\in\sigma_p}\sum_{l=0}^{m_j-1}\bar c_{j,l}\partial_k^l\big(k^p f_{\mathrm{out}}^{\pm}(k;x)F_{\mathrm{out}}^{\mp}(f)\big)\big(k=\bar\lambda_j\big),\end{aligned}\tag{2.99}$$

where the four terms involving $\Phi_{\mathrm{in(out)}}^{\pm}$ belong to $L^2_{\mathrm{loc}}(\mathbb{R}_x)$, and when f is not compactly supported, $\partial_k^l(F_{\mathrm{in(out)}}^{\pm}(f))(k=\lambda_j(=\bar\lambda_j))$ is defined by (2.88).

Remark. The terms involving $\Phi_{\mathrm{in(out)}}^{\pm}(\ldots)$ make sense thanks to the previous lemmas. We shall see that

$$\kappa^p\big[W_{\mathrm{in(out)}}\big(\kappa+(-)\mathrm{i}0\big)\big]^{-1}F_{\mathrm{in(out)}}^{\pm}(f)\in\mathcal{E}'(\mathbb{R}_\kappa)+L^2_{1-p}(\mathbb{R}_\kappa).\tag{2.100}$$

Therefore Lemma 2.10 assures that all the terms are well defined in $L^2_{\mathrm{loc}}(\mathbb{R}_x)$.

Proof of Theorem 2.12. Given $f \in C_0^\infty(\mathbb{R}_x)$ we write for $p = 0, 1$:

$$pf(x) = \frac{1}{2i\pi} \oint_{\Gamma(R)} \frac{f(x)}{k^{2-p}}\,dk,$$
$$0 < R,\ \Gamma(R) := \{k \in \mathbb{C};\ |k| = R\}\ \text{(positively oriented)}. \tag{2.101}$$

We introduce the Green function $G(k; x, y)$:

$$\Im k > 0 \quad \Rightarrow \quad G(k; x, y) := G_{\text{in}}(k; x, y),$$
$$\Im k < 0 \quad \Rightarrow \quad G(k; x, y) := G_{\text{out}}(k; x, y). \tag{2.102}$$

We have for $k, \bar{k} \notin \sigma_p$:

$$f(x) = \int_{-\infty}^{\infty} G(k; x, y)\big[f''(y) + \big(k - A(y)\big)^2 f(y) - V(y)f(y)\big]\,dy. \tag{2.103}$$

Replacing (2.103) into the right-hand side of (2.101), we get:

$$2i\pi pf(x) = \int_{-\infty}^{\infty} \big[f''(y) + \big(A^2(y) - V(y)\big)f(y)\big]\left(\oint_{\Gamma(R)} k^{p-2}G(k; x, y)\,dk\right)dy$$
$$- \int_{-\infty}^{\infty} 2A(y)f(y)\left(\oint_{\Gamma(R)} k^{p-1}G(k; x, y)\,dk\right)dy$$
$$+ \int_{-\infty}^{\infty} f(y)\left(\oint_{\Gamma(R)} k^{p}G(k; x, y)\,dk\right)dy. \tag{2.104}$$

We deduce from (2.10), (2.11), (2.48), that

$$\big|G(k; x, y)\big| \leqslant C(x, y)\frac{e^{-|\Im k||x-y|}}{1 + |k|}, \tag{2.105}$$

where $C(x, y)$ is a locally bounded function that does not depend on k. Since,

$$\int_0^{\pi/2} e^{-|x-y|R\sin\theta}\,d\theta \leqslant \int_0^{\pi/2} e^{-2|x-y|R\theta/\pi}\,d\theta = O\left(\frac{1}{R|x-y|}\right),$$

the two first integrals of the right-hand side of (2.104) tend to 0 as $R \to \infty$, and we conclude that

$$pf(x)=\frac{1}{2i\pi}\lim_{R\to\infty}\int_{-\infty}^{\infty}f(y)\left(\oint_{\Gamma_R}k^pG(k;x,y)\,dk\right)dy. \tag{2.106}$$

For $0<\varepsilon<\inf(\Im\lambda_j;\ \lambda_j\in\sigma_p)$, $R>\sup(|\lambda_j|;\ \lambda_j\in\sigma_p)$, we write

$$\Gamma(R)=\Gamma_{\text{in}}(R,\varepsilon)\cup\Gamma_{\text{out}}(R,\varepsilon)\cup\Gamma_0(R,\varepsilon)$$

with

$$\begin{aligned}\Gamma_{\text{in(out)}}(R,\varepsilon)&:=\left\{k\in\mathbb{C};\ |k|=R,\ +(-)\Im k\geqslant\varepsilon\right\}\\&\quad\cup\left\{k\in\mathbb{C};\ |k|\leqslant R,\ \Im k=+(-)\varepsilon\right\}\quad\text{positively oriented,}\end{aligned} \tag{2.107}$$

$$\begin{aligned}\Gamma_0(R,\varepsilon)&:=\left\{k\in\mathbb{C};\ |k|=R,\ |\Im k|\leqslant\varepsilon\right\}\\&\quad\cup\left\{k\in\mathbb{C};\ |k|\leqslant R,\ |\Im k|=\varepsilon\right\}\quad\text{negatively oriented.}\end{aligned} \tag{2.108}$$

We invoke Lemma 2.7 and the Cauchy theorem to obtain:

$$\begin{aligned}pf(x)&=\frac{1}{2i\pi}\lim_{R\to\infty}\sum_{\text{in,out}}\oint_{\Gamma_{\text{in/out}}(R,\varepsilon)}\frac{k^p}{W_{\text{in/out}}(k)}f^{\pm}_{\text{in/out}}(k;x)\left(\int_{-\infty}^{\infty}f^{\mp}_{\text{in/out}}(k;y)f(y)\,dy\right)dk\\&\quad+\int_{-\infty}^{\infty}f(y)\left(\oint_{\Gamma_0(R,\varepsilon)}k^pG(k;x,y)\,dk\right)dy.\end{aligned} \tag{2.109}$$

Then the residue theorem gives:

$$\begin{aligned}&\frac{1}{2i\pi}\lim_{R\to\infty}\sum_{\text{in,out}}\oint_{\Gamma_{\text{in/out}}(R,\varepsilon)}\frac{k^p}{W_{\text{in/out}}(k)}f^{\pm}_{\text{in/out}}(k;x)\left(\int_{-\infty}^{\infty}f^{\mp}_{\text{in/out}}(k;y)f(y)\,dy\right)dk\\&=\sum_{\lambda_j\in\sigma_p}\lim_{k\to\lambda_j}\frac{\partial^{m_j-1}}{\partial k^{m_j-1}}\left[\frac{(k-\lambda_j)^{m_j}k^p}{(m_j-1)!W_{\text{in}}(k)}f^{\pm}_{\text{in}}(k;x)\int_{-\infty}^{\infty}f^{\mp}_{\text{in}}(k;y)f(y)\,dy\right]\\&\quad+\lim_{k\to\bar\lambda_j}\frac{\partial^{m_j-1}}{\partial k^{m_j-1}}\left[\frac{(k-\bar\lambda_j)^{m_j}k^p}{(m_j-1)!W_{\text{out}}(k)}f^{\pm}_{\text{out}}(k;x)\int_{-\infty}^{\infty}f^{\mp}_{\text{out}}(k;y)f(y)\,dy\right]\\&=\sum_{\lambda_j\in\sigma_p}\sum_{l=0}^{m_j-1}c_{j,l}\partial_k^l\big(k^pf^{\pm}_{\text{in}}(k;x)F^{\mp}_{\text{in}}(f)\big)(k=\lambda_j)\\&\quad+\sum_{\lambda_j\in\sigma_p}\sum_{l=0}^{m_j-1}\bar c_{j,l}\partial_k^l\big(k^pf^{\pm}_{\text{out}}(k;x)F^{\mp}_{\text{out}}(f)\big)\big(k=\bar\lambda_j\big),\end{aligned} \tag{2.110}$$

with

$$c_{j,l} := \frac{1}{l!(m_j-1-l)!} \lim_{k\to\lambda_j} \frac{\mathrm{d}^{m_j-1-l}}{\mathrm{d}k^{m_j-1-l}} \left(\frac{(k-\lambda_j)^{m_j}}{W_{\mathrm{in}}(k)} \right), \tag{2.111}$$

and it is clear that $c_{j,m_j-1} \neq 0$.

We deduce from (2.71) that $|F^{\mp}_{\mathrm{in(out)}}(f)(\kappa+(-)\mathrm{i}\lambda)| \to 0$ as $|\kappa| \to \infty$, and

$$\sup_{\kappa} \sup_{0\leqslant\lambda} \left| F^{\mp}_{\mathrm{in(out)}}(f)(\kappa+(-)\mathrm{i}\lambda) \right| < \infty.$$

Moreover by Proposition 2.1 and (2.48), we also have:

$$\sup_{\kappa\in\mathbb{R}\setminus I_K} \sup_{0\leqslant\lambda\leqslant\varepsilon} \left| \frac{(\kappa+(-)\mathrm{i}\lambda)^p}{W_{\mathrm{in(out)}}(\kappa+(-)\mathrm{i}\lambda)} F^{\mp}_{\mathrm{in(out)}}(f)\big(\kappa+(-)\mathrm{i}\lambda\big) \right| < \infty,$$

therefore we conclude that

$$\begin{aligned} &\lim_{R\to\infty} \int_{-\infty}^{\infty} f(y) \left(\oint_{\Gamma_0(R,\varepsilon)} k^p G(k;x,y)\,\mathrm{d}k \right) \mathrm{d}y \\ &= \int_{-\infty}^{\infty} f^{\pm}_{\mathrm{in}}(\kappa+\mathrm{i}\varepsilon;x) \frac{(\kappa+\mathrm{i}\varepsilon)^p}{W_{\mathrm{in}}(\kappa+\mathrm{i}\varepsilon)} F^{\mp}_{\mathrm{in}}(f)(\kappa+\mathrm{i}\varepsilon)\,\mathrm{d}\kappa \\ &\quad - \int_{-\infty}^{\infty} f^{\pm}_{\mathrm{out}}(\kappa-\mathrm{i}\varepsilon;x) \frac{(\kappa-\mathrm{i}\varepsilon)^p}{W_{\mathrm{out}}(\kappa-\mathrm{i}\varepsilon)} F^{\mp}_{\mathrm{out}}(f)(\kappa-\mathrm{i}\varepsilon)\,\mathrm{d}\kappa. \end{aligned} \tag{2.112}$$

Given χ_K satisfying (2.80), we write:

$$\begin{aligned} &\int_{-\infty}^{\infty} f^{\pm}_{\mathrm{in(out)}}\big(\kappa+(-)\mathrm{i}\varepsilon;x\big) \frac{(\kappa+(-)\mathrm{i}\varepsilon)^p}{W_{\mathrm{in(out)}}(\kappa+(-)\mathrm{i}\varepsilon)} F^{\mp}_{\mathrm{in(out)}}(f)\big(\kappa+(-)\mathrm{i}\varepsilon\big)\,\mathrm{d}\kappa \\ &= \int_{-\infty}^{\infty} \big(1-\chi_K(\kappa)\big) f^{\pm}_{\mathrm{in(out)}}\big(\kappa+(-)\mathrm{i}\varepsilon;x\big) \frac{(\kappa+(-)\mathrm{i}\varepsilon)^p}{W_{\mathrm{in(out)}}(\kappa+(-)\mathrm{i}\varepsilon)} F^{\mp}_{\mathrm{in(out)}}(f)\big(\kappa+(-)\mathrm{i}\varepsilon\big)\,\mathrm{d}\kappa \\ &\quad + \Big\langle \chi_K(\kappa)\big[f^{\pm}_{\mathrm{in(out)}}\big(\kappa+(-)\mathrm{i}\varepsilon;x\big) - f^{\pm}_{\mathrm{in(out)}}(\kappa;x)\big], \\ &\qquad \frac{(\kappa+(-)\mathrm{i}\varepsilon)^p F^{\mp}_{\mathrm{in(out)}}(f)(\kappa+(-)\mathrm{i}\varepsilon)}{W_{\mathrm{in(out)}}(\kappa+(-)\mathrm{i}\varepsilon)} \Big\rangle_{H^{\nu}(\mathbb{R}_\kappa),H^{-\nu}(\mathbb{R}_\kappa)} \\ &\quad + \Phi^{\pm}_{\mathrm{in(out)}}\left(\chi_K(\kappa) \frac{(\kappa+(-)\mathrm{i}\varepsilon)^p}{W_{\mathrm{in(out)}}(\kappa+(-)\mathrm{i}\varepsilon)} F^{\mp}_{\mathrm{in(out)}}(f)\big(\kappa+(-)\mathrm{i}\varepsilon\big) \right). \end{aligned} \tag{2.113}$$

Proposition 2.1 and (2.48) show that

$$\big(1-\chi_K(\kappa)\big) f^{\pm}_{\text{in(out)}}\big(\kappa+(-)\mathrm{i}\varepsilon;x\big)\frac{(\kappa+(-)\mathrm{i}\varepsilon)^p}{W_{\text{in(out)}}(\kappa+(-)\mathrm{i}\varepsilon)}F^{\mp}_{\text{in(out)}}(f)\big(\kappa+(-)\mathrm{i}\varepsilon\big)$$
$$\to \big(1-\chi_K(\kappa)\big) f^{\pm}_{\text{in(out)}}(\kappa;x)\frac{\kappa^p}{W_{\text{in(out)}}(\kappa)}F^{\mp}_{\text{in(out)}}(f)(\kappa) \quad \text{in } L^1(\mathbb{R}_\kappa),\ \varepsilon\to 0^+, \tag{2.114}$$
$$\chi_K(\kappa)\big[f^{\pm}_{\text{in(out)}}\big(\kappa+(-)\mathrm{i}\varepsilon;x\big)-f^{\pm}_{\text{in(out)}}(\kappa;x)\big]\to 0 \quad \text{in } H^{\nu}(\mathbb{R}_\kappa),\ \varepsilon\to 0^+. \tag{2.115}$$

Lemma 2.8 and Lemma 3.15 yield:

$$\frac{(\kappa+(-)\mathrm{i}\varepsilon)^p}{W_{\text{in(out)}}(\kappa+(-)\mathrm{i}\varepsilon)}F^{\mp}_{\text{in(out)}}(f)\big(\kappa+(-)\mathrm{i}\varepsilon\big)\to\frac{\kappa^p}{W_{\text{in(out)}}(\kappa+(-)\mathrm{i}0)}F^{\mp}_{\text{in(out)}}(f)(\kappa)$$
$$\text{in } H^{-\nu}(\mathbb{R}_\kappa),\ \varepsilon\to 0^+. \tag{2.116}$$

Therefore we conclude with Lemma 2.10 that

$$\int_{-\infty}^{\infty} f^{\pm}_{\text{in(out)}}\big(\kappa+(-)\mathrm{i}\varepsilon;x\big)\frac{(\kappa+(-)\mathrm{i}\varepsilon)^p}{W_{\text{in(out)}}(\kappa+(-)\mathrm{i}\varepsilon)}F^{\mp}_{\text{in(out)}}(f)\big(\kappa+(-)\mathrm{i}\varepsilon\big)\,\mathrm{d}\kappa$$
$$\to \Phi^{\pm}_{\text{in(out)}}\bigg(\frac{\mathrm{i}\kappa^p}{W_{\text{in(out)}}(\kappa+(-)\mathrm{i}0)}F^{\mp}_{\text{in(out)}}(f)\bigg)(x),\quad \varepsilon\to 0^+, \tag{2.117}$$

and we get the following spectral expansion:

$$pf(x)=\Phi^{\pm}_{\text{in}}\bigg(\frac{\mathrm{i}\kappa^p}{W_{\text{in}}(\kappa+\mathrm{i}0)}F^{\mp}_{\text{in}}(f)\bigg)(x)-\Phi^{\pm}_{\text{out}}\bigg(\frac{\mathrm{i}\kappa^p}{W_{\text{out}}(\kappa-\mathrm{i}0)}F^{\mp}_{\text{out}}(f)\bigg)(x)$$
$$+\sum_{\lambda_j\in\sigma_p}\sum_{l=0}^{m_j-1}c_{j,l}\partial_k^l\big(k^p f^{\pm}_{\text{in}}(k;x)F^{\mp}_{\text{in}}(f)\big)(k=\lambda_j)$$
$$+\sum_{\lambda_j\in\sigma_p}\sum_{l=0}^{m_j-1}\bar{c}_{j,l}\partial_k^l\big(k^p f^{\pm}_{\text{out}}(k;x)F^{\mp}_{\text{out}}(f)\big)\big(k=\bar{\lambda}_j\big),\quad p=0,1. \tag{2.118}$$

We remark that this formula is still valid for $f\in L^2_s(\mathbb{R}_x)$, $s>\max(\nu,1)-1/2$. Firstly the terms $\partial_k^l(F^{\mp}_{\text{in(out)}}(f))(k=\lambda_j(k=\bar{\lambda}_j))$ are well defined continuous functions of f, by (2.88). Secondly, Proposition 2.9 and (2.81) assure that $F^{\mp}_{\text{in(out)}}(f)$ is H^{-s} near $\kappa=0,a$, and belongs to $H^s(\mathbb{R}_\kappa\setminus\omega_{0,a})$, where $\omega_{0,a}$ is a neighborhood of $\{0,a\}$. Since $0,a\notin\sigma_{ss}$ we obtain (2.100) with Lemma 3.15 and (2.48), and we conclude thanks to Lemma 2.10. □

We can slightly relax the assumption on f:

Corollary 2.13. *Given functions* $\chi^{\pm}_{\rho}$ *satisfying* (2.78), *we have for all* $f\in L^2_s(\mathbb{R}_x)$, $s\geqslant 0$, $s>\nu-1/2$:

$$
\begin{aligned}
f &= \Phi_{\text{in}}^{+}\left(\frac{\mathrm{i}(\kappa-a)}{W_{\text{in}}(\kappa+\mathrm{i}0)}F_{\text{in}}^{-}(\chi_\rho^- f)\right)+\Phi_{\text{in}}^{-}\left(\frac{\mathrm{i}\kappa}{W_{\text{in}}(\kappa+\mathrm{i}0)}F_{\text{in}}^{+}(\chi_\rho^+ f)\right)\\
&\quad-\Phi_{\text{out}}^{+}\left(\frac{\mathrm{i}(\kappa-a)}{W_{\text{out}}(\kappa-\mathrm{i}0)}F_{\text{out}}^{-}(\chi_\rho^- f)\right)-\Phi_{\text{out}}^{-}\left(\frac{\mathrm{i}\kappa}{W_{\text{out}}(\kappa-\mathrm{i}0)}F_{\text{out}}^{+}(\chi_\rho^+ f)\right)\\
&\quad+\sum_{\lambda_j\in\sigma_p}\sum_{l=0}^{m_j-1}c_{j,l}\frac{\partial^l}{\partial k^l}\left[(k-a)f_{\text{in}}^{+}(k;x)F_{\text{in}}^{-}(\chi_\rho^- f)+kf_{\text{in}}^{-}(k;x)F_{\text{in}}^{+}(\chi_\rho^+ f)\right](k=\lambda_j)\\
&\quad+\bar c_{j,l}\frac{\partial^l}{\partial k^l}\left[(k-a)f_{\text{out}}^{+}(k;x)F_{\text{out}}^{-}(\chi_\rho^- f)+kf_{\text{out}}^{-}(k;x)F_{\text{out}}^{+}(\chi_\rho^+ f)\right]\big(k=\bar\lambda_j\big),
\end{aligned}
\tag{2.119}
$$

where the four terms involving $\Phi_{\text{in(out)}}^{\pm}$ belong to $L^2_{-s}(\mathbb{R}_x)$. Moreover we have:

$$
\begin{aligned}
0 &= \Phi_{\text{in}}^{+}\left(\frac{\mathrm{i}}{W_{\text{in}}(\kappa+\mathrm{i}0)}F_{\text{in}}^{-}(\chi_\rho^- f)\right)+\Phi_{\text{in}}^{-}\left(\frac{\mathrm{i}}{W_{\text{in}}(\kappa+\mathrm{i}0)}F_{\text{in}}^{+}(\chi_\rho^+ f)\right)\\
&\quad-\Phi_{\text{out}}^{+}\left(\frac{\mathrm{i}}{W_{\text{out}}(\kappa-\mathrm{i}0)}F_{\text{out}}^{-}(\chi_\rho^- f)\right)-\Phi_{\text{out}}^{-}\left(\frac{\mathrm{i}}{W_{\text{out}}(\kappa-\mathrm{i}0)}F_{\text{out}}^{+}(\chi_\rho^+ f)\right)\\
&\quad+\sum_{\lambda_j\in\sigma_p}\sum_{l=0}^{m_j-1}c_{j,l}\frac{\partial^l}{\partial k^l}\left[f_{\text{in}}^{+}(k;x)F_{\text{in}}^{-}(\chi_\rho^- f)+f_{\text{in}}^{-}(k;x)F_{\text{in}}^{+}(\chi_\rho^+ f)\right](k=\lambda_j)\\
&\quad+\bar c_{j,l}\frac{\partial^l}{\partial k^l}\left[f_{\text{out}}^{+}(k;x)F_{\text{out}}^{-}(\chi_\rho^- f)+f_{\text{out}}^{-}(k;x)F_{\text{out}}^{+}(\chi_\rho^+ f)\right]\big(k=\bar\lambda_j\big),
\end{aligned}
\tag{2.120}
$$

where the four terms involving $\Phi_{\text{in(out)}}^{\pm}$ belong to $C^0(\mathbb{R}_x)$.

Proof. Given $f\in C_0^\infty(\mathbb{R}_x)$, substracting (2.118) for $p=0$ to (2.118) to $p=1$, we deduce that

$$
\begin{aligned}
f(x) &= \Phi_{\text{in}}^{+}\left(\frac{\mathrm{i}(\kappa-a)}{W_{\text{in}}(\kappa+\mathrm{i}0)}F_{\text{in}}^{-}(f)\right)(x)-\Phi_{\text{out}}^{+}\left(\frac{\mathrm{i}(\kappa-a)}{W_{\text{out}}(\kappa-\mathrm{i}0)}F_{\text{out}}^{-}(f)\right)(x)\\
&\quad+\sum_{\lambda_j\in\sigma_p}\sum_{l=0}^{m_j-1}c_{j,l}\partial_k^l\big((k-a)f_{\text{in}}^{+}(k;x)F_{\text{in}}^{-}(f)\big)(k=\lambda_j)\\
&\quad+\sum_{\lambda_j\in\sigma_p}\sum_{l=0}^{m_j-1}\bar c_{j,l}\partial_k^l\big((k-a)f_{\text{out}}^{+}(k;x)F_{\text{out}}^{-}(f)\big)\big(k=\bar\lambda_j\big)\\
&= \Phi_{\text{in}}^{-}\left(\frac{\mathrm{i}\kappa}{W_{\text{in}}(\kappa+\mathrm{i}0)}F_{\text{in}}^{+}(f)\right)(x)-\Phi_{\text{out}}^{-}\left(\frac{\mathrm{i}\kappa}{W_{\text{out}}(\kappa-\mathrm{i}0)}F_{\text{out}}^{+}(f)\right)(x)\\
&\quad+\sum_{\lambda_j\in\sigma_p}\sum_{l=0}^{m_j-1}c_{j,l}\partial_k^l\big(kf_{\text{in}}^{-}(k;x)F_{\text{in}}^{+}(f)\big)(k=\lambda_j)
\end{aligned}
$$

$$+\sum_{\lambda_j\in\sigma_p}\sum_{l=0}^{m_j-1}\bar{c}_{j,l}\partial_k^l\big(kf_{\text{out}}^-(k;x)F_{\text{out}}^+(f)\big)\big(k=\bar{\lambda}_j\big). \tag{2.121}$$

We write $f=\chi_\rho^- f+\chi_\rho^+ f$ and we apply the previous equalities to get (2.119) and (2.120) for $f\in C_0^\infty(\mathbb{R}_x)$. To extend the result to $f\in L_s^2(\mathbb{R}_x)$, we get from (2.71) that $F_{\text{in(out)}}^\mp(\chi_\rho^\mp f)\in H^s(\mathbb{R}_\kappa)$, hence

$$\frac{\chi_K(\kappa)}{W_{\text{in(out)}}(\kappa+(-)\text{i}0)}F_{\text{in(out)}}^\mp(\chi_\rho^\mp f)\in H^{-s}(\mathbb{R}_\kappa),$$
$$\frac{((1-\chi_K(\kappa))}{W_{\text{in(out)}}(\kappa+(-)\text{i}0)}F_{\text{in(out)}}^\mp(\chi_\rho^\mp f)\in L_1^2(\mathbb{R}_\kappa), \tag{2.122}$$

and we conclude from (2.93) and (2.92) that we have:

$$\Phi_{\text{in(out)}}^+\left(\frac{\text{i}(\kappa-a)}{W_{\text{in(out)}}(\kappa+(-)\text{i}0)}F_{\text{in(out)}}^-(\chi_\rho^- f)\right),$$
$$\Phi_{\text{in(out)}}^-\left(\frac{\text{i}\kappa}{W_{\text{in(out)}}(\kappa+(-)\text{i}0)}F_{\text{in(out)}}^+(\chi_\rho^+ f)\right)\in L_{-s}^2(\mathbb{R}_x).$$

This proves the L_{-s}^2-regularity of the terms of formula (2.119). At last, the C^0-regularity for (2.120) is given by (2.122) and (2.92) with $p=1$ since $H^1\subset C^0$. □

3. Scattering

In this section, we investigate the asymptotic behaviours in time of the solutions of the charged Klein–Gordon equation:

$$\big(\partial_t-\text{i}A(x)\big)^2u-\partial_x^2u+V(x)u=0,\quad t\in\mathbb{R},\ x\in\mathbb{R}, \tag{3.1}$$
$$u(t=0,x)=u_0(x),\qquad \partial_tu(t=0,x)=u_1(x), \tag{3.2}$$

where the assumptions on the potentials A and V are of type (1.7) and (1.8). More precisely we assume that there exists $a_+\in\mathbb{R}$ such that $(A-a_+,V)$ satisfies (2.2)–(2.4) for some $a=a_--a_+\in\mathbb{R}\setminus\{0\}$, $\alpha>0$. By putting $u(t,x)=\text{e}^{\text{i}ta_+}v(t,x)$, it is sufficient to investigate the case $a_+=0$. For the sake of simplicity, this is this case that we shall consider from here on. It is convenient to introduce the following Hilbert space of initial amplitude:

$$H_s^1(I):=\big\{f\in L_s^2(I),\ f'\in L_s^2(I)\big\},$$
$$s\in\mathbb{R},\ I\subset\mathbb{R},\ \|f\|_{H_s^1(I)}^2:=\|f\|_{L_s^2(I)}^2+\|f'\|_{L_s^2(I)}^2. \tag{3.3}$$

The Cauchy problem is easily solved:

Lemma 3.1. (1) *For any* $u_0 \in H^1_s(\mathbb{R})$, $u_1 \in L^2_s(\mathbb{R})$, $s \in \mathbb{R}$, *there exists a unique solution* $u \in C^0(\mathbb{R}_t; H^1_s(\mathbb{R}_x)) \cap C^1(\mathbb{R}_t; L^2_s(\mathbb{R}_x))$ *of* (3.1), (3.2).

(2) *If* $s \geqslant 0$, *the energy of this solution is conserved*:

$$\int_{-\infty}^{\infty} \left|\partial_t u(t,x)\right|^2 + \left|\partial_x u(t,x)\right|^2 + \left[V(x) - A^2(x)\right]\left|u(t,x)\right|^2 \mathrm{d}x$$

$$= \int_{-\infty}^{\infty} \left|u_1(x)\right|^2 + \left|u_0(x)\right|^2 + \left[V(x) - A^2(x)\right]\left|u_0(x)\right|^2 \mathrm{d}x. \tag{3.4}$$

(3) *If* u_0 *and* u_1 *are supported in* $]r', r''[$, *then* $u(t,\cdot)$ *is supported in* $]r' - |t|, r'' + |t|[$.

(4) *For any* $u_0 \in H^1_{\mathrm{loc}}(\mathbb{R})$, $u_1 \in L^2_{\mathrm{loc}}(\mathbb{R})$, *there exists a unique solution* $u \in C^0(\mathbb{R}_t; H^1_{\mathrm{loc}}(\mathbb{R}_x)) \cap C^1(\mathbb{R}_t; L^2_{\mathrm{loc}}(\mathbb{R}_x))$ *of* (3.1), (3.2).

Remark. Because of the weak regularity assumptions for A, the solution does not belong a priori to $C^2(\mathbb{R}_t; L^2(\mathbb{R}_x))$, even if $(u_0, u_1) \in H^2(\mathbb{R}) \times H^1(\mathbb{R})$.

Proof of Lemma 3.1. We introduce the group $G_0(t)$ defined on $C_0^\infty(\mathbb{R}_x) \times C_0^\infty(\mathbb{R}_x)$, associated with the free Klein–Gordon equation $\partial_t^2 v - \partial_x^2 v + v = 0$: $G_0(t) : \binom{v(0,\cdot)}{\partial_t v(0,\cdot)} \mapsto \binom{v(t,\cdot)}{\partial_t v(t,\cdot)}$. We remark that

$$\left\| \mathrm{e}^{\mathrm{i}t\langle\xi\rangle} \varphi(\xi) \right\|_{H^s(\mathbb{R}_\xi)} \leqslant C_s \langle t\rangle^{|s|} \|\varphi\|_{H^s(\mathbb{R}_\xi)}.$$

The proof is direct for $s \in \mathbb{N}$, the case $s \geqslant 0$ follows by interpolation, and then the case $s < 0$ by duality. Hence, since $f \in L^2_s(\mathbb{R}_x)$ iff $\hat{f}(\xi) \in H^s(\mathbb{R}_\xi)$, and $f \in H^1_s(\mathbb{R}_x)$ iff $\langle\xi\rangle \hat{f}(\xi) \in H^s(\mathbb{R}_\xi)$, an easy Fourier analysis shows that $G_0(t)$ is a strongly continuous group on $H^1_s(\mathbb{R}_x) \times L^2_s(\mathbb{R}_x)$, with

$$\left\| G_0(t) \right\|_{\mathcal{L}(H^1_s(\mathbb{R}_x) \times L^2_s(\mathbb{R}_x))} \leqslant C_s \langle t\rangle^{|s|}. \tag{3.5}$$

We consider the integral equation:

$$U(t) = G_0(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} + \int_0^t G_0(t-\tau)\begin{pmatrix} 0 & 0 \\ A^2 - V + 1 & 2\mathrm{i}A \end{pmatrix} U(\tau)\,\mathrm{d}\tau. \tag{3.6}$$

We solve (3.6) by Picard iteration. We introduce:

$$U_0(t) := G_0(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \qquad U_{n+1}(t) := \int_0^t G_0(t-\tau)\begin{pmatrix} 0 & 0 \\ A^2 - V + 1 & 2\mathrm{i}A \end{pmatrix} U_n(\tau)\,\mathrm{d}\tau.$$

Since $A, V \in L^\infty(\mathbb{R}_x)$, we deduce from (3.5) that there exists $C > 0$ such that for any $n \in \mathbb{N}$ we have:

$$\left\| U_n(t) \right\|_{H_s^1(\mathbb{R}_x)\times L_s^2(\mathbb{R}_x)} \leqslant \frac{1}{n!}\left(C\langle t\rangle^{|s|}\right)^{n+1}|t|^n.$$

Therefore

$$U(t) := \sum_{n=0}^{\infty} U_n(t)$$

is solution of (3.6) in $C^0(\mathbb{R}_t; H_s^1(\mathbb{R}_x) \times L_s^2(\mathbb{R}_x))$ and satisfies:

$$\left\| U(t) \right\|_{H_s^1(\mathbb{R}_x)\times L_s^2(\mathbb{R}_x)} \leqslant C\langle t\rangle^{|s|}\mathrm{e}^{C\langle t\rangle^{|s|+1}}. \tag{3.7}$$

Its first component is solution of (3.1), (3.2) in $C^0(\mathbb{R}_t; H_s^1(\mathbb{R}_x)) \cap C^1(\mathbb{R}_t; L_s^2(\mathbb{R}_x))$.

To prove the conservation of the energy when $s \geqslant 0$, we choose $\theta \in C_0^\infty(\mathbb{R}_t)$, $0 \leqslant \theta(t)$, $\int \theta(t)\,\mathrm{d}y = 1$, and we put $\theta_n(t) := n\theta(nt)$. Then $u_n(t,x) := \int \theta(s)u(t-s,x)\,\mathrm{d}s$ is solution of (3.1) in $C^\infty(\mathbb{R}_t; H^1(\mathbb{R}_x))$, therefore in $C^\infty(\mathbb{R}_t; H^2(\mathbb{R}_x))$. Then an integration by parts yields the conservation of the energy of $u_n(t,x)$. Since $u_n \to u$ in $C^0(\mathbb{R}_t; H^1(\mathbb{R}_x)) \cap C^1(\mathbb{R}_t; L^2(\mathbb{R}_x))$ as $n \to \infty$, the energy of u is also conserved.

Thanks to the property of propagation with finite velocity for the free Klein–Gordon equation, $U_0(t)$ is supported in $]r' - |t|, r'' + |t|[$ when u_0 and u_1 are supported in $]r', r''[$; moreover if $U_n(t)$ is supported in $]r' - |t|, r'' + |t|[$, then $U_{n+1}(t)$ is supported in $]r' - |t|, r'' + |t|[$ again, and we get the same for $U(t)$.

To prove uniqueness we consider the solution $v(t,x)$ obtained by the previous method with initial data at time $T \in \mathbb{R}$, $v(T,x) = 0$, $\partial_t v(T,x) = \varphi(x) \in C_0^\infty(\mathbb{R}_x)$. Since $A, V \in L^\infty(\mathbb{R})$, any solution of (3.1) in $C^0(\mathbb{R}_t; H^1_{\mathrm{loc}}(\mathbb{R}_x)) \cap C^1(\mathbb{R}_t; L^2_{\mathrm{loc}}(\mathbb{R}_x))$ belongs to $C^2(\mathbb{R}_t; H^{-1}_{\mathrm{loc}}(\mathbb{R}_x))$. Then if $u \in C^0(\mathbb{R}_t; H^1_{\mathrm{loc}}(\mathbb{R}_x)) \cap C^1(\mathbb{R}_t; L^2_{\mathrm{loc}}(\mathbb{R}_x))$ is solution with $u(t=0) = \partial_t u(t=0) = 0$, we may consider the Wronskian $w(t) := \int u(t,x)\partial_t v(t,x) - v(t,x)\partial_t u(t,x)\,\mathrm{d}x \in C^1(\mathbb{R}_t)$. We have $w'(t) = 0$, $w(0) = 0$, therefore $w(T) = 0$ and we conclude that $u = 0$.

At last the existence of local solutions is obtained in the usual manner. Given $u_0 \in H^1_{\mathrm{loc}}(\mathbb{R}_x)$, $u_1 \in L^2_{\mathrm{loc}}(\mathbb{R}_x)$, we choose $\chi \in C_0^\infty(\mathbb{R}_x)$, $\chi(x) = 1$ for $|x| < 1$. Let $u_n(t,x)$ the solution with initial data $u_n(0,x) = \chi(x/n)u_0(x)$, $\partial_t u_n(0,x) = \chi(x/n)u_1(x)$. The finite velocity propagation assures that the sequence $u_n(t,x)$ is locally stationary, hence we obtain the unique local solution by putting $u(t,x) := \lim_{n\to\infty} u_n(t,x)$. □

We define the propagator:

$$G(t): \begin{pmatrix} u(0,\cdot) \\ \partial_t u(0,\cdot) \end{pmatrix} \mapsto \begin{pmatrix} u(t,\cdot) \\ \partial_t u(t,\cdot) \end{pmatrix}, \tag{3.8}$$

where u is solution of (3.1), (3.2). The previous lemma assures that $G(t)$ is a strongly continuous group on $H_s^1(\mathbb{R}) \times L_s^2(\mathbb{R})$, $H^1(\mathbb{R}) \cap \mathcal{E}'(\mathbb{R}) \times L^2(\mathbb{R}) \cap \mathcal{E}'(\mathbb{R})$ and on

$H^1_{\text{loc}}(\mathbb{R}) \times L^2_{\text{loc}}(\mathbb{R})$. We now investigate its asymptotic properties. In order to represent the solutions using the distorted Fourier transforms, we introduce the operators:

$$G^{\pm}_{\text{in(out)}} : \varphi \in C_0^{\infty}(\mathbb{R}_\kappa) \mapsto G^{\pm}_{\text{in(out)}}(\varphi)(t,x) := \Phi^{\pm}_{\text{in(out)}}\big(\mathrm{e}^{\mathrm{i}\kappa t}\varphi\big)(x). \tag{3.9}$$

Lemma 3.2. *The operators $G^{\pm}_{\text{in(out)}}$ can be extended as bounded operators from $L^2_1(\mathbb{R}_\kappa)$ to $C^1(\mathbb{R}_t; L^2_{\text{loc}}(\mathbb{R}_x)) \cap C^0(\mathbb{R}_t; H^1_{\text{loc}}(\mathbb{R}_x))$. For any $\varphi \in L^2_1(\mathbb{R}_\kappa)$, $G^{\pm}_{\text{in(out)}}(\varphi)$, is a solution of* (3.1) *and satisfies for any $R \in \mathbb{R}$*:

$$\mathbf{1}_{I_R^{\pm}} G^{\pm}_{\text{in(out)}}(\varphi) \in C^2\big(\mathbb{R}_t; H^{-1}\big(I_R^{\pm}\big)\big) \cap C^1\big(\mathbb{R}_t; L^2\big(I_R^{\pm}\big)\big) \cap C^0\big(\mathbb{R}_t; H^1\big(I_R^{\pm}\big)\big), \tag{3.10}$$

$$\big\|G^{\pm}_{\text{in(out)}}(\varphi)(t)\big\|_{H^1(I_R^{\pm})} + \big\|\partial_t G^{\pm}_{\text{in(out)}}(\varphi)(t)\big\|_{L^2(I_R^{\pm})} \leqslant C(R)\|\varphi\|_{L^2_1(\mathbb{R}_\kappa)}, \tag{3.11}$$

$$\begin{aligned}&\big\|G^{+}_{\text{in(out)}}\big(\langle\kappa\rangle^{-1}(\kappa-a)\varphi\big)(t)\big\|_{H^1(\mathbb{R}_x)} + \big\|\partial_t G^{+}_{\text{in(out)}}\big(\langle\kappa\rangle^{-1}(\kappa-a)\varphi\big)(t)\big\|_{L^2(\mathbb{R}_x)}\\ &\quad\leqslant C\|\varphi\|_{L^2_1(\mathbb{R}_\kappa)},\end{aligned} \tag{3.12}$$

$$\big\|G^{-}_{\text{in(out)}}\big(\langle\kappa\rangle^{-1}\kappa\varphi\big)(t)\big\|_{H^1(\mathbb{R}_x)} + \big\|\partial_t G^{-}_{\text{in(out)}}\big(\langle\kappa\rangle^{-1}\kappa\varphi\big)(t)\big\|_{L^2(\mathbb{R}_x)} \leqslant C\|\varphi\|_{L^2_1(\mathbb{R}_\kappa)}, \tag{3.13}$$

$$\begin{aligned}&\big\|G^{+}_{\text{in(out)}}(\varphi)(t,x) - \big(\mathcal{F}^{-1}\varphi\big)\big(+(-)x+t\big)\big\|_{H^1(\mathbb{R}_x^+)}\\ &\quad+ \big\|\partial_t G^{+}_{\text{in(out)}}(\varphi)(t,x) - \big(\mathcal{F}^{-1}\varphi\big)'\big(+(-)x+t\big)\big\|_{L^2(\mathbb{R}_x^+)} \to 0, \quad t \to -(+)\infty,\end{aligned} \tag{3.14}$$

$$\begin{aligned}&\big\|G^{-}_{\text{in(out)}}(\varphi)(t,x) - \mathrm{e}^{+(-)\mathrm{i}ax}\big(\mathcal{F}^{-1}\varphi\big)\big(-(+)x+t\big)\big\|_{H^1(\mathbb{R}_x^-)}\\ &\quad+ \big\|\partial_t G^{-}_{\text{in(out)}}(\varphi)(t,x) - \mathrm{e}^{+(-)\mathrm{i}ax}\big(\mathcal{F}^{-1}\varphi\big)'\big(-(+)x+t\big)\big\|_{L^2(\mathbb{R}_x^-)} \to 0,\\ &t \to -(+)\infty,\end{aligned} \tag{3.15}$$

$$\big\|G^{\pm}_{\text{in(out)}}(\varphi)(t,x)\big\|_{H^1(\mathbb{R}_x^{\pm})} + \big\|\partial_t G^{\pm}_{\text{in(out)}}(\varphi)(t,x)\big\|_{L^2(\mathbb{R}_x^{\pm})} \to 0, \quad t \to +(-)\infty. \tag{3.16}$$

Moreover for any $\varphi \in H^1_p(\mathbb{R}_\kappa)$, $p = 0, 1$, we have:

$$\big\|\Phi^{\pm}_{\text{in(out)}}(\varphi)\big\|_{H_1^p(I_R^{\pm})} \leqslant C(R)\|\varphi\|_{H^1_p(\mathbb{R}_\kappa)}, \tag{3.17}$$

$$\big\|\Phi^{+}_{\text{in(out)}}\big(\langle\kappa\rangle(\kappa-a)\varphi\big)\big\|_{H_1^p(\mathbb{R})} + \big\|\Phi^{-}_{\text{in(out)}}\big(\langle\kappa\rangle\kappa\varphi\big)\big\|_{H_1^p(\mathbb{R})} \leqslant C\|\varphi\|_{H^1_p(\mathbb{R}_\kappa)}, \tag{3.18}$$

$$\big\|G^{\pm}_{\text{in(out)}}(\varphi)(t)\big\|_{H^1_1(I_R^{\pm})} + \big\|\partial_t G^{\pm}_{\text{in(out)}}(\varphi)(t)\big\|_{L^2_1(I_R^{\pm})} \leqslant C(R)\langle t\rangle\|\varphi\|_{H^1_1(\mathbb{R}_\kappa)}, \tag{3.19}$$

$$\big\|G^{\pm}_{\text{in(out)}}(\varphi)(t)\big\|_{H^1_1(I_R^{\pm})} + \big\|\partial_t G^{\pm}_{\text{in(out)}}(\varphi)(t)\big\|_{L^2_1(I_R^{\pm})} \to 0, \quad t \to +(-)\infty. \tag{3.20}$$

Here we have denoted by $\mathcal{F}^{-1}$ the inverse Fourier transform,

$$\mathcal{F}^{-1}(\varphi)(x) := \frac{1}{2\pi}\mathcal{F}(\varphi)(-x).$$

Proof. When $\varphi \in L^2_1(\mathbb{R}_\kappa)$, the map $t \in \mathbb{R} \mapsto \mathrm{e}^{\mathrm{i}\kappa t}\varphi(\kappa)$ belongs to

$$\bigcap_{p=-1}^{1} C^{1-p}\big(\mathbb{R}_t; L^2_p(\mathbb{R}_\kappa)\big).$$

Then (2.90) and (2.92) assure that $G^{\pm}_{\mathrm{in(out)}}(\varphi)$ is well defined, satisfies (3.10)–(3.13) with

$$\partial_t G^{\pm}_{\mathrm{in(out)}}(\varphi) = G^{\pm}_{\mathrm{in(out)}}(\mathrm{i}k\varphi)$$

and equation (3.1) since $f^{\pm}_{\mathrm{in(out)}}$ is solution of (2.1). Thanks to (3.11), to prove the scattering properties (3.14) to (3.16), it is sufficient to consider $\varphi \in C^\infty_0(\mathbb{R}_\kappa)$. We write:

$$G^{+}_{\mathrm{in(out)}}(\varphi)(t,x) = \big(\mathcal{F}^{-1}\varphi\big)\big(+(-)x+t\big) + R^+_1(t,x) + R^+_2(t,x),$$

with

$$R^+_1(t,x) := \big(\mathrm{e}^{+(-)\mathrm{i}\int_x^\infty A(y)\,\mathrm{d}y} - 1\big)\big(\mathcal{F}^{-1}\varphi\big)\big(+(-)x+t\big),$$

$$R^+_2(t,x) := \frac{\mathrm{i}}{2\pi t}\mathrm{e}^{+(-)\mathrm{i}\int_x^\infty A(y)\,\mathrm{d}y}\int \mathrm{e}^{\mathrm{i}\kappa t}\frac{\mathrm{d}}{\mathrm{d}\kappa}\big(\mathrm{e}^{+(-)\mathrm{i}\kappa x}\varphi(\kappa)\varepsilon^+_{\mathrm{in(out)}}(\kappa;x)\big)\,\mathrm{d}\kappa.$$

It is clear that $\|(\mathcal{F}^{-1}\varphi)(+(-)x+t)\|_{H^1(\mathbb{R}^+_x)} \to 0$ as $t \to +(-)\infty$, and $\|R^+_1(t,x)\|_{H^1(\mathbb{R}^+_x)} + \|\partial_t R^+_1(t,x)\|_{L^2(\mathbb{R}^+_x)} \to 0$ as $|t| \to \infty$. By (2.10) we get $|R^+_2(t,x)| + |\partial_x R^+_2(t,x)| + |\partial_t R^+_2(t,x)| \leqslant C(R,\beta)\mathrm{e}^{-\beta x}|t|^{-1}$. We conclude that (3.14) and (3.16) for exponent $+$ are proved. We write also:

$$G^{-}_{\mathrm{in(out)}}(\varphi)(t,x) = \mathrm{e}^{+(-)\mathrm{i}ax}\big(\mathcal{F}^{-1}\varphi\big)\big(-(+)x+t\big) + R^-_1(t,x) + R^-_2(t,x),$$

with

$$R^-_1(t,x) := \mathrm{e}^{+(-)\mathrm{i}ax}\big(\mathrm{e}^{+(-)\mathrm{i}\int_{-\infty}^x [A(y)-a]\,\mathrm{d}y} - 1\big)\big(\mathcal{F}^{-1}\varphi\big)\big(-(+)x+t\big),$$

$$R^-_2(t,x) := \frac{\mathrm{e}^{+(-)\mathrm{i}ax}}{2\pi t}\mathrm{e}^{+(-)\mathrm{i}\int_{-\infty}^x [A(y)-a]\,\mathrm{d}y}\int \mathrm{e}^{\mathrm{i}\kappa t}\frac{\mathrm{d}}{\mathrm{d}\kappa}\big(\mathrm{e}^{-(+)\mathrm{i}\kappa x}\varphi(\kappa)\varepsilon^-_{\mathrm{in(out)}}(\kappa;x)\big)\,\mathrm{d}\kappa.$$

As previously we have $\|(\mathcal{F}^{-1}\varphi)(-(+)x+t)\|_{H^1(\mathbb{R}^-_x)} \to 0$ as $t \to +(-)\infty$, and $\|R^-_1(t,x)\|_{H^1(\mathbb{R}^-_x)} + \|\partial_t R^-_1(t,x)\|_{L^2(\mathbb{R}^-_x)} \to 0$ as $|t| \to \infty$. Moreover (2.11) implies that $|R^-_2(t,x)| + |\partial_x R^-_2(t,x)| + |\partial_t R^-_2(t,x)| \leqslant C(R,\beta)\mathrm{e}^{\beta x}|t|^{-1}$. Therefore (3.15), (3.16) are proved. (3.20) is established in the same way. Now (3.17) is a consequence of (2.91) and (2.94), and implies (3.19). Finally we deduce (3.18) from (3.17) and (2.95). □

To take the hyperradiant modes into account, we have to define $G^{\pm}_{\mathrm{in(out)}}$ for singular distributions.

Lemma 3.3. *The operators $G^{\pm}_{\text{in(out)}}$ can be extended as bounded operators from $\mathcal{E}'(\mathbb{R}_\kappa)$ to $C^\infty(\mathbb{R}_t; H^2_{\text{loc}}(\mathbb{R}_x))$, and for $\varphi \in \mathcal{E}'(\mathbb{R}_\kappa)$, $G^{\pm}_{\text{in(out)}}(\varphi)$ is solution of* (3.1)*. Moreover for any compact $K_0 \subset I_K$, $N \in \mathbb{N}$, $l \leqslant 2$, $m \in \mathbb{N}$, there exists $C(K_0, l, m, N) > 0$ such that for any $\varphi \in H^{-N}(\mathbb{R}_\kappa)$, compactly supported in K_0, we have*:

$$\left\| \partial_x^l \partial_t^m G^{\pm}_{\text{in(out)}}(\varphi)(t) \right\|_{L^2_{-N}(\mathbb{R}_x)} \leqslant C(K_0, l, m, N)\langle t\rangle^N \|\varphi\|_{H^{-N}(\mathbb{R}_\kappa)}, \tag{3.21}$$

and for any $s \in \mathbb{R}$, we have:

$$G^{+}_{\text{in(out)}}(\varphi)\big(t, x = s - (+)t\big) \to \mathcal{F}^{-1}(\varphi)\big(+(-)s\big), \quad t \to -(+)\infty, \tag{3.22}$$

$$\mathrm{e}^{-\mathrm{i}at} G^{-}_{\text{in(out)}}(\varphi)\big(t, x = s + (-)t\big) \to \mathrm{e}^{+(-)\mathrm{i}as} \mathcal{F}^{-1}(\varphi)\big(-(+)s\big), \quad t \to -(+)\infty. \tag{3.23}$$

Proof. Given $\varphi \in \mathcal{E}'(\mathbb{R}_\kappa)$, there exists $N \in \mathbb{N}$, $R > 0$, such that $\varphi \in H^{-N}(\mathbb{R}_\kappa)$, $\text{supp}(\varphi) \subset [-R, R]$. Thus $t \in \mathbb{R} \mapsto \mathrm{e}^{\mathrm{i}\kappa t}\varphi(\kappa)$ belongs to $C^\infty(\mathbb{R}_t; H^{-N}(\mathbb{R}_\kappa))$, hence $G^{\pm}_{\text{in(out)}}(\varphi)$ is solution of (3.1) in $C^\infty(\mathbb{R}_t; H^2_{\text{loc}}(\mathbb{R}_x))$ by Lemma 2.10. Now for $\varphi \in H^{-N}(\mathbb{R}_\kappa)$ compactly supported in I_K, (2.93) shows that

$$\begin{aligned}\left\| G^{\pm}_{\text{in(out)}}(\varphi)(t) \right\|_{L^2_{-N}(\mathbb{R}_x)} &\leqslant C(K_0) \left\| \chi_K(\kappa) \mathrm{e}^{\mathrm{i}\kappa t} \varphi(\kappa) \right\|_{H^{-N}(\mathbb{R}_\kappa)} \\ &\leqslant C(K_0, N)\langle t\rangle^N \left\| \varphi(\kappa) \right\|_{H^{-N}(\mathbb{R}_\kappa)}.\end{aligned}$$

Then we deduce from (2.96) that

$$\left\| \partial_x^2 G^{\pm}_{\text{in(out)}}(\varphi)(t) \right\|_{L^2_{-N}(\mathbb{R}_x)} \leqslant C(K_0, N)\langle t\rangle^N \left\| \varphi(\kappa) \right\|_{H^{-N}(\mathbb{R}_\kappa)}.$$

Now $u, u'' \in L^2_{-N}(\mathbb{R}_x)$ iff $\hat{u}, \xi^2\hat{u} \in H^{-N}(\mathbb{R}_\xi)$. We write:

$$\xi\hat{u}(\xi) = \big[\xi\chi(\xi)\big]\hat{u}(\xi) + \frac{1 - \chi(\xi)}{\xi}\big[\xi^2\hat{u}(\xi)\big],$$

where we have chosen some $\chi \in C_0^\infty(\mathbb{R}_\kappa)$ equal to 1 near $\xi = 0$. Thus $u' \in L^2_{-N}(\mathbb{R}_x)$ and we obtain (3.21) for $l = 0, 1, 2$, $m = 0$. To get the result for any m, we replace $\varphi(\kappa)$ by $\kappa^m\varphi(\kappa)$. To establish the asymptotic behaviours, we write:

$$G^{\pm}_{\text{in(out)}}(\varphi)\big(t, x = s \mp (\pm)t\big) = \big\langle \varphi(\kappa), \mathrm{e}^{\mathrm{i}\kappa t} f^{\pm}_{\text{in(out)}}\big(\kappa; s \mp (\pm)t\big)\big\rangle_{\mathcal{E}'(\mathbb{R}_\kappa), \mathcal{E}(\mathbb{R}_\kappa)},$$

and obtain (3.22) and (3.23), since Proposition 2.1 gives:

$$\mathrm{e}^{\mathrm{i}\kappa t} f^{+}_{\text{in(out)}}\big(\kappa; s - (+)t\big) \to \mathrm{e}^{+(-)\mathrm{i}\kappa s} \quad \text{in } \mathcal{E}(\mathbb{R}_\kappa),\ t \to -(+)\infty,$$

$$\mathrm{e}^{\mathrm{i}(\kappa - a)t} f^{-}_{\text{in(out)}}\big(\kappa; s + (-)t\big) \to \mathrm{e}^{+(-)\mathrm{i}(\kappa - a)s} \quad \text{in } \mathcal{E}(\mathbb{R}_\kappa),\ t \to -(+)\infty. \qquad \square$$

It will be useful to introduce the operators:

$$E^{\pm}_{\mathrm{in(out)}}: (u_0, u_1) \mapsto E^{\pm}_{\mathrm{in(out)}}(u_0, u_1)(k) := kF^{\pm}_{\mathrm{in(out)}}(u_0) - \mathrm{i}F^{\pm}_{\mathrm{in(out)}}(u_1 - 2\mathrm{i}Au_0). \quad (3.24)$$

Lemma 3.4. *There exists $C > 0$ such that*

$$\left\| E^{\pm}_{\mathrm{in(out)}}(u_0, u_1) \right\|_{L^2(\mathbb{R}_\kappa)} \leqslant C\left(\|u_0\|_{H^1 \cap L^2_1(\mathbb{R}_x)} + \|u_1\|_{L^2_1(\mathbb{R}_x)} \right), \quad (3.25)$$

$$\left\| \langle\kappa\rangle^{-1}(\kappa - a)E^{+}_{\mathrm{in(out)}}(u_0, u_1) \right\|_{H^1(\mathbb{R}_\kappa)} \leqslant C\left(\|u_0\|_{H^1_1(\mathbb{R}_x)} + \|u_1\|_{L^2_1(\mathbb{R}_x)} \right), \quad (3.26)$$

$$\left\| \langle\kappa\rangle^{-1}\kappa E^{-}_{\mathrm{in(out)}}(u_0, u_1) \right\|_{H^1(\mathbb{R}_\kappa)} \leqslant C\left(\|u_0\|_{H^1_1(\mathbb{R}_x)} + \|u_1\|_{L^2_1(\mathbb{R}_x)} \right). \quad (3.27)$$

For any $R \in \mathbb{R}$, there exists $C(R) > 0$ such that for any $u_0^{\pm}, u_1^{\pm}$ supported in $I_R^{\pm}$ we have:

$$\left\| E^{\pm}_{\mathrm{in(out)}}\left(u_0^{\pm}, u_1^{\pm}\right) \right\|_{H^1(\mathbb{R}_\kappa)} \leqslant C(R)\left(\|u_0^{\pm}\|_{H^1_1(\mathbb{R}_x)} + \|u_1^{\pm}\|_{L^2_1(\mathbb{R}_x)} \right). \quad (3.28)$$

For any $s \geqslant 0$, $E^{+(-)}_{\mathrm{in/out}}$ is bounded from $L^2_s(\mathbb{R}_x) \times L^2_s(\mathbb{R}_x)$ to $H^s_{\mathrm{loc}}(\mathbb{R}_\kappa \setminus \{a\})$ (to $H^s_{\mathrm{loc}}(\mathbb{R}_\kappa \setminus \{0\})$).

Proof. The third assertion of Proposition 2.9 implies (3.25). We show (3.26), the proof of (3.27) is analogous. Firstly (2.73) and (2.74) assure that $\langle\kappa\rangle^{-1}(\kappa - a)\kappa F^{+}_{\mathrm{in(out)}}(u_0) \in H^1_{\mathrm{loc}} \cap L^2(\mathbb{R}_\kappa)$ and $\langle\kappa\rangle^{-1}(\kappa - a)F^{+}_{\mathrm{in(out)}}(u_1 - 2\mathrm{i}Au_0) \in H^1(\mathbb{R}_\kappa)$ for $u_0 \in H^1 \cap L^2_1(\mathbb{R}_x)$, $u_1 \in L^2_1(\mathbb{R}_x)$. Thus to prove (3.26), it is sufficient to show that for $R > |a|$, we have:

$$\left\| \kappa\left(F^{+}_{\mathrm{in}}(u_0)\right)' \right\|_{L^2(|\kappa| > R)} \leqslant C\|u_0\|_{H^1_1(\mathbb{R}_x)}. \quad (3.29)$$

We take the first derivative of (2.77) to get:

$$\begin{aligned}
&\kappa\left(F^{\pm}_{\mathrm{in(out)}}\left(\chi^{\pm}_\rho u_0\right)\right)' \\
&\quad = -2F^{\pm}_{\mathrm{in(out)}}(\chi^{\pm}_\rho u_0) + \kappa^{-1}F^{\pm}_{\mathrm{in(out)}}\left(\mathrm{i}(\chi^{\pm}_\rho u_0)' + 2A\chi^{\pm}_\rho u_0\right) \\
&\qquad + \left(F^{\pm}_{\mathrm{in(out)}}\left(\mathrm{i}(\chi^{\pm}_\rho u_0)' + 2A\chi^{\pm}_\rho u_0\right)\right)' \\
&\qquad + \kappa^{-1}\left(F^{\pm}_{\mathrm{in(out)}}\left(\left(V - A^2\right)\chi^{\pm}_\rho u_0 - \mathrm{i}A(\chi^{\pm}_\rho u_0)'\right)\right)' \\
&\qquad + \kappa^{-1}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\kappa x}\left[\mathrm{i}\partial_x\varepsilon^{\pm}_{\mathrm{in(out)}}(\kappa; x)x(\chi^{\pm}_\rho u_0)'(x) + \partial_\kappa\partial_x\varepsilon^{\pm}_{\mathrm{in(out)}}(\kappa; x)(\chi^{\pm}_\rho u_0)'(x)\right]\mathrm{d}x.
\end{aligned}$$

Since $u_0 \in H^1_1(\mathbb{R}_x)$, (2.74) and (2.10), (2.11) assure that each term of the right-hand side belongs to $L^2(|\kappa| > R)$, hence

$$\left\| \kappa\left(F^{\pm}_{\mathrm{in(out)}}(\chi^{\pm}_\rho u_0)\right)' \right\|_{L^2(|\kappa| > R)} \leqslant C\|u_0\|_{H^1_1(\mathbb{R}_x)}.$$

We now get (3.29) by writing,

$$F^{+}_{\mathrm{in(out)}}(u_0) = F^{+}_{\mathrm{in(out)}}(\chi^{+}_\rho u_0) + \rho^{+}_{\mathrm{in(out)}}(\kappa)F^{-}_{\mathrm{in(out)}}(\chi^{-}_\rho u_0) + \tau^{+}_{\mathrm{in(out)}}(\kappa)F^{-}_{\mathrm{out(in)}}(\chi^{-}_\rho u_0),$$

and taking Lemma 2.5 into account. Then we deduce (3.28) from (2.71) and (3.29). Finally, when $u_0, u_1 \in L^2_s(\mathbb{R}_x)$, (2.74) assures that $E^{+(-)}_{\mathrm{in/out}}(u_0,u_1) \in H^s_{\mathrm{loc}}(\mathbb{R}_\kappa \setminus \{a(0)\})$. $\square$

When u_0 and u_1 are not compactly supported, we cannot define $E^{\pm}_{\mathrm{in(out)}}(u_0,u_1)(k)$ for non real k, nevertheless, for $\lambda_j \in \sigma_p$, we may introduce the quantities denoted $(\frac{\mathrm{d}^l}{\mathrm{d}^l_k}) E^{\pm}_{\mathrm{in(out)}}(u_0,u_1)(k=\lambda_j(=\bar{\lambda}_j))$ using the convention (2.88). Thus we put:

$$\begin{aligned}
&\frac{\mathrm{d}^l}{\mathrm{d}k^l} E^{\pm}_{\mathrm{in(out)}}(u_0,u_1)\big(k=\lambda_j\big(=\bar{\lambda}_j\big)\big)\\
&\quad := \int_{-\infty}^{\infty} \big(k\partial_k^l f^{\pm}_{\mathrm{in(out)}} + l\partial_k^{l-1} f^{\pm}_{\mathrm{in(out)}}\big)\big(k=\lambda_j\big(=\bar{\lambda}_j\big);x\big)u_0(x)\,\mathrm{d}x\\
&\qquad - \mathrm{i}\int_{-\infty}^{\infty} \partial_k^l f^{\pm}_{\mathrm{in(out)}}\big(k=\lambda_j\big(=\bar{\lambda}_j\big);x\big)\big(u_1(x)-2\mathrm{i}A(x)u_0(x)\big)\,\mathrm{d}x.
\end{aligned} \tag{3.30}$$

We are now ready to give a representation of the solution of (3.1) involving the distorted Fourier transforms. We introduce the Hilbert space of initial data:

$$X := H^1_{\max(\nu,1)}(\mathbb{R}_x) \times L^2_{\max(\nu,1)}(\mathbb{R}_x), \tag{3.31}$$

where ν defined by (2.97) is the largest multiplicity of the hyperradiant modes.

Proposition 3.5. *For any $(u_0,u_1) \in X$, the unique solution u of* (3.1), (3.2) *that belongs to $C^0(\mathbb{R}_t; H^1_{\max(\nu,1)}(\mathbb{R}_x)) \cap C^1(\mathbb{R}_t; L^2_{\max(\nu,1)}(\mathbb{R}_x))$, is expressed by*:

$$\begin{aligned}
u(t) &= G^{\mp}_{\mathrm{in}}\left(\frac{\mathrm{i}}{W_{\mathrm{in}}(\kappa+\mathrm{i}0)} E^{\pm}_{\mathrm{in}}(u_0,u_1)\right)(t) - G^{\mp}_{\mathrm{out}}\left(\frac{\mathrm{i}}{W_{\mathrm{out}}(\kappa-\mathrm{i}0)} E^{\pm}_{\mathrm{out}}(u_0,u_1)\right)(t)\\
&\quad + \sum_{\lambda_j\in\sigma_p}\sum_{l=0}^{m_j-1} c_{j,l}\,\partial_k^l\big(\mathrm{e}^{\mathrm{i}kt} f^{\mp}_{\mathrm{in}}(k;x)E^{\pm}_{\mathrm{in}}(u_0,u_1)\big)(k=\lambda_j)\\
&\quad + \sum_{\lambda_j\in\sigma_p}\sum_{l=0}^{m_j-1} \bar{c}_{j,l}\,\partial_k^l\big(\mathrm{e}^{\mathrm{i}kt} f^{\mp}_{\mathrm{out}}(k;x)E^{\pm}_{\mathrm{out}}(u_0,u_1)\big)\big(k=\bar{\lambda}_j\big),
\end{aligned} \tag{3.32}$$

where the two terms involving $G^{\pm}_{\mathrm{in(out)}}$ belong to $C^1(\mathbb{R}_t; L^2_{\mathrm{loc}}(\mathbb{R}_x)) \cap C^0(\mathbb{R}_t; H^1_{\mathrm{loc}}(\mathbb{R}_x))$, and the constants $c_{j,l}$ are defined in Theorem 2.12.

Proof. Since $\sigma_{ss} \cap \{0,a\} = \emptyset$, Lemmas 3.15 and 3.4 imply that

$$\frac{\mathrm{i}}{W_{\mathrm{in(out)}}(\kappa+(-)\mathrm{i}0)} E^{\pm}_{\mathrm{in(out)}}(u_0,u_1) \in H^t \cap \mathcal{E}'(\mathbb{R}_\kappa) + L^2_1(\mathbb{R}_\kappa), \quad t < -\nu + \frac{1}{2}. \tag{3.33}$$

Thus Lemmas 3.2 and 3.3 assure that the right-hand side of (3.32), that we denote $u^{\pm}(t)$, is solution of (3.1) in $C^1(\mathbb{R}_t; L^2_{\text{loc}}(\mathbb{R}_x)) \cap C^0(\mathbb{R}_t; H^1_{\text{loc}}(\mathbb{R}_x))$. Taking $t = 0$, Theorem 2.12 gives $u^{\pm}(0) = u_0$, and

$$\partial_t u^{\pm}(0) = u_1 - 2\mathrm{i}Au_0 - \Phi^{\mp}_{\text{in}}\left(\frac{\kappa^2}{W_{\text{in}}(\kappa + \mathrm{i}0)}\big[F^{\pm}_{\text{in}}(u_0)\big]\right) + \Phi^{\mp}_{\text{out}}\left(\frac{\kappa^2}{W_{\text{out}}(\kappa - \mathrm{i}0)}\big[F^{\pm}_{\text{out}}(u_0)\big]\right)$$
$$- \sum_{\lambda_j \in \sigma_p} \sum_{l=0}^{m_j - 1} c_{j,l}\partial^l_k\big(k^2 f^{\mp}_{\text{in}}(k;x)F^{\pm}_{\text{in}}(u_0)\big)(k = \lambda_j)$$
$$+ \bar{c}_{j,l}\partial^l_k\big(k^2 f^{\mp}_{\text{out}}(k;x)F^{\pm}_{\text{out}}(u_0)\big)\big(k = \bar{\lambda}_j\big). \tag{3.34}$$

Since

$$k^2 f^{\pm}_{\text{in/out}}(k;x) = -\partial^2_x f^{\pm}_{\text{in/out}}(k;x) + 2kA(x) f^{\pm}_{\text{in/out}}(k;x)$$
$$+ \big[V(x) - A^2(x)\big] f^{\pm}_{\text{in/out}}(k;x), \tag{3.35}$$

and for any $\varphi \in C^\infty_0(\mathbb{R}_\kappa)$:

$$\Phi^{\pm}_{\text{in/out}}\big(\kappa^2\varphi\big)(x) = -\partial^2_x \Phi^{\pm}_{\text{in/out}}(\varphi)(x) + 2A(x)\Phi^{\pm}_{\text{in/out}}(\kappa\varphi)(x)$$
$$+ \big[V(x) - A^2(x)\big]\Phi^{\pm}_{\text{in/out}}(\varphi)(x), \tag{3.36}$$

we conclude, with Theorem 2.12, that $\partial_t u^{\pm}(0) = u_1$. □

To be able to investigate the peculiar role of the hyperradiant modes, we need a microlocalization near these frequencies and as $x \to \pm\infty$, $t \to \pm\infty$.

Lemma 3.6. *For $k \in \sigma_p \cup \sigma_{ss}$, $0 \leqslant l \leqslant m(k) - 1$, there exists $g^{\pm}_{\text{in(out)}}(k,l;x) \in C^\infty_0(\mathbb{R}_x)$, such that for any $k, k' \in \sigma_p \cup \sigma_{ss}$, $l, l' \in \mathbb{N}$, $l \leqslant m(k) - 1$, $l' \leqslant m(k') - 1$:*

$$\frac{\mathrm{d}^l}{\mathrm{d}k^l} F^{\pm}_{\text{in}}\big(g^{\pm}_{\text{in}}(k',l')\big)(k) = \frac{\mathrm{d}^l}{\mathrm{d}k^l} F^{\pm}_{\text{out}}\big(g^{\pm}_{\text{out}}(k',l')\big)\big(\bar{k}\big) = \delta^{(k',l')}_{(k,l)} \ (\textit{Kroneker symbol}), \tag{3.37}$$

$$\frac{\mathrm{d}^l}{\mathrm{d}k^l} F^{\pm}_{\text{in}}\big(g^{\pm}_{\text{out}}(k',l')\big)(k) = \frac{\mathrm{d}^l}{\mathrm{d}k^l} F^{\pm}_{\text{out}}\big(g^{\pm}_{\text{in}}(k',l')\big)\big(\bar{k}\big) = 0. \tag{3.38}$$

Proof. We only construct the family $g^{+}_{\text{in(out)}}$. We can treat $g^{-}_{\text{in(out)}}$ in a similar way. We show that the map $\Theta : g \in C^\infty_0(\mathbb{R}_x) \mapsto (\partial^l_k F^{+}_{\text{in}}(g)(k), \partial^l_k F^{+}_{\text{out}}(g)(\bar{k}))_{k \in \sigma_p \cup \sigma_{ss},\ l \leqslant m(k)-1} \in \mathbb{C}^M$, $M = \sum_{k \in \sigma_p \cup \sigma_{ss}} \sum_{l \leqslant m(k)-1} 2$, is onto. If $\alpha = (\alpha^{\text{in}}_{k,l}, \alpha^{\text{out}}_{k,l})_{k,l} \in (\operatorname{Ran}\Theta)^{\perp} \subset \mathbb{C}^M$, we have:

$$\sum_{k \in \sigma_p \cup \sigma_{ss}} \sum_{l \leqslant m(k)-1} \alpha^{\text{in}}_{k,l}\partial^l_k f^{+}_{\text{in}}(k;\cdot) + \alpha^{\text{out}}_{k,l}\partial^l_k f^{+}_{\text{out}}\big(\bar{k};\cdot\big) = 0.$$

We prove that $\alpha = 0$. It is convenient to introduce some sets of indices:

$$E_0 := \big\{(k,l);\ k \in \sigma_p \cup \sigma_{ss},\ 0 \leqslant l \leqslant m(k)-1\big\}, \qquad E_{n+1} := E_n \setminus D_n,$$
$$\gamma_n := \min\big\{\Im k;\ \exists l,\ (k,l) \in E_n\big\}, \qquad L_n := \max\big\{l;\ \exists k,\ \Im k = \gamma_n,\ (k,l) \in E_n\big\},$$
$$D_n := \big\{(k, L_n) \in E_n;\ \Im k = \gamma_n\big\}.$$

To show that α is null, it is sufficient to prove that when

$$\sum_{(k,l)\in E_n} \alpha^{\text{in}}_{k,l} \partial^l_k f^+_{\text{in}}(k;\cdot) + \alpha^{\text{out}}_{k,l} \partial^l_k f^+_{\text{out}}\big(\bar{k};\cdot\big) = 0, \tag{3.39}$$

we have:

$$\forall (k,l) \in D_n, \quad \alpha^{\text{in}}_{k,l} = \alpha^{\text{out}}_{k,l} = 0. \tag{3.40}$$

We get from the asymptotic behaviours of the Jost functions as $x \to +\infty$ given by (2.5), and from (2.10),

$$\sum_{(k,l)\in E_n} \alpha^{\text{in}}_{k,l} \partial^l_k f^+_{\text{in}}(k;\cdot) + \alpha^{\text{out}}_{k,l} \partial^l_k f^+_{\text{out}}\big(\bar{k};\cdot\big) = \mathrm{e}^{-\gamma_n x} (\mathrm{i}x)^{L_n} \sum_{(k,l)\in D_n} \alpha^{\text{in}}_{k,l} \mathrm{e}^{\mathrm{i}x\Re k}$$
$$+ (-1)^{L_n} \alpha^{\text{out}}_{k,l} \mathrm{e}^{-\mathrm{i}x\Re k} + \mathrm{o}\big(\mathrm{e}^{-\gamma_n x} (x)^{L_n}\big).$$

Thus equality (3.39) becomes:

$$\sum_{(k,l)\in D_n} \alpha^{\text{in}}_{k,l} \mathrm{e}^{\mathrm{i}x\Re k} + (-1)^{L_n} \alpha^{\text{out}}_{k,l} \mathrm{e}^{-\mathrm{i}x\Re k} = 0,$$

and we deduce that

$$\forall (k,l) \in D_n, \quad \Re k \neq 0 \ \Rightarrow\ \alpha^{\text{in}}_{k,l} = \alpha^{\text{out}}_{k,l} = 0. \tag{3.41}$$

We now express $f^+_{\text{in(out)}}$ with $f^-_{\text{in(out)}}$, using (2.22), (2.23), and the fact that

$$\partial^l_k f^+_{\text{in(out)}}\big(k\big(\bar{k}\big);\cdot\big) = \partial^l_k \big(\rho^+_{\text{in(out)}}\big(k\big(\bar{k}\big)\big) f^-_{\text{in(out)}}\big(k\big(\bar{k}\big);\cdot\big)\big)$$

for any $k \in \sigma_p \cup \sigma_{ss}$, $l \leqslant m(k)-1$. Hence we get as $x \to -\infty$,

$$\sum_{(k,l)\in E_n} \alpha^{\text{in}}_{k,l} \partial^l_k f^+_{\text{in}}(k;\cdot) + \alpha^{\text{out}}_{k,l} \partial^l_k f^+_{\text{out}}\big(\bar{k};\cdot\big)$$
$$= \mathrm{e}^{\gamma_n x} (-\mathrm{i}x)^{L_n} \sum_{(k,l)\in D_n} \alpha^{\text{in}}_{k,l} \rho^+_{\text{in}}(k) \mathrm{e}^{-\mathrm{i}x(\Re k - a)}$$
$$+ (-1)^{L_n} \alpha^{\text{out}}_{k,l} \rho^+_{\text{out}}\big(\bar{k}\big) \mathrm{e}^{\mathrm{i}x(\Re k - a)} + \mathrm{o}\big(\mathrm{e}^{\gamma_n x} (x)^{L_n}\big).$$

Thus equality (3.39) becomes:

$$\sum_{(k,l)\in D_n} \alpha_{k,l}^{\text{in}}\rho_{\text{in}}^{+}(k)\mathrm{e}^{-\mathrm{i}x(\Re k-a)} + (-1)^{L_n}\alpha_{k,l}^{\text{out}}\rho_{\text{out}}^{+}(\bar{k})\mathrm{e}^{\mathrm{i}x(\Re k-a)} = 0,$$

and we deduce that

$$\forall (k,l)\in D_n, \quad \Re k \neq a \;\Rightarrow\; \alpha_{k,l}^{\text{in}} = \alpha_{k,l}^{\text{out}} = 0, \tag{3.42}$$

and (3.40) follows from (3.41) and (3.42) since $a\neq 0$. □

We now construct solutions of finite energy with polynomial behaviour in time (1.10). We denote $v_{\text{in(out)}}^{\pm}(k,l;t,x)$ the solutions of (3.1), (3.2), with $u(t=0)=0$, $\partial_t u(t=0) = g_{\text{in(out)}}^{\pm}(k,l)$, for $k\in\sigma_{ss}\cup\sigma_p$, $l\leqslant m(k)-1$.

Theorem 3.7. *For all $\kappa_j\in\sigma_{ss}$, $l\leqslant n_j-1$, there exist $c_{\text{in(out)}}^{\pm}\neq 0$, $C>0$, such that for any $t,x\in\mathbb{R}$, we have*:

$$v_{\text{in(out)}}^{\pm}(\kappa_j,l;t,x) = c_{\text{in(out)}}^{\pm} t^{n_j-l-1}\mathrm{e}^{\mathrm{i}\kappa_j t} f_{\text{in(out)}}^{\mp}(\kappa_j;x) + \mathrm{o}\big(t^{n_j-l-1}\big),$$
$$t\to -(+)\infty, \tag{3.43}$$

$$v_{\text{in(out)}}^{\pm}(\kappa_j,l;t,x) = \mathrm{o}\big(t^{n_j-l-1}\big), \quad t\to +(-)\infty, \tag{3.44}$$

$$\big|\partial_x^p\partial_t^q v_{\text{in(out)}}^{\pm}(\kappa_j,l;t,x)\big| \leqslant C\langle t\rangle^{n_j-l-1}\min\big(\langle x\rangle,\langle t\rangle\big), \quad p+q\leqslant 1, \tag{3.45}$$

$$\big\|v_{\text{in(out)}}^{\pm}(\kappa_j,l;t)\big\|_{H_s^1(\mathbb{R}_x)} + \big\|\partial_t v_{\text{in(out)}}^{\pm}(\kappa_j,l;t)\big\|_{L_s^2(\mathbb{R}_x)} \leqslant C\langle t\rangle^{n_j-l+1/2+s}, \quad 0\leqslant s. \tag{3.46}$$

Proof. For $\kappa_j\in\sigma_{ss}$, $0\leqslant l\leqslant n_j-1$, we consider $v_{\text{in}}^{+}(\kappa_j,l;t)$. The other cases are treated in a similar way. We note that

$$F_{\text{in(out)}}^{\pm}\big(g_{\text{in}}^{+}(\kappa_j,l)\big) \in \bigcap_n H^n\cap L_n^2(\mathbb{R}_\kappa). \tag{3.47}$$

By the previous lemma, and from the fact that for $k\in\sigma_p\cup\sigma_{ss}$, $l\leqslant m(k)-1$, we have:

$$\partial_k^l\big(f_{\text{in(out)}}^{-}\big(k\big(\bar{k}\big);x\big)\big) = \partial_k^l\big(\rho_{\text{in(out)}}^{-} f_{\text{in(out)}}^{+}\big)\big(k\big(\bar{k}\big);x\big), \tag{3.48}$$

we deduce that

$$k\in\sigma_p\cup\sigma_{ss}, \quad l'\leqslant m(k)-1 \;\Rightarrow\; \partial_k^{l'}F_{\text{out}}^{\pm}\big(g_{\text{in}}^{+}(\kappa_j,l)\big)\big(\bar{k}\big) = 0, \tag{3.49}$$

$$k\in\sigma_p\cup\sigma_{ss}\setminus\{\kappa_j\}, \quad l'\leqslant m(k)-1 \;\Rightarrow\; \partial_k^{l'}F_{\text{in}}^{\pm}\big(g_{\text{in}}^{+}(\kappa_j,l)\big)(k) = 0, \tag{3.50}$$

$$l'\leqslant l-1 \quad\Rightarrow\quad \partial_k^{l'}F_{\text{in}}^{\pm}\big(g_{\text{in}}^{+}(\kappa_j,l)\big)(\kappa_j) = 0, \tag{3.51}$$

$$\partial_k^{l}F_{\text{in}}^{+}\big(g_{\text{in}}^{+}(\kappa_j,l)\big)(\kappa_j) = 1, \quad \partial_k^{l}F_{\text{in}}^{-}\big(g_{\text{in}}^{+}(\kappa_j,l)\big)(\kappa_j) = \rho_{\text{in}}^{-}(\kappa_j)\neq 0. \tag{3.52}$$

Therefore we get for some $\psi^{\pm}_{\mathrm{in(out)},h} \in C^{\infty}(\mathbb{R}_\kappa)$:

$$F^{\pm}_{\mathrm{in}}\big(g^{+}_{\mathrm{in}}(\kappa_j,l)\big)(\kappa) = (\kappa-\kappa_j)^l \psi^{\pm}_{\mathrm{in},j}(\kappa), \quad \psi^{\pm}_{\mathrm{in},j}(\kappa_j) \neq 0, \tag{3.53}$$

$$i \neq j \quad \Rightarrow \quad F^{\pm}_{\mathrm{in}}\big(g^{+}_{\mathrm{in}}(\kappa_j,l)\big)(\kappa) = (\kappa-\kappa_i)^{n_i} \psi^{\pm}_{\mathrm{in},i}(\kappa), \tag{3.54}$$

$$\forall \kappa_i \in \sigma_{ss}, \quad F^{\pm}_{\mathrm{out}}\big(g^{+}_{\mathrm{in}}(\kappa_j,l)\big)(\kappa) = (\kappa-\kappa_j)^{n_i} \psi^{\pm}_{\mathrm{out,i}}(\kappa). \tag{3.55}$$

With (3.49) and (3.50), Proposition 3.5 gives:

$$v^{+}_{\mathrm{in}}(\kappa_j,l;t) = G^{\mp}_{\mathrm{in}}\left(\frac{1}{W_{\mathrm{in}}(\kappa+\mathrm{i}0)} F^{\pm}_{\mathrm{in}}\big(g^{+}_{\mathrm{in}}(\kappa_j,l)\big)(\kappa)\right)(t) - G^{\mp}_{\mathrm{out}}\left(\frac{1}{W_{\mathrm{out}}(\kappa-\mathrm{i}0)} F^{\pm}_{\mathrm{out}}\big(g^{+}_{\mathrm{in}}(\kappa_j,l)\big)(\kappa)\right)(t). \tag{3.56}$$

We introduce:

$$\varphi^{\pm}_{\mathrm{out}}(\kappa) := -\frac{1}{W_{\mathrm{out}}(\kappa-\mathrm{i}0)} F^{\pm}_{\mathrm{out}}\big(g^{+}_{\mathrm{in}}(\kappa_j,l)\big)(\kappa) \in \bigcap_n H^n \cap L^2_n(\mathbb{R}_\kappa), \tag{3.57}$$

and we can write

$$\frac{1}{W_{\mathrm{in}}(\kappa+\mathrm{i}0)} F^{\pm}_{\mathrm{in}}\big(g^{+}_{\mathrm{in}}(\kappa_j,l)\big)(\kappa) = \frac{\theta^{\pm}(\kappa)}{(\kappa-\kappa_j+\mathrm{i}0)^{n_j-l}} + \varphi^{\pm}_{\mathrm{in}}(\kappa), \tag{3.58}$$

with

$$\theta^{\pm} \in C^{\infty}_0(I_K), \quad \theta^{\pm}(\kappa_j) \neq 0, \quad \varphi^{\pm}_{\mathrm{in}} \in \bigcap_n H^n \cap L^2_n(\mathbb{R}_\kappa). \tag{3.59}$$

Hence we obtain:

$$v^{+}_{\mathrm{in}}(\kappa_j,l;t) = G^{\mp}_{\mathrm{in}}\left(\frac{\theta^{\pm}(\kappa)}{(\kappa-\kappa_j+\mathrm{i}0)^{n_j-l}}\right)(t) + G^{\mp}_{\mathrm{in}}\big(\varphi^{\pm}_{\mathrm{in}}\big)(t) + G^{\mp}_{\mathrm{out}}\big(\varphi^{\pm}_{\mathrm{out}}\big)(t). \tag{3.60}$$

Lemma 3.2 assures that for all $x \in \mathbb{R}$, $n \in \mathbb{N}$,

$$G^{\mp}_{\mathrm{in}}\big(\varphi^{\pm}_{\mathrm{in}}\big)(t,x) + G^{\mp}_{\mathrm{out}}\big(\varphi^{\pm}_{\mathrm{out}}\big)(t,x) \to 0, \quad |t| \to \infty, \tag{3.61}$$

$$\sup_t \big\| \partial^n_t G^{\mp}_{\mathrm{in(out)}}(\varphi^{\pm}_{\mathrm{in(out)}})(t,\cdot) \big\|_{H^1(I^{\mp}_R)} < \infty, \tag{3.62}$$

and using (3.1) we deduce that

$$\sup_t \big\| \partial^2_x G^{\mp}_{\mathrm{in(out)}}\big(\varphi^{\pm}_{\mathrm{in(out)}}\big)(t,\cdot) \big\|_{H^1(I^{\mp}_R)} < \infty. \tag{3.63}$$

We also have for $p = 0, 1$:

$$\partial_x^p \partial_t^n G_{\mathrm{in}}^{\mp}\left(\frac{\theta^{\pm}(\kappa)}{(\kappa-\kappa_j+\mathrm{i}0)^{n_j-l}}\right)(t,x)$$
$$=\sum_{m=0}^{n_j-l-1} \frac{(n_j-l-1)!(it)^m}{m!(n_j-l-1-m)!}$$
$$\times \left\langle \mathrm{P.V.}\left(\frac{1}{\kappa-\kappa_j}\right) - \mathrm{i}\pi\delta_{\kappa_j}; \mathrm{e}^{\mathrm{i}\kappa t}(\mathrm{i}\kappa)^n \frac{\partial^{n_j-l-1-m}}{\partial\kappa^{n_j-l-1-m}}\left(\partial_x^p f_{\mathrm{in}}^{\mp}(\kappa;x)\theta^{\pm}(k)\right)\right\rangle_{\mathcal{D}'(\mathbb{R}_\kappa),\mathcal{D}(\mathbb{R}_\kappa)}. \tag{3.64}$$

We note that for $\varphi \in C_0^\infty(]-r,r[)$, we have:

$$\left\langle \mathrm{P.V.}\left(\frac{1}{\kappa-\kappa_j}\right); \mathrm{e}^{\mathrm{i}\kappa t}\varphi(\kappa)\right\rangle_{\mathcal{D}'(\mathbb{R}_\kappa),\mathcal{D}(\mathbb{R}_\kappa)}$$
$$= \mathrm{e}^{\mathrm{i}\kappa_j t}\varphi(\kappa_j)2\mathrm{i}\int_0^{rt} \frac{\sin(s)}{s}\,\mathrm{d}s + \mathrm{e}^{\mathrm{i}\kappa_j t}\int_{-r}^{r} \mathrm{e}^{\mathrm{i}\sigma t}\left(\int_0^1 \varphi'(\kappa_j+\sigma s)\,\mathrm{d}s\right)\mathrm{d}\sigma, \tag{3.65}$$

hence

$$\left\langle \mathrm{P.V.}\left(\frac{1}{\kappa-\kappa_j}\right); \mathrm{e}^{\mathrm{i}\kappa t}\varphi(\kappa)\right\rangle_{\mathcal{D}'(\mathbb{R}_\kappa),\mathcal{D}(\mathbb{R}_\kappa)} = \frac{t}{|t|}\mathrm{i}\pi\mathrm{e}^{\mathrm{i}\kappa_j t}\varphi(\kappa_j) + \mathrm{o}(1), \quad |t|\to\infty, \tag{3.66}$$
$$\left|\left\langle \mathrm{P.V.}\left(\frac{1}{\kappa-\kappa_j}\right); \mathrm{e}^{\mathrm{i}\kappa t}\varphi(\kappa)\right\rangle_{\mathcal{D}'(\mathbb{R}_\kappa),\mathcal{D}(\mathbb{R}_\kappa)}\right| \leqslant C(r)\left(\|\varphi\|_{L^\infty(\mathbb{R}_\kappa)} + \|\varphi'\|_{L^\infty(\mathbb{R}_\kappa)}\right). \tag{3.67}$$

Then, recalling that $\theta^{\pm}(\kappa_j) \neq 0$, we deduce from (3.66) that

$$\partial_x^p \partial_t^n G_{\mathrm{in}}^{\mp}\left(\frac{\theta^{\pm}(\kappa)}{(\kappa-\kappa_j+\mathrm{i}0)^{n_j-l}}\right)(t,x) = C_n^{\pm} t^{n_j-l-1}\mathrm{e}^{\mathrm{i}\kappa_j t}\partial_x^p f_{\mathrm{in}}^{\mp}(\kappa_j;x) + \mathrm{o}\left(t^{n_j-l-1}\right),$$
$$t\to-\infty,\ C_n^{\pm}\neq 0, \tag{3.68}$$
$$\partial_x^p \partial_t^n G_{\mathrm{in}}^{\mp}\left(\frac{\theta^{\pm}(\kappa)}{(\kappa-\kappa_j+\mathrm{i}0)^{n_j-l}}\right)(t,x) = \mathrm{o}\left(t^{n_j-l-1}\right), \quad t\to+\infty, \tag{3.69}$$

therefore (3.43) and (3.44) follow from (3.60), (3.61), (3.68) and (3.69). Moreover (3.67) and the estimates of Proposition 2.1 assure that for $n\in\mathbb{N}$, $p=0,1$, $R\in\mathbb{R}$, there exists $C(R,n)>0$ such that

$$x\in I_R^{\mp} \quad\Rightarrow\quad \left|\partial_x^p \partial_t^n G_{\mathrm{in}}^{\mp}\left(\frac{\theta^{\pm}(\kappa)}{(\kappa-\kappa_j+\mathrm{i}0)^{n_j-l}}\right)(t,x)\right| \leqslant C(R,n)\langle x\rangle\left(\langle t\rangle+\langle x\rangle\right)^{n_j-l-1}. \tag{3.70}$$

Now if $g_{\mathrm{in}}^{+}(\kappa_j,l)$ is supported in $[-r,+r]$, then

$$|x| \geqslant r + |t| \quad \Rightarrow \quad v_{\text{in}}^{+}(\kappa_j, l; t, x) = 0. \tag{3.71}$$

Thus

$$x \in I_R^{\mp} \quad \Rightarrow \quad \left| \partial_x^p \partial_t^n G_{\text{in}}^{\mp}\left(\frac{\theta^{\pm}(\kappa)}{(\kappa - \kappa_j + \mathrm{i}0)^{n_j - l}} \right)(t, x) \right| \leqslant C(R, n) \langle t \rangle^{n_j - l - 1} \min\big(\langle x \rangle, \langle t \rangle\big), \tag{3.72}$$

and (3.45) follows from (3.62), (3.63) and (3.72). Finally (3.46) is consequence of (3.45) and (3.71). □

Lemma 3.8. *There exists $C(R) > 0$ such that for any $\lambda \in \sigma_p$, $0 \leqslant l \leqslant m(\lambda) - 1$, we have*:

$$\left\| v_{\text{in(out)}}^{\pm}(\lambda, l; t) \right\|_{H^1(\mathbb{R}_x)} + \left\| \partial_t v_{\text{in(out)}}^{\pm}(\lambda, l; t) \right\|_{L^2(\mathbb{R}_x)} \leqslant C \langle t \rangle^{m(\lambda) - l - 1} \mathrm{e}^{-(+)\Im(\lambda) t}. \tag{3.73}$$

Proof. We only treat the case of $v_{\text{in}}^{+}(\lambda_{j_*}, l_*; t, x)$ with $\lambda_{j_*} \in \sigma_p$, $0 \leqslant l_* \leqslant m(\lambda_{j_*}) - 1$. From Lemma 3.6 and (3.48), we get:

$$k \in \sigma_p \cup \sigma_{ss},\ l \leqslant m(k) - 1 \quad \Rightarrow \quad \partial_k^l F_{\text{out}}^{\pm}\big(g_{\text{in}}^{+}(\lambda_{j_*}, l_*)\big)\big(\bar{k}\big) = 0, \tag{3.74}$$

$$k \in \sigma_p \cup \sigma_{ss} \setminus \{\lambda_{j_*}\},\ l \leqslant m(k) - 1 \quad \Rightarrow \quad \partial_k^l F_{\text{in}}^{\pm}\big(g_{\text{in}}^{+}(\lambda_{j_*}, l_*)\big)(k) = 0, \tag{3.75}$$

$$l \leqslant l_* - 1 \quad \Rightarrow \quad \partial_k^l F_{\text{in}}^{\pm}\big(g_{\text{in}}^{+}(\lambda_{j_*}, l_*)\big)(\lambda_{j_*}) = 0, \tag{3.76}$$

and Proposition 3.5 gives:

$$\begin{aligned}
v_{\text{in}}^{+}(\lambda_{j_*}, l_*; t) &= G_{\text{in}}^{-}\left(\frac{1}{W_{\text{in}}(\kappa + \mathrm{i}0)} F_{\text{in}}^{+}\big(g_{\text{in}}^{+}(\lambda_{j_*}, l_*)\big) \right)(t) \\
&\quad - G_{\text{out}}^{-}\left(\frac{1}{W_{\text{out}}(\kappa - \mathrm{i}0)} F_{\text{out}}^{+}\big(g_{\text{in}}^{+}(\lambda_{j_*}, l_*)\big) \right)(t) \\
&\quad - \mathrm{i} \sum_{l = l_*}^{m(\lambda_{j_*}) - 1} \frac{l!}{l_*!(l - l_*)!} c_{j_*, l} \partial_k^{l - l_*}\big(\mathrm{e}^{\mathrm{i}kt} f_{\text{in}}^{-}(k; x)\big)(\lambda_{j_*}) \\
&= G_{\text{in}}^{+}\left(\frac{1}{W_{\text{in}}(\kappa + \mathrm{i}0)} F_{\text{in}}^{-}\big(g_{\text{in}}^{+}(\lambda_{j_*}, l_*)\big) \right)(t) \\
&\quad - G_{\text{out}}^{+}\left(\frac{1}{W_{\text{out}}(\kappa - \mathrm{i}0)} F_{\text{out}}^{-}\big(g_{\text{in}}^{+}(\lambda_{j_*}, l_*)\big) \right)(t) \\
&\quad - \mathrm{i} \sum_{l = l_*}^{m(\lambda_{j_*}) - 1} \frac{l!}{l_*!(l - l_*)!} c_{j_*, l} \partial_k^{l - l_*}\big(\mathrm{e}^{\mathrm{i}kt} \rho_{\text{in}}^{-}(k) f_{\text{in}}^{+}(k; x)\big)(\lambda_{j_*}),
\end{aligned} \tag{3.77}$$

with

$$\frac{1}{W_{\text{in(out)}}(\kappa+(-)\mathrm{i}0)}F^{\pm}_{\text{in(out)}}\big(g^{+}_{\text{in}}(\lambda_{j_*},l_*)\big)\in L^2_1(\mathbb{R}_\kappa). \tag{3.78}$$

Therefore (3.73) follows from Lemma 2.4 and (3.11). □

The energy estimates for the solutions are described in the following:

Theorem 3.9. *There exist $C>0$, $N\in\mathbb{N}$, such that for any $(u_0,u_1)\in X$, we have,*

$$\begin{aligned}
&\left\|G(t)\begin{pmatrix}u_0\\u_1\end{pmatrix}\right\|_{H^1(\mathbb{R}_x)\times L^2(\mathbb{R}_x)}\\
&\quad\leqslant C\Bigg(\left\|\begin{pmatrix}u_0\\u_1\end{pmatrix}\right\|_X+\sum_{\kappa\in\sigma_{ss}}\sum_{l=0}^{m(\kappa)-1}\langle t\rangle^{m(\kappa)-l+1/2}\sum_{\substack{\sharp=+,-\\ \flat=\text{in},\text{out}}}\left|\frac{\mathrm{d}^l}{\mathrm{d}k^l}E^{\sharp}_{\flat}(u_0,u_1)(\kappa)\right|\\
&\qquad+\sum_{\lambda\in\sigma_p}\sum_{l=0}^{m(\lambda)-1}\langle t\rangle^{m(\lambda)-l-1}\sum_{\sharp=+,-}\left|\frac{\mathrm{d}^l}{\mathrm{d}k^l}E^{\sharp}_{\text{in}}(u_0,u_1)(\lambda)\right|\mathrm{e}^{-\Im(\lambda)t}\\
&\qquad+\left|\frac{\mathrm{d}^l}{\mathrm{d}k^l}E^{\sharp}_{\text{out}}(u_0,u_1)\big(\bar\lambda\big)\right|\mathrm{e}^{\Im(\lambda)t}\Bigg),
\end{aligned}\tag{3.79}$$

$$\big\|G(t)\big\|_{\mathcal{L}(X)}\leqslant C\langle t\rangle^N\mathrm{e}^{\gamma|t|},\quad \gamma:=\max_{\lambda\in\sigma_p}\Im\lambda. \tag{3.80}$$

Proof. We introduce:

$$v^{\pm}_1(x):=\mathrm{i}\sum_{k\in\sigma_p\cup\sigma_{ss}}\sum_{l=0}^{m(k)-1}\partial_k E^{\pm}_{\text{in}}(u_0,u_1)(k)g^{\pm}_{\text{in}}(k,l;x)+\partial_k E^{\pm}_{\text{out}}(u_0,u_1)\big(\bar k\big)g^{\text{out}}_{\text{in}}(k,l;x). \tag{3.81}$$

We easily check that for any $k_*\in\sigma_p\cup\sigma_{ss}$, $0\leqslant l_*\leqslant m(k_*)-1$, we have:

$$\partial^{l_*}_k E^{\pm}_{\text{in(out)}}\big(u_0,u_1-v^{\pm}_1\big)\big(k=k_*\big(\bar k_*\big)\big)=0. \tag{3.82}$$

Hence the solution $u(t,x)$ of (3.1), (3.2) is given by:

$$\begin{aligned}
u(t)={}&G^{\mp}_{\text{in}}\left(\frac{\mathrm{i}}{W_{\text{in}}(\kappa+\mathrm{i}0)}E^{\pm}_{\text{in}}\big(u_0,u_1-v^{\pm}_1\big)\right)(t)\\
&-G^{\mp}_{\text{out}}\left(\frac{\mathrm{i}}{W_{\text{out}}(\kappa-\mathrm{i}0)}E^{\pm}_{\text{out}}\big(u_0,u_1-v^{\pm}_1\big)\right)(t)\\
&+\mathrm{i}\sum_{k\in\sigma_p\cup\sigma_{ss}}\sum_{l=0}^{m(k)-1}\partial^l_k E^{\pm}_{\text{in}}(u_0,u_1)(k)v^{\pm}_{\text{in}}(k,l;t)\\
&+\partial^l_k E^{\pm}_{\text{out}}(u_0,u_1)\big(\bar k\big)v^{\text{out}}_{\text{in}}(k,l;t),
\end{aligned}\tag{3.83}$$

and thanks to (3.82) we have:

$$\frac{\mathrm{i}}{W_{\mathrm{in(out)}}(\kappa+(-)\mathrm{i}0)}E^{\pm}_{\mathrm{in(out)}}\big(u_0,u_1-v_1^{\pm}\big)\in L^2_1(\mathbb{R}_\kappa). \tag{3.84}$$

Therefore (3.79) follows from (3.11), (3.46) and (3.73).

To prove (3.80) we firstly consider the case:

$$A(x)=a\mathbf{1}_{]-\infty,0]}(s),\qquad V(x)=0, \tag{3.85}$$

for which $\sigma_p=\emptyset$, $\sigma_{ss}=\{a/2\}$ and $m(a/2)=1$. The solution is easily written, for $|x|>|t|$,

$$x>|t|\quad\Rightarrow\quad u(t,x)=\frac{1}{2}\left(u_0(x+t)+u_0(x-t)+\int_{-t}^{+t}u_1(x+\tau)\,\mathrm{d}\tau\right), \tag{3.86}$$

$$x<-|t|\quad\Rightarrow\quad u(t,x)=\frac{\mathrm{e}^{\mathrm{i}at}}{2}\left(u_0(x+t)+u_0(x-t)+\int_{-t}^{+t}(u_1-\mathrm{i}au_0)(x+\tau)\,\mathrm{d}\tau\right). \tag{3.87}$$

We deduce that for $s\geqslant 0$ there exists $C_s>0$ such that

$$\begin{aligned}&\big\|\big(u(t),\partial_t u(t)\big)\big\|_{H^1_s\times L^2_s(]-\infty,-|t|[)}+\big\|\big(u(t),\partial_t u(t)\big)\big\|_{H^1_s\times L^2_s(]|t|,\infty[)}\\&\quad\leqslant C_s\langle t\rangle^{s+1}\big\|(u_0,u_1)\big\|_{H^1_s\times L^2_s(\mathbb{R})}.\end{aligned} \tag{3.88}$$

Since (4.14) assures that

$$\big\|\big(u(t),\partial_t u(t)\big)\big\|_{H^1_s\times L^2_s(]-|t|,|t|[)}\leqslant C\langle t\rangle^{s+3/2}\big\|(u_0,u_1)\big\|_{H^1_s\times L^2_s(\mathbb{R})}, \tag{3.89}$$

we conclude that the propagator $G_a(t)$ associated to (3.85) satisfies:

$$\big\|G_a(t)\big\|_{\mathcal{L}(X)}\leqslant C\langle t\rangle^{3/2+\max(\nu,1)}. \tag{3.90}$$

Since we have:

$$G(t)=G_a(t)+\int_0^t G_a(t-\tau)\begin{pmatrix}0 & 0\\ A^2-a^2\mathbf{1}_{]-\infty,0[}-V & 2\mathrm{i}(A-a\mathbf{1}_{]-\infty,0[})\end{pmatrix}G(\tau)\,\mathrm{d}\tau, \tag{3.91}$$

and A, V satisfy (2.3), we get the following estimate for the solution $u(t,x)$ of (3.1), (3.2):

$$\left\| \left(u(t), \partial_t u(t)\right) \right\|_X \leqslant C \langle t \rangle^{3/2+\max(\nu,1)} \left(\left\| (u_0, u_1) \right\|_X + \int_{(0,t)} \left\| \left(u(\tau), \partial_t u(\tau)\right) \right\|_{H^1(\mathbb{R}_x) \times L^2(\mathbb{R}_x)} \mathrm{d}\tau \right). \tag{3.92}$$

(3.79) implies that

$$\left\| \left(u(t), \partial_t u(t)\right) \right\|_{H^1(\mathbb{R}_x) \times L^2(\mathbb{R}_x)} \leqslant C \langle t \rangle^{\beta} \mathrm{e}^{\gamma |t|} \left\| (u_0, u_1) \right\|_X, \tag{3.93}$$

with

$$\beta = \max_{\lambda \in \sigma_p} \left(\nu + \frac{1}{2}, m(\lambda) - 1 \right), \qquad \gamma = \max_{\lambda \in \sigma_p} \left(\Im(\lambda) \right).$$

Therefore (3.80) follows from (3.92), (3.93). □

To investigate the scattering states, we must avoid the usual modes and the hyperradiant ones. Hence we introduce the following subspaces of finite codimension in X:

$$X_{\text{in(out)}} := \left\{ (u_0, u_1) \in X;\ \forall k \in \sigma_p \cup \sigma_{ss},\ \forall l \leqslant m(k) - 1, \frac{\mathrm{d}^l}{\mathrm{d}k^l} E^{\pm}_{\text{in(out)}}(u_0, u_1)\left(k(\bar{k})\right) = 0 \right\}, \tag{3.94}$$

$$X_{\text{scatt}} := X_{\text{in}} \cap X_{\text{out}}. \tag{3.95}$$

Lemma 3.10. *$X_{\text{in(out)}}$ and X_{scatt} are well defined, and they are Hilbert subspaces of X, invariant under the action of the group $G(t)$. The map*

$$(u_0, u_1) \mapsto \left(\bar{u}_0, -\bar{u}_1\right) \tag{3.96}$$

is an isometry from X_{in} onto X_{out}.

Proof. Thanks to (3.30), $\frac{\mathrm{d}^l}{\mathrm{d}k^l} E^{\pm}_{\text{in}}(u_0, u_1)(\lambda_j)$ is well defined for any $\lambda_j \in \sigma_p$. Moreover Lemma 3.4 assures that $E^{\pm}_{\text{in}}(u_0, u_1)(\kappa)$ is $H^{\max(\nu,1)}$ on a neighbourhood of σ_{ss}. Thus $E^{\pm}_{\text{in}}(u_0, u_1)(\kappa)$ is $C^{\max(\nu-1,0)}$ near $\kappa_j \in \sigma_{ss}$, and $(u_0, u_1) \mapsto \frac{\mathrm{d}^l}{\mathrm{d}k^l} E^{\pm}_{\text{in}}(u_0, u_1)(\kappa_j)$ is a continuous linear form on X. Hence X_{in} is a Hilbert subspace of X of finite codimension. Now given $k_* \in \sigma_p \cup \sigma_{ss}$, $l \leqslant m(k_*) - 1$, we put:

$$h^{\pm}_l(t) := \frac{\mathrm{d}^l}{\mathrm{d}k^l} E^{\pm}_{\text{in}}\left(u(t), \partial_t u(t)\right)(k_*). \tag{3.97}$$

Using Eqs. (2.1) and (3.1), and an integration by parts, we check that

$$\frac{\mathrm{d}}{\mathrm{d}t}h_0^{\pm}(t) = \mathrm{i}k_* h_0^{\pm}(t), \tag{3.98}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}h_l^{\pm}(t) = \mathrm{i}k_* h_l^{\pm}(t) + \mathrm{i}l h_{l-1}^{\pm}(t), \quad 1 \leqslant l \leqslant m(k_*) - 1. \tag{3.99}$$

If we assume that $h_l^{\pm}(0) = 0$ for $0 \leqslant l \leqslant m(k_*) - 1$, we deduce that $h_l^{\pm}(t) = 0$ for any t, and we conclude that $G(t)$ leaves X_{in} invariant. The proof for X_{out} is similar. To see that (3.96) maps X_{in} onto X_{out} it is sufficient to note that (2.8) implies that

$$E_{\mathrm{in}}^{\pm}(u_0, u_1)(k) = \overline{E_{\mathrm{out}}^{\pm}\big(\bar{u}_0, -\bar{u}_1\big)\big(\bar{k}\big)}. \qquad \square$$

We now arrive at an important result of this work: the solutions, with Cauchy data in X_{scatt}, are asymptotically free. We emphasize that the conserved energy of such solutions given by (1.9) can be negative. In fact we do not use this conservation law to get our scattering theory.

Theorem 3.11. *For any* $(u_0, u_1) \in X_{\mathrm{scatt}}$ *there exists unique* $u_{\mathrm{in(out)}}^{\pm} \in H^1(\mathbb{R})$ *such that*

$$\begin{aligned}&\big\|u(t,x) - \big(\mathrm{e}^{\mathrm{i}ax}u_{\mathrm{in}}^{-}(t-x) + u_{\mathrm{in}}^{+}(t+x)\big)\big\|_{H^1(\mathbb{R}_x)}\\ &\quad + \big\|\partial_t u(t,x) - \big(\mathrm{e}^{\mathrm{i}ax}(u_{\mathrm{in}}^{-})'(t-x) + (u_{\mathrm{in}}^{+})'(t+x)\big)\big\|_{L^2(\mathbb{R}_x)} \to 0, \quad t \to -\infty,\end{aligned} \tag{3.100}$$

$$\begin{aligned}&\big\|u(t,x) - \big(\mathrm{e}^{-\mathrm{i}ax}u_{\mathrm{out}}^{-}(t+x) + u_{\mathrm{out}}^{+}(t-x)\big)\big\|_{H^1(\mathbb{R}_x)}\\ &\quad + \big\|\partial_t u(t,x) - \big(\mathrm{e}^{-\mathrm{i}ax}(u_{\mathrm{out}}^{-})'(t+x) + (u_{\mathrm{out}}^{+})'(t-x)\big)\big\|_{L^2(\mathbb{R}_x)} \to 0,\\ &t \to +\infty.\end{aligned} \tag{3.101}$$

u_0, u_1, $u_{\mathrm{in(out)}}^{\pm}$ *are bound by the following relations*:

$$u_{\mathrm{in(out)}}^{\pm} = +(-)\mathcal{F}^{-1}\left(\frac{\mathrm{i}}{W_{\mathrm{in(out)}}(\kappa + (-)\mathrm{i}0)}E_{\mathrm{in(out)}}^{\mp}(u_0, u_1)\right), \tag{3.102}$$

$$u_0 = \Phi_{\mathrm{in}}^{\pm}\big(\mathcal{F}\big(u_{\mathrm{in}}^{\pm}\big)\big) + \Phi_{\mathrm{out}}^{\pm}\big(\mathcal{F}\big(u_{\mathrm{out}}^{\pm}\big)\big), \tag{3.103}$$

$$u_1 = \Phi_{\mathrm{in}}^{\pm}\big(\mathrm{i}\kappa\mathcal{F}\big(u_{\mathrm{in}}^{\pm}\big)\big) + \Phi_{\mathrm{out}}^{\pm}\big(\mathrm{i}\kappa\mathcal{F}\big(u_{\mathrm{out}}^{\pm}\big)\big), \tag{3.104}$$

$$\big\|u_{\mathrm{in(out)}}^{\pm}\big\|_{H^1(\mathbb{R})} \leqslant C\big\|(u_0, u_1)\big\|_X, \tag{3.105}$$

$$\forall\kappa \in \mathbb{R}\setminus\sigma_{ss}, \quad \begin{pmatrix}\mathcal{F}(u_{\mathrm{out}}^{+})(\kappa)\\ \mathcal{F}(u_{\mathrm{out}}^{-})(\kappa)\end{pmatrix} = \begin{pmatrix}R^{+}(\kappa) & T^{+}(\kappa)\\ T^{-}(\kappa) & R^{-}(\kappa)\end{pmatrix}\begin{pmatrix}\mathcal{F}(u_{\mathrm{in}}^{+})(\kappa)\\ \mathcal{F}(u_{\mathrm{in}}^{-})(\kappa)\end{pmatrix}, \tag{3.106}$$

where $R^{\pm}$ *and* $T^{\pm}$ *are the reflection and transmission coefficients given by* (2.52).

Proof. To establish the uniqueness of $u_{\mathrm{in(out)}}^{\pm}$, we remark that since $H^1(\mathbb{R}) \subset C^0(\mathbb{R})$, (3.100), (3.101) imply:

$$u^{+}_{\mathrm{in(out)}}(s)=\lim_{t\to-(+)\infty}u\big(t,-(+)t+(-)s\big),\tag{3.107}$$

$$u^{-}_{\mathrm{in(out)}}(s)=\lim_{t\to-(+)\infty}\mathrm{e}^{-\mathrm{i}a(t-s)}u\big(t,+(-)t-(+)s\big).\tag{3.108}$$

To prove the existence of $u^{\pm}_{\mathrm{in(out)}}$, we note that when $(u_0,u_1)\in X_{\mathrm{scatt}}$, Theorems 11.5 and 11.8 of [20] assure that $(\kappa-\kappa_j)^{-m(\kappa_j)}E^{\pm}_{\mathrm{in}}(u_0,u_1)(\kappa)$ is L^2 near $\kappa_j\in\sigma_{ss}$. Therefore by Lemma 3.4 and Proposition 3.5, we have:

$$u(t)=G^{\mp}_{\mathrm{in}}\left(\frac{\mathrm{i}}{W_{\mathrm{in}}(\kappa+\mathrm{i}0)}E^{\pm}_{\mathrm{in}}(u_0,u_1)\right)(t)-G^{\mp}_{\mathrm{out}}\left(\frac{\mathrm{i}}{W_{\mathrm{out}}(\kappa-\mathrm{i}0)}E^{\pm}_{\mathrm{out}}(u_0,u_1)\right)(t),$$

with

$$\left\|\frac{\mathrm{i}}{W_{\mathrm{in(out)}}(\kappa+(-)\mathrm{i}0)}E^{\pm}_{\mathrm{in(out)}}(u_0,u_1)\right\|_{L^2_1(\mathbb{R}_\kappa)}\leqslant C\big\|(u_0,u_1)\big\|_X.$$

Then (3.100)–(3.102) and (3.105) follow from Lemma 3.2. Now, sing (2.22) to (2.25), (2.27), (2.28) and (2.32), we calculate that for $\kappa\notin\sigma_{ss}$ and $\kappa\neq0,a$ we have:

$$f^{\pm}_{\mathrm{out}}(\kappa;x)=\frac{1}{\tau^{\mp}_{\mathrm{in}}(\kappa)}f^{\mp}_{\mathrm{in}}(\kappa;x)+\frac{\rho^{\pm}_{\mathrm{out}}(\kappa)}{\tau^{\pm}_{\mathrm{in}}(\kappa)}f^{\pm}_{\mathrm{in}}(\kappa;x).$$

Thus we deduce from (3.102) that

$$\mathcal{F}(u^{\pm}_{\mathrm{out}})(\kappa)=\frac{1}{\tau^{\pm}_{\mathrm{out}}(\kappa)}\mathcal{F}(u^{\mp}_{\mathrm{in}})(\kappa)+\frac{\rho^{\mp}_{\mathrm{out}}(\kappa)}{\tau^{\mp}_{\mathrm{out}}(\kappa)}\mathcal{F}(u^{\pm}_{\mathrm{in}})(\kappa),$$

and (3.106) follows from (2.52) and (2.53). (3.103) is directly obtained using (3.102) and Theorem 2.12 by taking $\frac{\mathrm{d}^l}{\mathrm{d}k^l}E^{\pm}_{\mathrm{in}}(u_0,u_1)(\lambda_j)=\frac{\mathrm{d}^l}{\mathrm{d}k^l}E^{\pm}_{\mathrm{out}}(u_0,u_1)(\bar\lambda_j)=0$ into account. In the same way, we evaluate $\Phi^{\pm}_{\mathrm{in}}(\mathrm{i}\kappa\mathcal{F}(u^{\pm}_{\mathrm{in}}))+\Phi^{\pm}_{\mathrm{out}}(\mathrm{i}\kappa\mathcal{F}(u^{\pm}_{\mathrm{out}}))$ to get (3.104). □

We introduce the Wave Operators:

$$\mathbb{W}_{\mathrm{in(out)}}:\begin{pmatrix}u_0\\u_1\end{pmatrix}\mapsto\begin{pmatrix}u^{+}_{\mathrm{in(out)}}\\u^{-}_{\mathrm{in(out)}}\end{pmatrix},\tag{3.109}$$

defined on the set $D(\mathbb{W}_{\mathrm{in(out)}})$ of Cauchy data (u_0,u_1) such that the limits (3.107), (3.108) exist. We now investigate their properties: domain, range, continuity, inverse. Firstly the previous theorem assures that these wave operators are well defined on X_{scatt}:

Corollary 3.12. $\mathbb{W}_{\mathrm{in(out)}}$ *is a one-to-one, continuous operator from* X_{scatt} *onto a subspace* $Y_{\mathrm{in(out)}}$ *of* $H^1(\mathbb{R}_s)\times H^1(\mathbb{R}_s)$. *Moreover the map*

$$\begin{pmatrix} u^+(s) \\ u^-(s) \end{pmatrix} \mapsto \begin{pmatrix} \overline{u^+(-s)} \\ \overline{u^-(-s)} \end{pmatrix} \tag{3.110}$$

is one-to-one from Y_{in} onto Y_{out}.

Proof. (3.105) assures the continuity of $\mathbb{W}_{\mathrm{in(out)}}$. If $u^+_{\mathrm{in(out)}} = u^-_{\mathrm{in(out)}} = 0$, (3.106) implies that $u^+_{\mathrm{out(in)}} = u^-_{\mathrm{out(in)}} = 0$, hence, by (3.103), (3.104), we deduce that $u_0 = u_1 = 0$, i.e., $\mathbb{W}_{\mathrm{in(out)}}$ is one-to-one. Now we remark that since A and V are real valued, $v(t,x) := \overline{u(-t,x)}$ is solution of (3.1) iff u is solution of this equation. Therefore we get from (3.107), (3.108): $v^{\pm}_{\mathrm{out}}(s) = \overline{u^{\pm}_{\mathrm{in}}(-s)}$ hence (3.110) is one to one from $Y_{\mathrm{in(out)}}$ onto $Y_{\mathrm{out(in)}}$. □

This result assures that $\mathbb{W}^{-1}_{\mathrm{in(out)}}$ is well defined at least on $Y_{\mathrm{in(out)}}$, but since we do not know if $Y_{\mathrm{in(out)}}$ is closed, the question of the continuity of $\mathbb{W}^{-1}_{\mathrm{in(out)}}$ remains open. Therefore we want to construct continuous inverse Wave Operators formally given by:

$$\boldsymbol{\Omega}_{\mathrm{in(out)}} : \begin{pmatrix} u^+_{\mathrm{in(out)}} \\ u^-_{\mathrm{in(out)}} \end{pmatrix} \mapsto \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}. \tag{3.111}$$

When $\sigma_p \neq \emptyset$, the modes associated with an eigenvalue are exponentially decreasing as $t \to +(-)\infty$, hence $\boldsymbol{\Omega}_{\mathrm{in(out)}}$ would be multivalued. Therefore it is natural to assume that there exists no such exponentially damped modes.

Proposition 3.13. *When $\sigma_p = \emptyset$, there exists $q \geqslant 1$, and bounded operators $\boldsymbol{\Omega}_{\mathrm{in(out)}}$ from $[H^1_{\max(\nu,1)}(\mathbb{R}) \cap H^1_q(\mathbb{R}^{-(+)})]^2$, to $X_{\mathrm{in(out)}} \cap D(\mathbb{W}_{\mathrm{in(out)}})$ such that*

$$\mathbb{W}_{\mathrm{in(out)}} \boldsymbol{\Omega}_{\mathrm{in(out)}} = \mathrm{Id} \quad \text{on } \left[H^1_{\max(\nu,1)}(\mathbb{R}) \cap H^1_q\left(\mathbb{R}^{-(+)}\right)\right]^2. \tag{3.112}$$

Proof. We use the Cook method. Given $(u^+_{\mathrm{in}}, u^-_{\mathrm{in}}) \in H^1_{\max(\nu,1)}(\mathbb{R}) \times H^1_{\max(\nu,1)}(\mathbb{R})$, we put:

$$W(t) := G(-t) \begin{pmatrix} \mathrm{e}^{\mathrm{i}ax} u^-_{\mathrm{in}}(t - \cdot) + u^+_{\mathrm{in}}(t + \cdot) \\ \mathrm{e}^{\mathrm{i}ax} (u^-_{\mathrm{in}})'(t - \cdot) + (u^+_{\mathrm{in}})'(t + \cdot) \end{pmatrix}. \tag{3.113}$$

We calculate:

$$\begin{aligned} \frac{\mathrm{d}W}{\mathrm{d}t}(t) &= G(-t) \\ &\quad \times \begin{pmatrix} 0 \\ (V + a^2 - A^2)\mathrm{e}^{\mathrm{i}ax}(u^-_{\mathrm{in}})'(t - \cdot)\mathrm{e}^{\mathrm{i}ax} u^-_{\mathrm{in}}(t - \cdot) + 2\mathrm{i}(a - A)\mathrm{e}^{\mathrm{i}ax}(u^-_{\mathrm{in}})'(t - \cdot) \end{pmatrix} \\ &\quad + G(-t) \begin{pmatrix} 0 \\ (V - A^2)u^+_{\mathrm{in}}(t + \cdot) - 2\mathrm{i}A(u^+_{\mathrm{in}})'(t + \cdot) \end{pmatrix}. \end{aligned} \tag{3.114}$$

Since $\sigma_p = \emptyset$, (3.80) gives:

$$\left\|\frac{\mathrm{d}W}{\mathrm{d}t}(t)\right\|_X \leqslant C\langle t\rangle^N\Big(\big\|\big(|V|+|A-a|\big)\big(\big|u_{\mathrm{in}}^-(t-\cdot)\big|+\big|(u_{\mathrm{in}}^-)'(t-\cdot)\big|\big)\big\|_{L^2_n(\mathbb{R})}$$
$$+\big\|\big(|V|+|A|\big)\big(\big|u_{\mathrm{in}}^+(t+\cdot)\big|+\big|(u_{\mathrm{in}}^+)'(t+\cdot)\big|\big)\big\|_{L^2_n(\mathbb{R})}\Big), \tag{3.115}$$

with $n := \max(\nu, 1)$. We need an auxiliary estimate. Given $\alpha > 0$, $P_\pm \in L^\infty(\mathbb{R})$, $0 \leqslant P_\pm$, satisfying

$$[[P_\pm]] := \int_{\mathbb{R}^\pm} P_\pm(x)\mathrm{e}^{\alpha|x|}\,\mathrm{d}x < \infty, \tag{3.116}$$

we evaluate for $u \in C_0^\infty(\mathbb{R})$:

$$I_\pm := \int_{-\infty}^{0} \langle t\rangle^{2m}\left(\int_{-\infty}^{\infty} \langle x\rangle^{2n} P_\pm(x)\big|u(t\pm x)\big|^2\,\mathrm{d}x\right)\mathrm{d}t. \tag{3.117}$$

We write:

$$\begin{aligned}
I_\pm &= \int_{-\infty}^{0} \big|u(z)\big|^2\left(\int_{z}^{0} \langle z-\xi\rangle^{2m}\langle\xi\rangle^{2n} P_\pm(\pm\xi)\,\mathrm{d}\xi\right)\mathrm{d}z \\
&\quad + \int_{-\infty}^{0} \big|u(z)\big|^2\left(\int_{0}^{\infty} \langle z-\xi\rangle^{2m}\langle\xi\rangle^{2n} P_\pm(\pm\xi)\,\mathrm{d}\xi\right)\mathrm{d}z \\
&\quad + \int_{0}^{\infty} \big|u(z)\big|^2\left(\int_{z}^{\infty} \langle z-\xi\rangle^{2m}\langle\xi\rangle^{2n} P_\pm(\pm\xi)\,\mathrm{d}\xi\right)\mathrm{d}z \\
&\leqslant \|P_\pm\|_{L^\infty(\mathbb{R})} \int_{-\infty}^{0} \big|u(z)\big|^2\langle z\rangle^{2m+2n+1}\,\mathrm{d}z \\
&\quad + \int_{-\infty}^{0} \big|u(z)\big|^2\langle z\rangle^{2m}\,\mathrm{d}z \int_{0}^{\infty} \langle\xi\rangle^{2m+2n} P_\pm(\pm\xi)\,\mathrm{d}\xi \\
&\quad + \int_{0}^{\infty} \big|u(z)\big|^2\,\mathrm{d}z \int_{0}^{\infty} \langle\xi\rangle^{2m+2n} P_\pm(\pm\xi)\,\mathrm{d}\xi,
\end{aligned} \tag{3.118}$$

hence we get:

$$I_\pm \leqslant C\big(\|P_\pm\|_{L^\infty(\mathbb{R})} + [[P_\pm]]\big)\big(\|u\|^2_{L^2_{m+n+1/2}(\mathbb{R}^-)} + \|u\|^2_{L^2(\mathbb{R}^+)}\big). \tag{3.119}$$

We apply this estimate to (3.115), and we get for $p>0$, $q=p+N+n+1/2$,

$$\int_{-\infty}^{0}\langle t\rangle^{2p}\left\|\frac{\mathrm{d}W}{\mathrm{d}t}(t)\right\|_X^2\mathrm{d}t\leqslant C\big(\|(u^+,u^-)\|^2_{H^1\times H^1(\mathbb{R}^+)}+\|(u^+,u^-)\|^2_{H^1_q\times H^1_q(\mathbb{R}^-)}\big). \tag{3.120}$$

Thus for $p>1/2$ and $u_{\mathrm{in}}^{\pm}\in H^1_n(\mathbb{R})\cap H^1_q(\mathbb{R}^-)$, we have:

$$\int_{-\infty}^{t}\left\|\frac{\mathrm{d}W}{\mathrm{d}t}(\tau)\right\|_X\mathrm{d}\tau\leqslant C\langle t\rangle^{1/2-p},$$

and we may define:

$$\boldsymbol{\Omega}_{\mathrm{in}}\begin{pmatrix}u^+_{\mathrm{in}}\\u^-_{\mathrm{in}}\end{pmatrix}:=\begin{pmatrix}\mathrm{e}^{\mathrm{i}ax}u^-_{\mathrm{in}}(-x)+u^+_{\mathrm{in}}(x)\\ \mathrm{e}^{\mathrm{i}ax}(u^-_{\mathrm{in}})'(-x)+(u^+_{\mathrm{in}})'(x)\end{pmatrix}-\int_{-\infty}^{0}\frac{\mathrm{d}W}{\mathrm{d}t}(\tau)\,\mathrm{d}\tau. \tag{3.121}$$

Then $(u_0,u_1):=\boldsymbol{\Omega}_{\mathrm{in}}(u^+_{\mathrm{in}},u^-_{\mathrm{in}})$ belongs to X and satisfies, for $t\leqslant 0$,

$$\left\|\begin{pmatrix}u_0(x)\\u_1(x)\end{pmatrix}-W(t)\right\|_X\leqslant C\langle t\rangle^{1/2-p}. \tag{3.122}$$

We deduce by (3.79) that

$$\left\|G(t)\begin{pmatrix}u_0\\u_1\end{pmatrix}-\begin{pmatrix}\mathrm{e}^{\mathrm{i}ax}u^-_{\mathrm{in}}(t-\cdot)+u^+_{\mathrm{in}}(t+\cdot)\\ \mathrm{e}^{\mathrm{i}ax}(u^-_{\mathrm{in}})'(t-\cdot)+(u^+_{\mathrm{in}})'(t+\cdot)\end{pmatrix}\right\|_X\leqslant C\langle t\rangle^{N-p},\quad t\leqslant 0, \tag{3.123}$$

hence we choose $p>\max(N,1/2)$. We now have to prove that $(u_0,u_1)\in X_{\mathrm{in}}$. Given $\kappa_*\in\sigma_{ss}$, $l_*\leqslant m(\kappa_*)-1$, we consider $h^{\pm}_{l_*}(t)$ defined by (3.97), and we introduce:

$$H^{\pm}_{l_*}(t):=\frac{\mathrm{d}^{l_*}}{\mathrm{d}k^{l_*}}E^{\pm}_{\mathrm{in}}\big(\mathrm{e}^{\mathrm{i}ax}u^-_{\mathrm{in}}(t-\cdot)+u^+_{\mathrm{in}}(t+\cdot),\mathrm{e}^{\mathrm{i}ax}(u^-_{\mathrm{in}})'(t-\cdot)+(u^+_{\mathrm{in}})'(t+\cdot)\big)(\kappa_*). \tag{3.124}$$

Since $\frac{\mathrm{d}^{l_*}}{\mathrm{d}k^{l_*}}E^{\pm}_{\mathrm{in}}(\cdot,\cdot)(\kappa_*)$ is continuous on X, (3.123) implies:

$$\big|H^{\pm}_{l_*}(t)-h^{\pm}_{l_*}(t)\big|\leqslant C'\langle t\rangle^{N-p},\quad t\leqslant 0. \tag{3.125}$$

We calculate:

$$H^{\pm}_{l_*}(t) = \int u^{+}_{\text{in}}(y)\partial_k^{l_*}\big[\big(k - 2A(y-t)\big) f^{\pm}_{\text{in}}(k;\, y-t) + \mathrm{i}\partial_x f^{\pm}_{\text{in}}(k;\, y-t)\big](\kappa_*)\,\mathrm{d}y$$
$$+ \mathrm{e}^{\mathrm{i}at}\int \mathrm{e}^{-\mathrm{i}ay} u^{-}_{\text{in}}(y)\partial_k^{l_*}$$
$$\times \big[\big(k - 2A(t-y) + a\big) f^{\pm}_{\text{in}}(k;\, t-y) - \mathrm{i}\partial_x f^{\pm}_{\text{in}}(k;\, t-y)\big](\kappa_*)\,\mathrm{d}y. \quad (3.126)$$

Thanks to Proposition 2.1 and (2.45) we have:

$$\big|\partial_k^{l_*}\big[k f^{\pm}_{\text{in}}(k;\, y-t) + \mathrm{i}\partial_x f^{\pm}_{\text{in}}(k;\, y-t)\big](\kappa_*)\big| \leqslant C\langle t\rangle^{\nu-1}\langle y\rangle^{\nu-1}\big(\big|A(y-t)\big| + \mathrm{e}^{-\beta|y-t|}\big),$$
$$\big|\partial_k^{l_*}\big[(k-a) f^{\pm}_{\text{in}}(k;\, t-y) + \mathrm{i}\partial_x f^{\pm}_{\text{in}}(k;\, t-y)\big](\kappa_*)\big|$$
$$\leqslant C\langle t\rangle^{\nu-1}\langle y\rangle^{\nu-1}\big(\big|A(t-y) - a\big| + \mathrm{e}^{-\beta|t-y|}\big),$$

hence:

$$\big|H^{\pm}_{l_*}(t)\big| \leqslant C\langle t\rangle^{\nu-1}\int \langle y\rangle^{\nu-1}\big(\big|u^{+}_{\text{in}}(y)\big| + \big|u^{-}_{\text{in}}(y)\big|\big)\big(\big|A(y-t)\big| + \big|A(t-y) - a\big|$$
$$+ \mathrm{e}^{-\beta|t-y|}\big)\,\mathrm{d}y. \quad (3.127)$$

For $t \leqslant 0$, $q > \nu - 1/2$, $0 < \beta < \alpha$, we have:

$$\int_{-\infty}^{t/2} \langle y\rangle^{\nu-1}\big(\big|u^{+}_{\text{in}}(y)\big| + \big|u^{-}_{\text{in}}(y)\big|\big)\big(\big|A(y-t)\big| + \big|A(t-y) - a\big| + \mathrm{e}^{-\beta|t-y|}\big)\,\mathrm{d}y$$
$$\leqslant C\big(1 + \|A\|_{L^\infty(\mathbb{R})}\big)\langle t\rangle^{\nu-q-1/2}\big(\big\|u^{+}_{\text{in}}\big\|_{L^2_q(\mathbb{R}^-)} + \big\|u^{-}_{\text{in}}\big\|_{L^2_q(\mathbb{R}^-)}\big),$$
$$\int_{t/2}^{\infty} \langle y\rangle^{\nu-1}\big(\big|u^{+}_{\text{in}}(y)\big| + \big|u^{-}_{\text{in}}(y)\big|\big)\big(\big|A(y-t)\big| + \big|A(t-y) - a\big| + \mathrm{e}^{-\beta|t-y|}\big)\,\mathrm{d}y$$
$$\leqslant C\big(1 + [[A,0]]\big)\mathrm{e}^{-\beta\frac{|t|}{4}}\big(\big\|u^{+}_{\text{in}}\big\|_{L^2_{\nu-1}(\mathbb{R})} + \ \big|u^{-}_{\text{in}}\big\|_{L^2_{\nu-1}(\mathbb{R})}\big).$$

We deduce that

$$\big|H^{\pm}_{l_*}(t)\big| \leqslant C\langle t\rangle^{2\nu-q-3/2}\big(\big\|u^{+}_{\text{in}}\big\|_{L^2_{\nu-1}(\mathbb{R})\cap L^2_q(\mathbb{R}^-)} + \big\|u^{-}_{\text{in}}\big\|_{L^2_{\nu-1}(\mathbb{R})\cap L^2_q(\mathbb{R}^-)}\big),$$

hence for q large enough ($q > \max(2N + n + 1/2,\, N + n + 1,\, 2\nu + 3/2)$), we get by (3.125) that

$$\lim_{t\to-\infty} h^{\pm}_{l_*}(t) = 0. \quad (3.128)$$

We conclude, by iteration on $l_* \leqslant m(\kappa_*) - 1$ using (3.98), (3.99), that $h_{l_*}^{\pm} = 0$, i.e., $(u_0, u_1) \in X_{\text{in}}$. Finally (3.123) assures that $(u_0, u_1) \in D(\mathbb{W}_{\text{in}})$ and $\mathbb{W}_{\text{in}}(u_0, u_1) = (u_{\text{in}}^+, u_{\text{in}}^-)$. The proof for $\boldsymbol{\Omega}_{\text{out}}$ is analogous. □

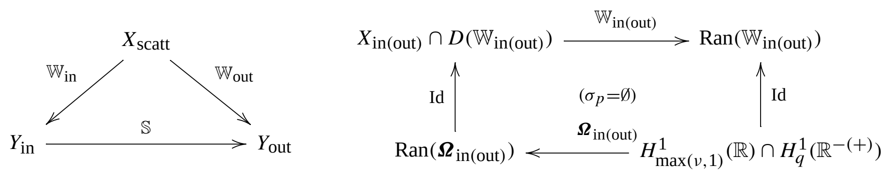

Fig. 1. Wave operators.

We summarize the construction of the waves operators in Fig. 1 where we have introduced the scattering operator:

$$\mathbb{S} = \mathbb{W}_{\text{out}}(\mathbb{W}_{\text{in}})^{-1}, \tag{3.129}$$

which is one-to-one from Y_{in} onto Y_{out}. The characterization of $Y_{\text{in(out)}}$ in terms of usual spaces, and the continuity of $\mathbb{W}_{\text{in(out)}}^{-1}$, and $\mathbb{S}$ are not clear in the general case. Nevertheless we can develop a complete scattering theory when there occurs no usual or hyperradiant mode. We need a subspace of X:

Lemma 3.14. *We assume $\sigma_{ss} = \sigma_p = \emptyset$. Then given $(u_0, u_1) \in X$, $E_{\text{in}}^+(u_0, u_1)$, $E_{\text{in}}^-(u_0, u_1)$ belong to $H^1(\mathbb{R}_\kappa)$ iff $E_{\text{out}}^+(u_0, u_1)$, $E_{\text{out}}^-(u_0, u_1)$ belong to $H^1(\mathbb{R}_\kappa)$. We put*:

$$X_1 := \left\{(u_0, u_1) \in X;\ E_{\text{in/out}}^{\pm}(u_0, u_1) \in H^1(\mathbb{R}_\kappa)\right\}, \tag{3.130}$$

$$\left\|(u_0, u_1)\right\|^2_{X_1,\text{in(out)}} := \left\|E_{\text{in(out)}}^+(u_0, u_1)\right\|^2_{H^1(\mathbb{R}_\kappa)} + \left\|E_{\text{in(out)}}^-(u_0, u_1)\right\|^2_{H^1(\mathbb{R}_\kappa)}; \tag{3.131}$$

$\|\cdot\|_{X_1,\text{in}}$ and $\|\cdot\|_{X_1,\text{out}}$ are two equivalent norms for which X_1 is a Hilbert space, invariant under the action of the group $G(t)$, and there exists $C > 0$ such that for all $(u_0, u_1) \in X_1$ we have:

$$\left\|(u_0, u_1)\right\|_X \leqslant C\left\|(u_0, u_1)\right\|_{X_1,\text{in(out)}}. \tag{3.132}$$

Moreover we have:

$$H^1 \cap \mathcal{E}'(\mathbb{R}_x) \times L^2 \cap \mathcal{E}'(\mathbb{R}_x) \subset X_1. \tag{3.133}$$

Proof. Since we have,

$$E_{\text{in(out)}}^{\pm} = \frac{1}{\tau_{\text{out(in)}}^{\mp}} E_{\text{out(in)}}^{\mp} - \frac{\rho_{\text{out(in)}}^{\mp}}{\tau_{\text{out(in)}}^{\mp}} E_{\text{out(in)}}^{\pm}, \tag{3.134}$$

Lemma 2.5 assures that there exists $C > 0$ such that

$$\left\| E^{\pm}_{\mathrm{in(out)}}(u_0, u_1) \right\|_{H^1(\mathbb{R}_\kappa)} \leqslant C \left(\left\| E^{+}_{\mathrm{out(in)}}(u_0, u_1) \right\|_{H^1(\mathbb{R}_\kappa)} + \left\| E^{-}_{\mathrm{out(in)}}(u_0, u_1) \right\|_{H^1(\mathbb{R}_\kappa)} \right). \tag{3.135}$$

On the other hand (3.103), (3.104) and (3.17) imply:

$$\|u_0\|_{H^1_1(\mathbb{R}^{\pm}_x)} + \|u_1\|_{L^2_1(\mathbb{R}^{\pm}_x)} \leqslant C_0 \left(\left\| E^{\mp}_{\mathrm{in}}(u_0, u_1) \right\|_{H^1(\mathbb{R}_\kappa)} + \left\| E^{\mp}_{\mathrm{out}}(u_0, u_1) \right\|_{H^1(\mathbb{R}_\kappa)} \right). \tag{3.136}$$

This proves (3.132) and, by (3.135), $\|\cdot\|_{X_1,\mathrm{in}}$ and $\|\cdot\|_{X_1,\mathrm{out}}$ are two equivalent norms. Now given $(u_0^n, u_1^n)_{n\in\mathbb{N}}$ a Cauchy sequence in X_1, (3.132) yields that $(u_0^n, u_1^n)_{n\in\mathbb{N}}$ converges in X to some (u_0, u_1). Then $E^{\pm}_{\mathrm{in(out)}}(u_0^n, u_1^n)$ tends to $E^{\pm}_{\mathrm{in(out)}}(u_0, u_1)$ in $L^2(\mathbb{R}_\kappa)$ as $n \to \infty$, and since $E^{\pm}_{\mathrm{in(out)}}(u_0^n, u_1^n)$ are Cauchy sequences in $H^1(\mathbb{R}_\kappa)$ we conclude that $(u_0^n, u_1^n)_{n\in\mathbb{N}}$ converges to (u_0, u_1) in X_1. To show that $G(t)$ leaves X_1 invariant, we remark that

$$(u_0, u_1) \in X_1 \quad \Rightarrow \quad u^{\pm}_{\mathrm{in(out)}} \in H^1_1(\mathbb{R}_s), \tag{3.137}$$

and by (3.107), (3.108), we have for any $T \in \mathbb{R}$:

$$\mathbb{W}_{\mathrm{in(out)}} G(T) = \mathcal{T}_T \mathbb{W}_{\mathrm{in(out)}}, \tag{3.138}$$

where $\mathcal{T}_T$ is the translation operator

$$\mathcal{T}_T \left(u^+(s), u^-(s) \right) = \left(u^+(s+T), u^-(s+T) \right). \tag{3.139}$$

Since $H^1_1(\mathbb{R}_s)$ is invariant under the T-translation, we conclude that $G(T)X_1 = X_1$. At last (3.28) gives (3.133). □

We introduce the Hilbert spaces:

$$K^+ := \left\{ u \in H^1(\mathbb{R}_x);\ u' \in L^2_1(\mathbb{R}_x) \right\}, \quad \|u\|^2_{K^+} := \|u\|^2_{H^1(\mathbb{R}_x)} + \|u'\|^2_{L^2_1(\mathbb{R}_x)}, \tag{3.140}$$

$$K^- := \left\{ u \in H^1(\mathbb{R}_x);\ \mathrm{i}u' + au \in L^2_1(\mathbb{R}_x) \right\},$$
$$\|u\|^2_{K^-} := \|u\|^2_{H^1(\mathbb{R}_x)} + \|\mathrm{i}u' + au\|^2_{L^2_1(\mathbb{R}_x)}. \tag{3.141}$$

The scattering theory, in the absence of modes, is described by the following:

Theorem 3.15. *We assume $\sigma_{ss} = \sigma_p = \emptyset$. Then $(u^+, u^-) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ belongs to $Y_{\mathrm{in(out)}}$ iff,*

$$u_0^{\mathrm{in(out)}} := \Phi^+_{\mathrm{out(in)}}\left(\frac{1}{\tau^+_{\mathrm{out(in)}}(\kappa)}\mathcal{F}(u^-)\right) + \Phi^-_{\mathrm{out(in)}}\left(\frac{1}{\tau^-_{\mathrm{out(in)}}(\kappa)}\mathcal{F}(u^+)\right) \in H^1_1(\mathbb{R}_x), \tag{3.142}$$

$$u_1^{\mathrm{in(out)}} := \Phi^+_{\mathrm{out(in)}}\left(\frac{\mathrm{i}\kappa}{\tau^+_{\mathrm{out(in)}}(\kappa)}\mathcal{F}(u^-)\right) + \Phi^-_{\mathrm{out(in)}}\left(\frac{\mathrm{i}\kappa}{\tau^-_{\mathrm{out(in)}}(\kappa)}\mathcal{F}(u^+)\right) \in L^2_1(\mathbb{R}_x), \tag{3.143}$$

and in this case we have:

$$\mathbb{W}_{\mathrm{in(out)}}\big(u_0^{\mathrm{in(out)}}, u_1^{\mathrm{in(out)}}\big) = (u^+, u^-). \tag{3.144}$$

Moreover we have:

$$H^1_1(\mathbb{R}) \times H^1_1(\mathbb{R}) \subset Y_{\mathrm{in(out)}} \subset K^+ \times K^-, \tag{3.145}$$

and $\mathbb{W}_{\mathrm{in(out)}}$ *are continuous, one-to-one, operators, from* X_1 *onto* $H^1_1(\mathbb{R}) \times H^1_1(\mathbb{R})$*, and from* X *onto* $Y_{\mathrm{in(out)}}$ *endowed with the norm of* $K^+ \times K^-$*. The scattering operator is a continuous, one-to-one operator from* $H^1_1(\mathbb{R}) \times H^1_1(\mathbb{R})$ *onto* $H^1_1(\mathbb{R}) \times H^1_1(\mathbb{R})$*, and from* Y_{in} *onto* Y_{out} *where* $Y_{\mathrm{in(out)}}$ *are endowed with the norm of* $K^+ \times K^-$*, or* $H^1(\mathbb{R}) \times H^1(\mathbb{R})$*. This operator has the form*:

$$\mathbb{S} = \mathcal{F}^{-1}\widehat{S}(\kappa)\mathcal{F}, \quad \widehat{S}(\kappa) := \begin{pmatrix} R^+(\kappa) & T^+(\kappa) \\ T^-(\kappa) & R^-(\kappa) \end{pmatrix}. \tag{3.146}$$

The scattering matrix $\widehat{S}(k)$ *is meromorphic on* $\omega := \{k \in \mathbb{C};\ |\Im k| < \alpha/2\}$ *and* $k \in \omega$ *is a pole of* $\widehat{S}$ *iff* $\bar{k}$ *belongs to the set of resonances* $\mathcal{R}$*. Furthermore the scattering is superradiant for the frequencies in the Klein zone*:

$$\kappa \in (0, a) \quad \Rightarrow \quad 1 < \big|R^{\pm}(\kappa)\big|, \big\|\widehat{S}(\kappa)\big\|_{\mathcal{L}(\mathbb{C}^2)}, \big\|\big(\widehat{S}(\kappa)\big)^{-1}\big\|_{\mathcal{L}(\mathbb{C}^2)}. \tag{3.147}$$

We make some comments on these results.

(1) We can easily see that

$$H^1_1(\mathbb{R}) \times H^1_1(\mathbb{R}) \underset{\neq}{\subset} Y_{\mathrm{in(out)}} \underset{\neq}{\subset} K^+ \times K^-, \tag{3.148}$$

when $A(x) = V(x) = 0$ for $x \geqslant 0$. If $Y_{\mathrm{in(out)}} = H^1_1 \times H^1_1$, then $X = X_1$ and the Banach theorem implies that the norms $\|\cdot\|_X$ and $\|\cdot\|_{X_1}$ are equivalent. Then $\mathbb{W}_{\mathrm{in(out)}}$ is continuous from X to $H^1_1(\mathbb{R}) \times H^1_1(\mathbb{R})$. We choose $f \in C^\infty_0(]a, b[)$, $0 < a < b$, $\int f(x)\,\mathrm{d}x = 1$. For $n \in \mathbb{N}$, let $u_n(t, x)$ be the solution of (3.1) with $u_n(0, x) = 0$, $\partial_t u_n(0, x) = f(x - n)$. We have:

$$\big\|\big(u_n(0), \partial_t u_n(0)\big)\big\|_X \sim n, \quad n \to \infty.$$

But since $A(x)=V(x)=0$ for $x\geqslant 0$, we also have $|u_n(t,x)|=1$ for $|t|\geqslant (b+n)/2$, $|t|\leqslant x\leqslant |t|+a+n$. Hence

$$\big\|\mathbb{W}_{\mathrm{in(out)}}\big(u_n(0),\partial_t u_n(0)\big)\big\|_{H_1^1\times H_1^1}\geqslant Cn^{3/2},\quad n\to\infty,$$

and we get a contradiction. Now if $Y_{\mathrm{in(out)}}=K^+\times K^-$, the open mapping theorem assures that $\mathbb{W}_{\mathrm{in(out)}}^{-1}$ is bounded from $K^+\times K^-$ to X. Then for $u^+\in C_0^\infty(]0,\infty[)$, we have:

$$\big\|W_{\mathrm{in}}^{-1}(u^+,0)\big\|_X\leqslant C\big(\|u^+\|_{H^1}+\big\|(u^+)'\big\|_{L_1^2}\big).$$

On the other hand, we have $\mathbb{W}_{\mathrm{in}}^{-1}(u^+,0)=(u^+,(u^+)')$, hence

$$\big\|\mathbb{W}_{\mathrm{in}}^{-1}(u^+,0)\big\|_X\geqslant \|u^+\|_{H_1^1},$$

and we get using the previous inequality:

$$\|u^+\|_{H_1^1}\leqslant C'\big(\|u^+\|_{H^1}+\big\|(u^+)'\big\|_{L_1^2}\big).$$

To see that we have obtained a contradiction again, we choose $u_n^+(x):=f(x/n)$, $f\in C_0^\infty(]0,\infty[)\setminus\{0\}$, and we easily estimate: $\|u_n^+\|_{H_1^1}\sim n^{3/2}$, $\|u_n^+\|_{H^1}\sim n^{1/2}$, $\|(u_n^+)'\|_{L_1^2}\sim n^{1/2}$.

(2) When $A=0$ and V is compactly supported, the Lax–Phillips theory assures that $\widehat{S}$ has a meromorphic continuation on $\mathbb{C}_k$, and solution a u has an asymptotic expansion:

$$u(t,x)\sim\sum_{\bar{k}\in\mathcal{R}}\sum_{n=0}^{m(k)}C(k,n,x)t^n\mathrm{e}^{\mathrm{i}tk},\quad t\to+\infty. \tag{3.149}$$

Several analogous results are known when V is a compactly supported, or short range potential, with an analytic continuation on a conic neighbourhood of $\mathbb{R}_x$ (e.g., [4,34]). We conjecture a similar expansion for the charged fields considered in this paper.

Proof of Theorem 3.15. Given $(u_{\mathrm{in}}^+,u_{\mathrm{in}}^-)=\mathbb{W}_{\mathrm{in}}(u_0,u_1)\in Y_{\mathrm{in}}$, we get (3.142) and (3.143), from (3.103), (3.104) and (3.106). Conversely, let (u^+,u^-) be in $H^1(\mathbb{R})\times H^1(\mathbb{R})$, satisfying (3.142) and (3.143) with exponent in. We shall prove (3.144) to show that $(u^+,u^-)\in Y_{\mathrm{in}}$. We introduce:

$$u(t):=G_{\mathrm{out}}^+\bigg(\frac{1}{\tau_{\mathrm{out}}^+(\kappa)}\mathcal{F}(u^-)\bigg)(t)+G_{\mathrm{out}}^-\bigg(\frac{1}{\tau_{\mathrm{out}}^-(\kappa)}\mathcal{F}(u^+)\bigg)(t). \tag{3.150}$$

We have:

$$u(t)=G_{\text{out}}^{+}\left(\frac{1}{\tau_{\text{out}}^{+}(\kappa)}\mathcal{F}(u^{-})+\frac{\rho_{\text{out}}^{-}(\kappa)}{\tau_{\text{out}}^{-}(\kappa)}\mathcal{F}(u^{+})\right)(t)+G_{\text{in}}^{+}\big(\mathcal{F}(u^{+})\big)(t)$$
$$+G_{\text{out}}^{-}\left(\frac{1}{\tau_{\text{out}}^{-}(\kappa)}\mathcal{F}(u^{+})+\frac{\rho_{\text{out}}^{+}(\kappa)}{\tau_{\text{out}}^{+}(\kappa)}\mathcal{F}(u^{-})\right)(t)+G_{\text{in}}^{-}\big(\mathcal{F}(u^{-})\big)(t). \tag{3.151}$$

Then Lemma 3.2 and Theorem 3.11 imply that

$$\mathcal{F}\big(u_{\text{out}}^{\pm}\big)(\kappa)=\frac{1}{\tau_{\text{out}}^{\pm}(\kappa)}\mathcal{F}\big(u^{\mp}\big)(\kappa)+\frac{\rho_{\text{out}}^{\mp}(\kappa)}{\tau_{\text{out}}^{\mp}(\kappa)}\mathcal{F}\big(u^{\pm}\big)(\kappa),$$

and we deduce with (3.106) that $u_{\text{in}}^{\pm}=u^{\pm}$.

We now prove that $H_1^1\times H_1^1\subset Y_{\text{in}}$. Given $(u^+,u^-)\in H_1^1\times H_1^1$ we pick a sequence $(u_n^+,u_n^-)\in C_0^\infty(\mathbb{R})\times C_0^\infty(\mathbb{R})$ tending to (u^+,u^-) in $H_1^1\times H_1^1$, as $n\to\infty$. According to Proposition 3.13, there exists $(u_{0,n},u_{1,n})=\boldsymbol{\Omega}_{\text{in}}(u_n^+,u_n^-)$. (3.18) and (3.142), (3.143) assure that $(u_{0,n},u_{1,n})$ converges to some (u_0,u_1) in X. Then $\mathbb{W}_{\text{in}}(u_{0,n},u_{1,n})=(u_n^+,u_n^-)\to\mathbb{W}_{\text{in}}(u_0,u_1)$ in $H^1\times H^1$. We conclude that $(u^+,u^-)=\mathbb{W}_{\text{in}}(u_0,u_1)\in Y_{\text{in}}$. Furthermore $\|(u_0,u_1)\|_{X_1}$ is equivalent to $\|\mathbb{W}_{\text{in(out)}}(u_0,u_1)\|_{H_1^1\times H_1^1}$, therefore $\mathbb{W}_{\text{in(out)}}$ is an isomorphism from X_1 onto $H_1^1(\mathbb{R})\times H_1^1(\mathbb{R})$.

(3.102), (3.26) and (3.27) show that $Y_{\text{in(out)}}\subset K^+\times K^-$ and $\mathbb{W}_{\text{in(out)}}$ is continuous from X onto $Y_{\text{in(out)}}$ endowed with the norm of $K^+\times K^-$.

(3.146) is a consequence of (3.106), and Lemma 2.6 assures that $\mathbb{S}$ is continuous with respect to the norms $H^1\times H^1$, $H_1^1\times H_1^1$ and $K^+\times K^-$. Since $|R^+R^- - T^+T^-|=1$, the same is true for $\mathbb{S}^{-1}$. Finally (2.58) implies (3.147). □

Since the asymptotic dynamics are $(\partial_t-\mathrm{i}a)^2-\partial_x^2$ as $x\to-\infty$ and $\partial_t^2-\partial_x^2$ as $x\to+\infty$, it is natural to study the scattering in the Hilbert spaces associated with the energy for these wave equations. Given $c\in\mathbb{R}$, we introduce the Beppo Levi type spaces $BL_{(c)}^1(\mathbb{R})$ defined as the closure of $C_0^\infty(\mathbb{R})$ in the norm:

$$f\in C_0^\infty(\mathbb{R}),\quad \|f\|_{BL_{(c)}^1}:=\|\mathrm{i}f'+cf\|_{L^2}. \tag{3.152}$$

These spaces are not spaces of distributions on $\mathbb{R}$ and the solutions of the wave equations have to be interpreted in the sense of the spectral calculus.

Corollary 3.16. *The operators $\mathbb{W}_{\text{in(out)}}$ can be extended into a bounded operator from $H^1(\mathbb{R})\times L^2(\mathbb{R})$ to $BL_{(0)}^1(\mathbb{R})\times BL_{(a)}^1(\mathbb{R})$; $\mathbb{S}$ can be extended into an isomorphism on $BL_{(0)}^1(\mathbb{R})\times BL_{(a)}^1(\mathbb{R})$ and denoting by $\|\cdot\|_e$ the norm of $\mathcal{L}(BL_{(0)}^1(\mathbb{R})\times BL_{(a)}^1(\mathbb{R}))$, we have*:

$$1<\|\mathbb{S}\|_e,\ \|\mathbb{S}^{-1}\|_e. \tag{3.153}$$

We emphasize that this extended scattering operator is of an unusual type: we do not know if the inverse wave operators $\mathbb{W}_{\text{in(out)}}^{-1}$, which are defined from $Y_{\text{in(out)}}$ onto X, can be extended from $BL_{(0)}^1(\mathbb{R})\times BL_{(a)}^1(\mathbb{R})$ to $H^1(\mathbb{R})\times L^2(\mathbb{R})$. This situation has already

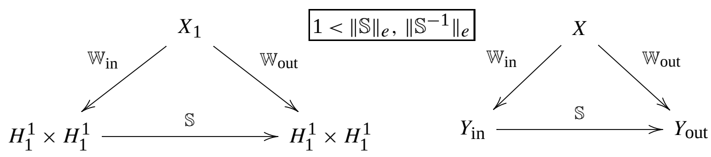

Fig. 2. Scattering when $\sigma_p = \sigma_{ss} = \emptyset$.

been encountered in the case of space-times with causality violation [3]. The root of this phenomenon is the same: the conserved energy is not definite positive.

Proof of Corollary 3.16. (2.73) assures that

$$\left\|(\kappa-a)\langle\kappa\rangle^{-1}E^{+}_{\text{in(out)}}(u_0,u_1)\right\|_{L^2}+\left\|\kappa\langle\kappa\rangle^{-1}E^{-}_{\text{in(out)}}(u_0,u_1)\right\|_{L^2}\leqslant C\big(\|u_0\|_{H^1}+\|u_1\|_{L^2}\big).$$

Therefore (3.102) show that $\mathbb{W}_{\text{in(out)}}$ can be extended into a bounded operator from $H^1(\mathbb{R})\times L^2(\mathbb{R})$ to $BL^1_{(0)}(\mathbb{R})\times BL^1_{(a)}(\mathbb{R})$. Now using (3.146), we write:

$$\kappa\mathcal{F}\big(u^{+}_{\text{out}}\big)(\kappa)=R^{+}(\kappa)\big[\kappa\mathcal{F}\big(u^{+}_{\text{in}}\big)(\kappa)\big]+\left(\kappa\frac{T^{+}(\kappa)}{\kappa-a}\right)\big[(\kappa-a)\mathcal{F}\big(u^{-}_{\text{in}}\big)(\kappa)\big],$$

$$(\kappa-a)\mathcal{F}\big(u^{-}_{\text{out}}\big)(\kappa)=R^{-}(\kappa)\big[(\kappa-a)\mathcal{F}\big(u^{-}_{\text{in}}\big)(\kappa)\big]+\left((\kappa-a)\frac{T^{-}(\kappa)}{\kappa}\right)\big[\kappa\mathcal{F}\big(u^{+}_{\text{in}}\big)(\kappa)\big].$$

Since Lemma 2.6 assures that

$$R^{\pm}(\kappa),\ \kappa\frac{T^{+}(\kappa)}{\kappa-a},\ (\kappa-a)\frac{T^{-}(\kappa)}{\kappa}\in L^{\infty}(\mathbb{R}_\kappa),$$

the continuity of $\mathbb{S}$ on $BL^1_{(0)}(\mathbb{R})\times BL^1_{(a)}(\mathbb{R})$ is established. Finally (3.147) gives (3.153). □

The scattering theory can be summarized by Fig. 2 where the arrows denote the one-to-one and onto continuous operators.

4. An application in general relativity

The asymptotic behaviours of classical fields on several important curved space-times of general relativity, have been the subject of numerous studies. We can mention the works on the scalar equations by the author [1,3], D. Häfner [12,13], J.-P. Nicolas [29], and on the Dirac system by the author [2], D. Häfner and J.-P. Nicolas [14], L.J. Mason and J.-P. Nicolas [23,24], F. Melnyk [25,26], J.-P. Nicolas [28,30]. As regards the propagation of the energy, there exists a deep difference between the bosons and the fermions: the L^2 norm of a field with half-integral spin, is conserved, while the conserved energy of the

Klein–Gordon field on a curved background is not necessarily positive. In such cases of indefinite conserved energy, the field is allowed to extract energy from a particular region of space-time, for instance the ergosphere of a Kerr black-hole, or the dyadosphere of a charged black-hole. This phenomenon has been described, for the first time, by R. Penrose who proved that a classical particle can enter the ergosphere of a rotating black hole, and come out again with more energy than it originally had. The corresponding effect for integral spin fields is called superradiance [11,32]. To our knowledge, a rigorous mathematical analysis is missing, and the present study is a first step in this direction since we can apply the results of the previous sections to the superradiant scattering of charged Klein–Gordon fields by a charged black-hole in an expanding universe.

The spin 0 field with mass $m \geqslant 0$, and charge $e \in \mathbb{R}$, on a Lorentzian manifold $(\mathcal{M}, g)$ endowed with an electromagnetic potential $A_\mu \, \mathrm{d}x^\mu$, obeys the Klein–Gordon equation:

$$(\nabla_\mu - \mathrm{i}eA_\mu)(\nabla^\mu - \mathrm{i}eA^\mu)\Phi + m^2\Phi + \xi R\Phi = 0, \tag{4.1}$$

where $R = g^{\mu\nu}R_{\mu\nu}$ is the scalar curvature, and $\xi \in \mathbb{R}$ is a numerical factor. This equation has the more explicit form:

$$\begin{aligned} &|g|^{-1/2}\partial_\mu\big(|g|^{1/2}g^{\mu\nu}\partial_\nu\Phi\big) - \mathrm{i}e\big[\partial_\mu(g^{\mu\nu}A_\nu\Phi) + A_\mu|g|^{-1/2}\partial_\nu\big(|g|^{1/2}\big)g^{\mu\nu}\Phi \\ &\quad + A_\mu g^{\mu\nu}(\partial_\nu\Phi - \mathrm{i}eA_\nu\Phi)\big] + m^2\Phi + \xi R\Phi = 0. \end{aligned} \tag{4.2}$$

We are concerned with the $(3+1)$-dimensional, spherically symmetric space-time $\mathbb{R}_t \times I_r \times S^2_\omega$, I being a real open interval, that describes a black hole in an expanding universe. In this case the metric can be written as:

$$g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = F(r)\,\mathrm{d}t^2 - \big[F(r)\big]^{-1}\,\mathrm{d}r^2 - r^2\,\mathrm{d}\omega^2, \tag{4.3}$$

where $F \in C^2([r_0, r_+])$, $0 < r_0 < r_+ < \infty$, is called the lapse function, and satisfies:

$$F(r_0) = F(r_+) = 0,\ r_0 < r < r_+ \quad \Rightarrow \quad 0 < F(r),\ 0 < F'(r_0),\ F'(r_+) < 0. \tag{4.4}$$

r_0 is the radius of the Horizon of the Black-Hole, r_+ is the radius of the Cosmological Horizon. The Ricci scalar is given by:

$$R = F'' + \frac{4}{r}F' + \frac{2}{r^2}(F - 1).$$

We assume that the electromagnetic potential is electrostatic and also spherically symmetric:

$$A_\mu\,\mathrm{d}x^\mu = A_t(r)\,\mathrm{d}t, \quad A_t \in C^1\big([r_0, r_+]\big),\ A_t(r_0) \neq A_t(r_+). \tag{4.5}$$

These hypotheses are satisfied, for a suitable choice of the physical parameters, in the important case of a charged black-hole in an expanding universe, for which the DeSitter–Reissner–Nordstrøm metric, and the Maxwell connection, are given by:

$$F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2, \quad A_t(r) = \frac{Q}{r}. \tag{4.6}$$

Here $0 < M$ and $Q \in \mathbb{R}$ are the mass and the charge of the black-hole, $\Lambda > 0$ is the cosmological constant (see, e.g., [2,25]).

It is convenient to push the horizons away to infinity by means of the tortoise coordinate:

$$x = \frac{1}{F'(r_0)}\left\{\ln|r - r_0| - \int_{r_0}^{r}\left[\frac{1}{r - r_0} - \frac{F'(r_0)}{F(r)}\right]\mathrm{d}r\right\}. \tag{4.7}$$

Then $u = r\Phi$ is solution of

$$\left(\partial_t - \mathrm{i}A(x)\right)^2 u - \partial_x^2 u - B(x)\Delta_{S^2} u + C(x)u = 0, \quad t \in \mathbb{R},\ x \in \mathbb{R},\ \omega \in S^2, \tag{4.8}$$

with

$$A(x) = eA_t(r), \qquad B(x) = \frac{1}{r^2}F(r),$$
$$C(x) = \left(\xi F''(r) + \frac{4\xi + 1}{r}F'(r) + \frac{2\xi}{r^2}F(r) - \frac{2\xi}{r^2} + m^2\right)F(r). \tag{4.9}$$

The conserved energy is given by:

$$E(u) := \int_{\mathbb{R}}\int_{S^2_\omega}\left(\left|\partial_t u(t,x,\omega)\right|^2 + \left|\partial_x u(t,x,\omega)\right|^2 + B(x)\left|\nabla_\omega u(t,x,\omega)\right|^2 \right.$$
$$\left. + \left[C(x) - A^2(x)\right]\left|u(t,x,\omega)\right|^2\right)\mathrm{d}x\,\mathrm{d}\omega. \tag{4.10}$$

The *dyadosphere* $\mathcal{D}_{e,m}$ is defined as the region outside the black hole horizon where the electrostatic energy, associated with the charge e of the field, exceeds the gravitational interacting energy associated with the mass m of the field:

$$\mathcal{D}_{e,m} := \left\{x \in \mathbb{R};\ A^2(x) > C(x)\right\} \times S^2_\omega. \tag{4.11}$$

We remark that, because of the existence of the cosmological horizon, unlike the case of the asymptotically flat space-time for which $F(r) \to 1$ as $r \to +\infty$, $\mathcal{D}_{e,m}$ is never empty, whatever the mass of the field and the gauge transform on A. Furthermore, if $|e|$ is large enough, we can have $\mathcal{D}_{e,m} = \mathbb{R}_x \times S^2_\omega$.

Taking advantage of the spherical symmetry, we expand $u(t,x,\cdot)$ on the basis of spherical harmonics $Y_{l,m}$ of $L^2(S^2_\omega)$:

$$u(t,x,\omega) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} u_{l,m}(t,x)Y_{l,m}(\omega). \tag{4.12}$$

Finally $u_{l,m}$ is solution of the gyroscopic Klein–Gordon equation:

$$\left(\partial_t - \mathrm{i}A(x)\right)^2 u_{l,m} - \partial_x^2 u_{l,m} + V(x)u_{l,m} = 0, \quad t \in \mathbb{R},\ x \in \mathbb{R}, \tag{4.13}$$

with

$$V(x) = l(l+1)B(x) + C(x). \tag{4.14}$$

Since A and V satisfy:

$$\left|A(x) - eA_t(r_0)\right| + \left|A'(x)\right| + \left|V(x)\right| \leqslant C\mathrm{e}^{F'(r_0)x}, \quad x \to -\infty,$$
$$\left|A(x) - eA_t(r_+)\right| + \left|A'(x)\right| + \left|V(x)\right| \leqslant C\mathrm{e}^{F'(r_+)x}, \quad x \to +\infty,$$

we may apply the results of the preceding sections to Eq. (4.13). In particular, Theorem 3.11 gives a rigorous explanation of the superradiance of charged black-holes, in terms of scattering of spin-0 charged fields [11,31]. We leave open the problem of the nature of σ_{ss} and σ_p for the DeSitter–Reissner–Nordstrøm metric (4.6). Several numerical experiments suggest that these sets are empty, hence we conjecture that there is no hyperradiance in this case.

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