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Global properties of the wave equation on non-globally hyperbolic manifolds

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Abstract

We introduce a class of four-dimensional Lorentzian manifolds with closed curves of null type or timelike. We investigate some global problems for the wave equation: uniqueness of solution with data on a changing type hypersurface; existence of resonant states; scattering by a violation of the chronology; global Cauchy problem and asymptotic completeness of the wave operators for the chronological but non-causal metrics. © 2002 Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

Résumé

On considère une classe de variétés lorentziennes de dimension quatre, admettant des courbes fermées de type nul ou de genre temps. On étudie quelques problèmes globaux pour l'équation des ondes : unicité de la solution avec données spécifiées sur une hypersurface de type changeant ; existence d'états résonants ; diffusion par une violation de la chronologie ; problème de Cauchy global et complétude asymptotique des opérateurs d'onde pour des métriques chronologiques mais non causales. © 2002 Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

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1. Introduction

The theory of the linear waves equations on globally hyperbolic manifolds has a long history since M. Riesz and J. Hadamard. It is impossible to cite all the important authors in the area, but we mention the fundamental works related to our study: the Cauchy problem investigated by J. Leray [25] and Y. Choquet-Bruhat [5] (see, e.g., the excellent monograph [10] by F.G. Friedlander), the scattering theory for a compactly supported perturbation by P. Lax and R. Phillips [24], the microlocal analysis of the solutions by L. Hörmander [18] and J.-M. Bony [3].

In opposite there are few works on the global hyperbolic problems on the *non*-globally hyperbolic spacetimes. Nevertheless the global hyperbolicity is an extremely strong hypothesis, which is not satisfied by a lot of solutions of the (in)homogeneous Einstein equations. The origin of the loss of global hyperbolicity can be a non-trivial topology, an elementary example is $S^1_t \times \mathbb{R}^3_x$ endowed with the Minkowski metric. Other examples are the Lorentzian wormholes [11,35], but since they lead to violations of the local energy conditions, these models are somewhat exotic. A deeper raison is linked with the non-linearity of the Einstein equations that can create some singularities of curvature, and also some closed time-like geodesics. In particular, the violation of the causality can be due to a fast rotation of the space-time that tilts over the light cones so strongly that some closed causal curves appear. This phenomenon is present in several important Einstein manifolds: the Van Stockum space-

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time [32], the Gödel universe [14], the Kerr black-hole (third Boyer–Lindquist block and fast Kerr) [23], the spinning cosmic string [8]. These Lorentzian manifolds belong to a wide range of stationnary, axisymmetric spacetimes that are described by the Papapetrou metric [28]

$$g_{\mu,\nu}\,\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu} = A(r,z)\bigl[\mathrm{d}t - C(r,z)\,\mathrm{d}\varphi\bigr]^2 - \frac{1}{A(r,z)}\bigl[r^2\,\mathrm{d}\varphi^2 + B(r,z)\bigl(\mathrm{d}r^2 + \mathrm{d}z^2\bigr)\bigr], \quad 0 < A, B,\ 0 \leqslant C, \tag{1.1}$$

on some $3\mathrm{D}+1$ manifold $\mathcal{M}$.

Our model consists by choosing $\mathcal{M} = \mathbb{R}^4$, $A = B = 1$, and for simplicity we assume that C is compactly supported. When we allow that $C(r,z) > r$ (resp. $C(r,z) = r$) for some (r,z), some *closed* time-like (resp. null) curves appear and this spacetime has the same properties that the previous Einstein manifolds of point of view of the causality. We investigate the wave equation:

$$|\det g|^{-\frac{1}{2}}\partial_{\mu}\bigl(|\det g|^{\frac{1}{2}} g^{\mu,\nu}\partial_{\nu}\bigr)u = \left(1 - \frac{C^2}{r^2}\right)\partial_t^2 u - \Delta_x u - 2\frac{C}{r^2}\partial_t\partial_{\varphi} u = 0. \tag{1.2}$$

We also consider the zero-order perturbation of the D'Alembertian by a potential, for instance the conformally invariant wave equation. Obviously the study of the solutions is difficult because of the presence of closed timelike/null curves: there exists no global Cauchy hypersurface. We can see how much intricated is the situation by formally expanding a solution of (1.2) in Fourier series with respect to φ:

$$u(t,\varphi,r,z) = \sum_{m\in\mathbb{Z}} r^{-\frac{1}{2}} u_m(t,r,z)\mathrm{e}^{\mathrm{i}m\varphi}.$$

Then u_m is solution of a changing type equation:

$$\left(1 - \frac{C^2}{r^2}\right)\partial_t^2 u_m - \bigl(\partial_r^2 + \partial_z^2\bigr)u_m - 2\mathrm{i}m\frac{C}{r^2}\partial_t u_m + \frac{m^2}{r^2}u_m = 0,$$

which is hyperbolic on $\{C < r\}$, elliptic on $\mathbb{T} := \{C > r\}$, and of Schrödinger type on $\Sigma := \{C = r\}$. In particular,

$$M_{t_0} := \{t = t_0\} \times \mathbb{R}_x^3$$

is not a Cauchy hypersurface for (1.2) when Σ is not empty. Another crucial point is that since ∂_t is a Killing vector field, there exists a conserved current for the sufficiently smooth solutions of (1.2):

$$E(u) := \frac{1}{2}\int_{\mathbb{R}^3}\left(1 - \frac{C^2}{r^2}\right)\bigl|\partial_t u(t,x)\bigr|^2 + \bigl|\nabla u(t,x)\bigr|^2\,\mathrm{d}x.$$

But this energy is *not* a positive form when the manifold is not chronological ($\mathbb{T} \neq \emptyset$).

We briefly describe our results. In Section 2, we investigate the causal structure of $\mathcal{M}$ that is not globally hyperbolic when $\Sigma \neq \emptyset$, and totally vicious when $\mathbb{T} \neq \emptyset$. We introduce the concept of *Non-Confining*, that is a non-trapping type condition, expressing that there exists no null geodesic included in $\Sigma \cap M_{t_0}$.

We study the properties of the solutions of (1.2) in Section 3. Since ∂_t is a Killing vector field, there exists a conserved energy, and it is natural to consider solutions u such that $\nabla_x u \in L^2_{\mathrm{loc}}(\mathbb{R}_t, L^2(\mathbb{R}_x^3))$. When $\mathbb{T} \neq \emptyset$, the energy is not non-negative and there is no control of $\partial_t u$ on Σ. Nevertheless, if the Non-Confining condition is satisfied, the microlocal analysis allows to prove an unexpected regularity: $\partial_t u \in L^2_{\mathrm{loc}}(\mathbb{R}_t, L^2(\mathbb{R}_x^3))$. Thanks to this key result, the traces of u and $(1 - \frac{C}{r})\partial_t u$ on M_{t_0} are well defined, and we may establish a uniqueness theorem which is not a consequence of the classical results of Calderon or Hörmander, since M_{t_0} is not non-characteristic, and Σ is nowhere strongly pseudo-convex.

In Section 4, we look for the solutions of type $u(t,x) = \mathrm{e}^{\lambda t}v(x)$, and v satisfies an outgoing condition. We prove that the set of resonances $\lambda \in \mathbb{C}$ is discrete, and when $\mathbb{T} \neq \emptyset$, there exists a sequence of resonances $0 < \lambda_n \to \infty$ with $v \in L^2(\mathbb{R}_x^3)$. Of physical point of view, this last fact means that the metric is instable, and partially justifies the conjecture of chronological protection of S. Hawking [16].

In Section 5, we construct asymptotically free global solutions u, with data of type regular wave packets, u_0^-, given at the null past infinity. Moreover u is asymptotic to a regular wave packet u_0^+ as t tends to $+\infty$. The scattering operator $S : u_0^- \mapsto u_0^+$, is a well defined isometry, even if the chronology is violated ($\mathbb{T} \neq \emptyset$), but in this case the wave operator $u_0^- \mapsto u$ is not causal. We investigate the link between the resonances and the poles of the meromorphic continuation of the scattering matrix.

In the last section, we consider the case where $\mathcal{M}$ is chronological ($\mathbb{T} = \emptyset$), but non-causal ($\Sigma \neq \emptyset$). The global Cauchy problem is well posed for a whole Hilbert space of finite energy data, including those vanishing on Σ. Moreover the local energy decaies, and we can prove the existence and asymptotic completeness of the wave operators describing the scattering by a violation of the causality. Although $\mathcal{M}$ is not causal, the scattering operator is causal.

It is without saying that this work is only a first incursion in the mathematically widely unexplored domain of the field equations on the non-globally hyperbolic manifolds (for a rather significant bibliography of the physical literature see, e.g., [8,10,12,13,16,22,33,35]). We have not dealed with many important questions such that: the asymptotic repartition of the resonances; the singularities of the scattering kernel; the existence of a "trace formula" making a link between some geometric quantities (e.g., the lenght of the closed null geodesics), and the spectral numbers; the Strichartz type estimates, etc. Last but not least, the field of the nonlinear wave equations on a non-causal space-time is *terra incognita*.

2. Geometrical framework

We consider the topologically trivial manifold:

$$\mathcal{M} := \mathbb{R}^4_{(x^0,x^1,x^2,x^3)} = \mathbb{R}_t \times \mathbb{R}^3_x \tag{2.1}$$

endowed with a Lorentzian metric g which is equal to the Minkowski metric outside a torus

$$\mathbb{R}_t \times \left\{\left(x^1,x^2,x^3\right);\, 0 < r_-^2 < \left|x^1\right|^2 + \left|x^2\right|^2 < r_+^2,\, z_- < x^3 < z_+\right\}.$$

We choose a particular case of the Papapetrou metric:

$$g_{\mu,\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \mathrm{d}t^2 - \left[r^2 - C^2(r,z)\right]\mathrm{d}\varphi^2 - 2C(r,z)\,\mathrm{d}t\,\mathrm{d}\varphi - \mathrm{d}r^2 - \mathrm{d}z^2, \tag{2.2}$$

where we have used the cylindrical coordinates $(t,\varphi,r,z) \in \mathbb{R} \times [0,2\pi[\times[0,\infty[\times\mathbb{R}$ given by

$$x^1 = r\cos\varphi, \qquad x^2 = r\sin\varphi, \qquad x^3 = z. \tag{2.3}$$

We assume that C satisfies

$$0 \leqslant C(r,z),\, C \in C^2\left(\mathbb{R}^2\right),\, (r,z) \notin [r_-,r_+] \times [z_-,z_+] \Rightarrow C(r,z) = 0, \tag{2.4}$$

and our geometrical framework is given by (2.1), (2.2), (2.4).

We note that t is a timelike coordinate and $(\mathcal{M},g)$ is naturally time oriented by the continuous, nowhere vanishing, timelike (and Killing) vector field ∂_t. Moreover r and z are spacelike coordinates. The interesting fact is that the nature of the Killing vector field ∂_φ is ambiguous: the crucial point is that φ is a *timelike* coordinate when $C > r$, thus we introduce:

$$\mathcal{T} := \mathbb{R}_t \times \mathcal{T}_0, \quad \mathcal{T}_0 := S^1 \times \left\{(r,z);\, C(r,z) > 0\right\}, \tag{2.5}$$

$$\mathbb{T} := \mathbb{R}_t \times \mathbb{T}_0, \quad \mathbb{T}_0 := S^1 \times \left\{(r,z);\, C(r,z) > r\right\}, \tag{2.6}$$

$$\Sigma := \mathbb{R}_t \times \Sigma_0, \quad \Sigma_0 := S^1 \times \left\{(r,z);\, C(r,z) = r > 0\right\}. \tag{2.7}$$

We shall need the hypersurfaces

$$M_t := \{t\} \times \mathbb{R}^3. \tag{2.8}$$

Its causal structure is complex. Since its normal is $\mathrm{d}t$, the nature of M_t is locally given by the sign of

$$g^{tt} = 1 - \frac{C^2}{r^2},$$

hence $M_t \cap (\mathbb{R}^3 \setminus (\mathbb{T} \cup \Sigma))$ is spacelike, $M_t \cap \Sigma$ is null, and $M_t \cap \mathbb{T}$ is timelike.

We shall be mainly concerned by the case where Σ is not empty. In this situation the causality is violated in a severe way: given $m_0 = (t_0,\varphi_0,r_0,z_0)$, the path

$$\tau \in \mathbb{R} \mapsto m(\tau) = (t_0,\varphi_0 - \tau,r_0,z_0) \in \mathcal{M}, \tag{2.9}$$

is a future directed closed null curve if $m_0 \in \Sigma$, and a future directed closed timelike curve if $m_0 \in \mathbb{T}$ since:

$$g\left(\frac{\mathrm{d}m}{\mathrm{d}\tau},\frac{\mathrm{d}m}{\mathrm{d}\tau}\right) = C^2(r_0,z_0) - r_0^2, \qquad g\left(\frac{\mathrm{d}m}{\mathrm{d}\tau},\frac{\partial}{\partial t}\right) = 2C(r_0,z_0) > 0.$$

More precisely, the causal structure of $\mathcal{M}$ is described by the following:

Proposition 2.1. *Let* $(\mathcal{M},g)$ *be the Lorentzian manifold defined by* (2.1), (2.2), (2.4).

(1) *If* $\Sigma = \emptyset$, $(\mathcal{M},g)$ *is globally hyperbolic*: M_t *is a Cauchy hypersurface for any* $t \in \mathbb{R}$.

(2) *If $\mathbb{T} = \emptyset$ and $\Sigma \neq \emptyset$, $(\mathcal{M}, g)$ is chronological but non-causal: there exists no closed timelike curve, but there exists a closed null geodesic.*
(3) *If $\mathbb{T} \neq \emptyset$, $(\mathcal{M}, g)$ is totally vicious, i.e. given $m_0, m_1 \in \mathcal{M}$, there exists a timelike future-pointing curve from m_0 to m_1.*

Proof. (1) If $\Sigma = \emptyset$, there exists $\varepsilon > 0$ such that

$$0 < \varepsilon^2 = \inf\bigl(r^2 - C^2\bigr).$$

Let $\tau \in \mathbb{R} \mapsto m(\tau) = (t(\tau), \varphi(\tau), r(\tau), z(\tau)) \in \mathcal{M}$ be a nonspacelike smooth curve. Since

$$\dot{t}^2 + \bigl(C^2 - r^2\bigr)\dot{\varphi}^2 - 2C\dot{t}\dot{\varphi} - \dot{r}^2 - \dot{z}^2 \geqslant 0, \quad \dot{m} \neq \vec{0},$$

$\dot{t}$ cannot vanish, for instance $t(\tau)$ is strictly increasing. If $t(\tau)$ is bounded as $\tau \to \pm\infty$, then $\dot{t}$ is integrable on $\mathbb{R}^{\pm}$. Moreover we have:

$$\varepsilon^2\dot{\varphi}^2 + \dot{r}^2 + \dot{z}^2 \leqslant \dot{t}^2 - 2\left(\frac{2}{\varepsilon}C\dot{t}\right)\frac{\varepsilon}{2}\dot{\varphi} \leqslant \left(1 + \frac{4C^2}{\varepsilon^2}\right)\dot{t}^2 + \frac{\varepsilon^2}{4}\dot{\varphi}^2.$$

Therefore $\dot{r}$, $\dot{z}$, $\dot{\varphi}$ are integrable on $\mathbb{R}^{\pm}$ and $m(\tau)$ is an extendible curve. We conclude that if $\Sigma = \emptyset$, any inextendible nonspacelike curve intersects exactly once M_t which is a Cauchy hypersurface. Therefore $(\mathcal{M}, g)$ is globally hyperbolic.

(2) Now the geodesics $\tau \in \mathbb{R} \mapsto m(\tau) = (t(\tau), \varphi(\tau), r(\tau), z(\tau)) \in \mathcal{M}$ are defined by the Euler–Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^{\mu}}\right) = \frac{\partial \mathcal{L}}{\partial x^{\mu}},$$

associated with the Lagrangian:

$$\mathcal{L} := \dot{t}^2 + \bigl(C^2 - r^2\bigr)\dot{\varphi}^2 - 2C\dot{t}\dot{\varphi} - \dot{r}^2 - \dot{z}^2. \tag{2.10}$$

The timelike Killing field $\partial/\partial t$ and the axial Killing field $\partial/\partial\varphi$ yield a conserved energy E, and a conserved angular momentum Ω:

$$E = \dot{t} - C(r,z)\dot{\varphi}, \tag{2.11}$$
$$\Omega = \bigl(C^2(r,z) - r^2\bigr)\dot{\varphi} - C(r,z)\dot{t}. \tag{2.12}$$

The two others geodesics equations are:

$$\ddot{r} = -\left(\frac{\Omega + C(r,z)E}{r^2}\right)\left(E\frac{\partial}{\partial r}C(r,z) - \frac{\Omega + C(r,z)E}{r}\right), \tag{2.13}$$
$$\ddot{z} = -\left(\frac{\Omega + C(r,z)E}{r^2}\right)E\frac{\partial}{\partial z}C(r,z). \tag{2.14}$$

Let (φ_0, r_0, z_0) be in Σ_0. Since $\mathbb{T} = \emptyset$, $C(r,z) - r \leqslant 0$ everywhere, hence we have:

$$C(r_0, z_0) = r_0, \qquad \partial_r C(r_0, z_0) = 1, \qquad \partial_z C(r_0, z_0) = 0, \tag{2.15}$$

and the path (2.9) satisfies (2.10), (2.11), (2.12), (2.13), (2.14), for $\mathcal{L} = 0$, $\Omega = 0$. Therefore it is a closed null geodesic: $(\mathcal{M}, g)$ is non-causal.

Now we consider a future pointing timelike curve $\tau \mapsto (t, \varphi, r, z)$:

$$\dot{t} \geqslant 0, \qquad \mathcal{L} > 0.$$

We deduce that:

$$\dot{t} - C\dot{\varphi} > \sqrt{r^2\dot{\varphi}^2 + \dot{r}^2 + \dot{z}^2}.$$

Therefore we have:

$$\dot{t} > (r - C)|\dot{\varphi}| \geqslant 0,$$

and the curve cannot be closed: $(\mathcal{M}, g)$ is chronological.

(3) In order to prove that $(\mathcal{M}, g)$ is totally vicious if $\mathbb{T} \neq \emptyset$, it is sufficient to construct, given $m_j = (t_j, \varphi_j, r_j, z_j) \in \mathcal{M}$, a C^1-piecewise curve from m_0 to m_1. We consider $m_* = (t_*, \varphi_*, r_*, z_*) \in \mathbb{T}$. First we define for $\alpha_0 > 0$

$$\tau \in [0,1] \mapsto m_0\bigl(\tau\bigr) = \bigl(t_0 + \alpha_0\tau, (1-\tau)\varphi_0 + \tau\varphi_*, (1-\tau)r_0 + \tau r_*, (1-\tau)z_0 + \tau z_*\bigr).$$

Since $r_* \leqslant r_+$, $|z_*| \leqslant \sup_\pm |z_\pm|$, and C is bounded, we have:

$$g\left(\dot m_0, \frac{\partial}{\partial t}\right) \geqslant \alpha_0 - A,$$
$$g(\dot m_0, \dot m_0) \geqslant \alpha_0^2 - A'\alpha_0 - A'',$$

where $0 < A, A', A''$ depend only of r_0 and z_0. We deduce that for $\alpha_0 = \alpha_0(r_0, z_0)$ large enough, $m_0(\tau)$ is a future pointing timelike path, from m_0 to $m_{*,0} := (t_0 + \alpha_0, \varphi_*, r_*, z_*) \in \mathbb{T}$. By the same way we construct a future pointing timelike path $m_1(\tau)$, from $m_{*,1} := (t_1 - \alpha_1, \varphi_*, r_*, z_*) \in \mathbb{T}$ to m_1. Now we show that there exists a future pointing timelike path, $p_*(\tau)$, from $m_{*,0}$ to $m_{*,1}$. If $t_0 + \alpha_0 \leqslant t_1 - \alpha_1$ we put:

$$p_*(\tau) = \big((1-\tau)(t_0 + \alpha_0) + \tau(t_1 - \alpha_1), \varphi_*, r_*, z_*\big).$$

If $t_0 + \alpha_0 > t_1 - \alpha_1$ we define for $k \in \mathbb{N}$:

$$p_*(\tau) = \big((1-\tau)(t_0 + \alpha_0) + \tau(t_1 - \alpha_1), \varphi_* - 2k\pi\tau, r_*, z_*\big).$$

We have:

$$g\left(\dot p_*, \frac{\partial}{\partial t}\right) = t_1 - \alpha_1 - t_0 - \alpha_0 + 2k\pi C(r_*, z_*),$$
$$g(\dot p_*, \dot p_*) = (t_1 - \alpha_1 - t_0 - \alpha_0)^2 + \big(C^2(r_*, z_*) - r_*^2\big)4k^2\pi^2 + 4k\pi C(r_*, z_*)(t_1 - \alpha_1 - t_0 - \alpha_0).$$

Since $C(r_*, z_*) > r_*$ we can choose k large enough in order to $p_*(\tau)$ is a future directed timelike path. Finally we glue $m_0(\tau)$, $p_*(\tau)$ and then $m_1(\tau)$ to join m_0 to m_1 by a C^1 piecewise, future going timelike curve. □

The previous proposition explains why, in the physical literature (see, e.g., [13,35]), $\mathbb{T}$ and Σ are respectively called, *time machine*, and *velocity-of-light surface*. This last term is somewhat misleading since $\partial(\mathcal{M} \setminus \mathbb{T}) \subset \Sigma$, but it can happen that $\partial(\mathcal{M} \setminus \mathbb{T}) \neq \Sigma$ and Σ is not necessarily a hypersurface. If there exists no (r_0, z_0) satisfying (2.15), the theorem of implicit functions immediately assures that Σ is a C^2-hypersurface that is timelike because its normal $N = (\partial_r C - 1)\,\mathrm{d}r + \partial_z C\,\mathrm{d}z$ is spacelike since $g^{\mu,\nu} N_{\mu,\nu} = -(\partial_r C - 1)^2 - (\partial_z C)^2 < 0$. Moreover, this is a sufficient and necessary condition on C for a geometrical property of non-trapping type:

Proposition 2.2. *Let $m \in C^2(\mathbb{R}_\tau; \mathcal{M})$ be a path. Then the following assertions are equivalent*:

(i) *m is a null geodesic and for some $T > 0$*:

$$m(\mathbb{R}) \subset [-T, +T]_t \times \Sigma_0, \tag{2.16}$$

(ii) *there exists $(t_0, \varphi_0, r_0, z_0)$, $\lambda \in \mathbb{R}^*$, such that*:

$$\begin{aligned} &C(r_0, z_0) = r_0 > 0, \qquad \partial_r C(r_0, z_0) = 1, \qquad \partial_z C(r_0, z_0) = 0,\\ &m(\tau) = (t_0, \varphi_0 + \lambda\tau, r_0, z_0). \end{aligned} \tag{2.17}$$

Proof. We have seen that the path (2.9) satisfying (2.17) is a null closed geodesic included in M_{t_0}. Conversely, the equations of a geodesic satisfying $m(\mathbb{R}) \subset \Sigma$ are:

$$\dot t = -\frac{\Omega}{r}, \qquad \dot\varphi = -\frac{\Omega}{r^2} - \frac{E}{r},$$
$$\ddot r = -\left(\frac{\Omega}{r^2} + \frac{E}{r}\right)\left[E(\partial_r C - 1) - \frac{\Omega}{r}\right], \qquad \ddot z = -\left(\frac{\Omega}{r^2} + \frac{E}{r}\right)E\,\partial_z C.$$

Thus (2.16) implies that $\Omega = 0$ since $0 < r_- < r < r_+$. If m is also a null geodesic we have:

$$0 = \mathcal{L} = -\dot r^2 - \dot z^2,$$

hence $r = r_0$, $z = z_0$, and since E cannot be zero, we deduce that $\partial_r C(r_0, z_0) = 1$, $\partial_z C(r_0, z_0) = 0$. In this case, the path (2.9) is a null geodesic, therefore m is given by (2.17). □

We say that Σ_0 is *Non-Confining* if there exists no null geodesic included in $\{t_0\} \times \Sigma_0$ for some t_0. Following the previous result, a necessary and sufficient condition is:

$$C(r_0, z_0) = r_0 > 0 \Longrightarrow \big(\partial_r C(r_0, z_0), \partial_z C(r_0, z_0)\big) \neq (1, 0), \tag{2.18}$$

and in this case Σ is a C^2 timelike hypersurface.

3. The wave equation

The D'Alembertian on a Lorentzian manifold $(\mathcal{M}, g)$ is defined by:

$$\Box_g := |\det g|^{-\frac{1}{2}}\partial_\mu\big(|\det g|^{\frac{1}{2}} g^{\mu,\nu}\partial_\nu\big).$$

For the space-time given by (2.1), (2.2), we obtain:

$$\Box_g = \left(1 - \frac{C^2}{r^2}\right)\partial_t^2 - \Delta_x - 2\frac{C}{r^2}\partial_t\partial_\varphi, \tag{3.1}$$

with

$$r^2 = \left|x^1\right|^2 + \left|x^2\right|^2, \qquad \Delta_x := \partial_{x^1}^2 + \partial_{x^3}^2 + \partial_{x^3}^2 = \partial_r^2 + \partial_z^2 + r^{-2}\partial_\varphi^2 + r^{-1}\partial_r, \qquad \partial_\varphi = x^1\partial_{x^2} - x^2\partial_{x^1}.$$

More generally we consider the scalar perturbations of the massless wave equation, compactly supported in x, invariant with respect to the both Killing vector fields ∂_t, ∂_φ:

$$L := \Box_g + V, \tag{3.2}$$

where

$$V \in C_0^0\big(\mathbb{R}_x^3; \mathbb{R}\big), \qquad \partial_\varphi V = 0. \tag{3.3}$$

These assumptions are fulfilled in the important case of the conformally invariant wave equation for which:

$$V = \frac{1}{6}R_g, \tag{3.4}$$

where R_g is the scalar curvature of $(\mathcal{M}, g)$. We use $R_0 > 0$ be such that

$$R_0 \leqslant |x| \Longrightarrow C(r,z) = V(x) = 0. \tag{3.5}$$

We know that the D'Alembertian on a Lorentzian curved space-time is strictly hyperbolic in a local sense (see, e.g., [10]). The global hyperbolicity is more delicate. We denote:

$$P_2(m,\xi) := g^{\mu,\nu}(m)\xi_\mu\xi_\nu, \quad m \in \mathcal{M},\ \xi \in T_m^*\mathcal{M}, \tag{3.6}$$

the principal symbol of L.

Proposition 3.1. (1) *Let α be in $\mathbb{R}$. Then, $P_2(m,\cdot)$ is (strictly) hyperbolic with respect to the covector* $\mathrm{d}t + \alpha\,\mathrm{d}\varphi$ *iff α satisfies*:

$$-C(m) - r < \alpha < r - C(m). \tag{3.7}$$

(2) *If $\Sigma \neq \emptyset$, there does not exist $F \in C^1(\mathcal{M}; \mathbb{R})$ such that L is hyperbolic with respect to the level surfaces of F.*

Proof. (1) Given $\xi = (\xi_t, \xi_\varphi, \xi_r, \xi_z) \in \mathbb{R}^4$ a covector, $N := \mathrm{d}t + \alpha\,\mathrm{d}\varphi$, $\tau \in \mathbb{R}$, we calculate:

$$\begin{aligned} P_2(m,N) &= 1 - \frac{(C+\alpha)^2}{r^2}, \\ P_2(m,\xi+\tau N) &= \left[1 - \frac{(C+\alpha)^2}{r^2}\right]\tau^2 + 2\left[\left(1 - \frac{C^2}{r^2}\right)\xi_t - \frac{\alpha\xi_\varphi}{r^2} - \frac{C}{r^2}(\alpha\xi_t + \xi_\varphi)\right]\tau \\ &\quad + \left(1 - \frac{C^2}{r^2}\right)\xi_t^2 - \xi_r^2 - \xi_z^2 - \frac{1}{r^2}\xi_\varphi^2 - \frac{2C}{r^2}\xi_t\xi_\varphi. \end{aligned}$$

The reduced discriminant of the equation $P_2(\xi + \tau N) = 0$ is equal to:

$$\Delta' = \frac{1}{r^2}(\alpha\xi_t - \xi_\varphi)^2 + \left[1 - \frac{(C+\alpha)^2}{r^2}\right]\big(\xi_r^2 + \xi_z^2\big).$$

We conclude that $P_2(m,\cdot)$ is hyperbolic with respect to N iff $[1 - \frac{(C+\alpha)^2}{r^2}] > 0$, and in the case $P_2(m,\cdot)$ is strictly hyperbolic.

(2) Let F be in $C^1(\mathcal{M}; \mathbb{R})$. We assume that

$$\forall m \in \mathcal{M}, \quad P_2\big(m,\ \mathrm{d}F(m)\big) \neq 0, \tag{3.8}$$

and that for every $m \in \mathcal{M}$ and $\xi \in T_m^*(\mathcal{M}) \setminus \mathbb{R}\,\mathrm{d}F(m)$, the roots of the characteristic equation

$$P_2\big(m, \xi + \tau\,\mathrm{d}F(m)\big) = 0$$

are real. We consider $m \in \Sigma$ and we choose $\xi = (\xi_t = 0, \xi_\varphi = 0, \xi_r, \xi_z) \in T_m^*(\mathcal{M}) \setminus \mathbb{R}\,\mathrm{d}F(m)$, $\xi_r, \xi_z \neq 0$. We have:

$$\begin{aligned} P_2\big(m, \xi + \tau\,\mathrm{d}F(m)\big) \;=\; & \tau^2\left[-\big|\partial_r F(m)\big|^2 - \frac{1}{r^2}\big|\partial_\varphi F(m)\big|^2 - \frac{2}{r}\partial_t F(m)\partial_\varphi F(m)\right] \\ & - 2\tau\big(\xi_r\partial_r F(m) + \xi_z\partial_z F(m)\big) - \xi_r^2 - \xi_z^2. \end{aligned}$$

The reduced discriminant is equal to:

$$\Delta' = \big(\xi_r^2 + \xi_z^2\big)\left(\big|\partial_t F(m)\big|^2 - \left(\frac{r\partial_t F(m) + \partial_\varphi F(m)}{r}\right)^2\right) - \xi_r^2\big|\partial_z F(m)\big|^2 - \xi_z^2\big|\partial_r F(m)\big|^2.$$

If $(\partial_r F(m), \partial_z F(m)) \neq (0,0)$, the condition $\Delta' \geqslant 0$ implies $\partial_\varphi F(m) \neq 0$. If $\partial_r F(m) = \partial_z F(m) = 0$, we have

$$P_2\big(m, \mathrm{d}F(m)\big) = -\frac{1}{r^2}\partial_\varphi F(m)\big(\partial_\varphi F(m) + 2r\partial_t F(m)\big),$$

hence we deduce from (3.8) that $\partial_\varphi F(m) \neq 0$ again. Now if $m = (t, \varphi, r, z) \in \Sigma$, then $\mathcal{C}_m := \{t\} \times S^1 \times \{r\} \times \{z\} \subset \Sigma$ and we conclude that $\partial_\varphi F \neq 0$ on $\mathcal{C}_m$. Obviously that is a contradiction. □

The previous result implies in particular that in the interesting case where $\mathbb{T} \neq \emptyset$, the initial value problem for L with data specified on $M_{t_0} = \{t_0\} \times \mathbb{R}^3$ is not well posed. (3.7) shows that the failure of the global hyperbolicity is due to the very fast rotation of the torus. Nevertheless, since ∂_t is a Killing vector field, it will be interesting to investigate the solutions of $Lu = 0$ as some distributions on $\mathbb{R}_t$, valued in some spaces of distributions on $\mathbb{R}_x^3$. In order to choose the functional framework, it is useful to note that since the time translation leaves the wave equation invariant, the Noether's theorem assures the existence of a conserved current. We formally obtain the conserved energy

$$E(u;t) := \frac{1}{2}\int_{\mathbb{R}^3}\left(1 - \frac{C^2}{r^2}\right)\big|\partial_t u(t,x)\big|^2 + \big|\nabla u(t,x)\big|^2 + V(x)\big|u(t,x)\big|^2\,\mathrm{d}x. \tag{3.9}$$

Therefore it is natural to look for the solutions of

$$Lu = 0, \quad u \in L^2_{\mathrm{loc}}\big(\mathbb{R}_t; W^1\big(\mathbb{R}_x^3\big)\big), \tag{3.10}$$

where $W^1(\mathbb{R}_x^3)$ is the Beppo–Levi space defined as the completion of $C_0^\infty(\mathbb{R}_x^3)$ with respect to the norm:

$$\|f\|_{W^1}^2 = \int_{\mathbb{R}^3}\big|\nabla f(x)\big|^2\,\mathrm{d}x, \quad \nabla := {}^t(\partial_{x^1}, \partial_{x^2}, \partial_z). \tag{3.11}$$

We recall the L^2-type estimate:

$$W^1\big(\mathbb{R}_x^3\big) \subset L^2_\rho\big(\mathbb{R}_x^3\big) := L^2\left(\mathbb{R}_x^3, \frac{1}{1+|x|^2}\,\mathrm{d}x\right), \qquad \|f\|_{L^2_\rho} \leqslant K\|f\|_{W^1}. \tag{3.12}$$

The choice of the regularity of $\partial_t u$ is less clear when $\mathcal{M}$ is not globally hyperbolic since $(1 - C^2/r^2)$ is negative on $\mathbb{T}_0$ and the energy is not a positive form. We introduce the space:

$$L^2_C\big(\mathbb{R}_x^3\big) := L^2\left(\mathbb{R}_x^3, \left|1 - \frac{C^2}{r^2}\right|\mathrm{d}x\right), \tag{3.13}$$

and we investigate the solutions u of (3.10) satisfying:

$$\partial_t u \in L^2_{\mathrm{loc}}\big(\mathbb{R}_t; L^2_C\big(\mathbb{R}_x^3\big)\big). \tag{3.14}$$

With this functional framework, we define usefull quantities associated with the wave equation: for $0 \leqslant R \leqslant \infty$, the local energy of u at time t is given by:

$$E_R(u;t) := \frac{1}{2}\int_{|x|\leqslant R}\left(1 - \frac{C^2}{r^2}\right)\big|\partial_t u(t,x)\big|^2 + \big|\nabla u(t,x)\big|^2 + V(x)\big|u(t,x)\big|^2\,\mathrm{d}x. \tag{3.15}$$

The Wronskian of u, v is defined by:

$$W(u,v;t) := \int_{\mathbb{R}^3} \left(1-\frac{C^2}{r^2}\right)\big(\partial_t u(t,x)\overline{v(t,x)} - u(t,x)\partial_t \overline{v(t,x)}\big) - \frac{C}{r^2}\big(\partial_\varphi u(t,x)\overline{v(t,x)} - u(t,x)\partial_\varphi \overline{v(t,x)}\big)\,\mathrm{d}x. \tag{3.16}$$

Lemma 3.2. *Given u, v satisfying* (3.10) *and* (3.14)*, we have for $R \geqslant R_0$, and almost all $t, s \in \mathbb{R}$:*

$$E_R(u,t) \leqslant E_{R+|t-s|}(u,s), \tag{3.17}$$

$$E_\infty(u,t) = E_\infty(u,s), \tag{3.18}$$

$$W(u,v;t) = W(u,v;s). \tag{3.19}$$

When $u, v \in C^0(\mathbb{R}_t; W^1(\mathbb{R}^3_x))$, $\partial_t u, \partial_t v \in C^0(\mathbb{R}_t; L^2_C(\mathbb{R}^3_x))$, (3.17), (3.18), (3.19) are satisfied for any $s, t \in \mathbb{R}$, and the conserved quantity

$$E(u) := E_\infty(u,t) \tag{3.20}$$

is the total energy of u. If $\mathbb{T}$ is not empty, this quadratic form is not definite positive.

Proof of Lemma 3.2. We choose $\theta(t) \in C_0^\infty(\mathbb{R}_t)$ such that $\int \theta(t)\,\mathrm{d}t = 1$. For $j \in \mathbb{N}$ we put:

$$u_j(t) = j \int_{-\infty}^{\infty} \theta(js) u(t-s)\,\mathrm{d}s. \tag{3.21}$$

v_j is defined by similar way. It is clear that u_j approximates u:

$$u_j \to u \quad \text{in } L^2_{\mathrm{loc}}\big(\mathbb{R}_t; W^1\big(\mathbb{R}^3_x\big)\big), \qquad \partial_t u_j \to \partial_t u \quad \text{in } L^2_{\mathrm{loc}}\big(\mathbb{R}_t; L^2_C\big(\mathbb{R}^3_x\big)\big) \quad \text{as } j \to \infty,$$

and we easily check that:

$$E_R(u_j,t) \to E_R(u,t), \quad W(u_j,v_j;t) \to W(u,v;t) \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}_t), \quad j \to \infty.$$

Therefore it is sufficient to prove that (3.18), (3.17) and (3.19) are true for u_j for all t, s to get that these estimates are satisfied for u and almost all t, s. For that, we note that u_j is a solution, smooth by respect to t:

$$Lu_j = 0, \quad u_j \in C^\infty\big(\mathbb{R}_t; W^1\big(\mathbb{R}^3_x\big)\big), \qquad \partial_t u_j \in C^\infty\big(\mathbb{R}_t; L^2\big(\mathbb{R}^3_x\big)\big). \tag{3.22}$$

Moreover, by using the equation $Lu = 0$ and the embedding of the Sobolev spaces, we have

$$u_j \in C^\infty(\mathcal{M}). \tag{3.23}$$

For such solutions, we may derivate the Wronskian or the total energy by respect to t, and by using the equation and an integration by parts, we get (3.19) and (3.18) for all t, s. As regards the local energy estimates, we check that

$$\partial_\mu P^\mu = (Lu_j)\overline{\partial_t u_j} = 0,$$

where the Pointing vector P is defined by:

$$2P^t = \left(1-\frac{C^2}{r^2}\right)|\partial_t u_j|^2 + |\nabla_x u_j|^2 + V|u_j|^2,$$

$$P^x = -\Re \partial_t u_j \overline{\partial_x u_j} + Cr^{-2} y |\partial_t u_j|^2, \qquad P^y = -\Re \partial_t u_j \overline{\partial_y u_j} - Cr^{-2} x |\partial_t u_j|^2, \qquad P^z = -\Re \partial_t u_j \overline{\partial_z u_j}.$$

We evaluate

$$0 = 2\int_{\mathcal{D}} \partial_\mu P^\mu(t,x)\,\mathrm{d}t\,\mathrm{d}x, \quad \text{on } \mathcal{D} = \big\{(t,x);\ |x| \leqslant T - t + R\big\},$$

and we get:

$$E_{R+T}(u_j,0) - E_R(u_j,T) = \frac{1}{2}\int_{|x|=T-t+R} |\partial_t u_j|^2 + |\nabla_x u_j|^2 - 2\Re \partial_t u_j \overline{\nabla_x u_j} \cdot \frac{x}{|x|}\,\mathrm{d}\sigma \geqslant 0. \qquad \square$$

We could only consider solutions of (3.10) such that $\partial_t u \in L^2_{\mathrm{loc}}(\mathbb{R}_t; L^2_C(\mathbb{R}^3_x))$, but if Σ_0 is Non-Confining, $\partial_t u$ is much more regular:

Theorem 3.3. *We assume that Σ_0 is Non-Confining. Let u be such that*

$$u \in L^2_{\mathrm{loc}}\big(\mathbb{R}_t; W^1\big(\mathbb{R}^3_x\big)\big), \quad Lu \in L^2_{\mathrm{loc}}\big(\mathbb{R}_t; L^2\big(\mathbb{R}^3_x\big)\big). \tag{3.24}$$

Then we have:

$$\partial_t u \in L^2_{\mathrm{loc}}\big(\mathbb{R}_t; L^2\big(\mathbb{R}^3_x\big)\big). \tag{3.25}$$

Proof. We shall use the results of microlocal regularity and of propagation of singularities, which are due to L. Hörmander [18] when C and V are C^∞, and J.-M. Bony [3] when C and V are C^2 (see so [2]). Since $u \in L^2_{\mathrm{loc}}(\mathbb{R}_t; W^1(\mathbb{R}^3_x))$, it is microlocally H^1 near $(m,\xi) \in T^*(\mathcal{M})$ for $\xi \notin \mathbb{R}\,\mathrm{d}t$. On the other hand since $P_2(m, \mathrm{d}t) = 1 - C^2 r^{-2}$, L is microlocally elliptic near $(m, \mathrm{d}t)$ for $m \notin \Sigma$ hence $u \in H^2$ microlocally near these points, and finally $u \in H^1_{\mathrm{loc}}(\mathcal{M} \setminus \Sigma)$. Therefore to prove that $u \in H^1_{\mathrm{loc}}(\mathcal{M})$ we have to establish that u is microlocally H^1 near $(m_0, \mathrm{d}t)$ for $m_0 \in \Sigma$. Let $\tau \in \mathbb{R} \mapsto \gamma(\tau) = (m(\tau), \xi(\tau)) \in T^*\mathcal{M}$ be a bicharacteristic curve with

$$\gamma(0) = (m_0, \mathrm{d}t). \tag{3.26}$$

The equations for γ are:

$$\dot{x}^\mu = \frac{\partial P_2}{\partial \xi_\mu}, \qquad \dot{\xi}_\mu = -\frac{\partial P_2}{\partial x^\mu},$$
$$P_2(\xi) = \mathcal{L}(\dot{m}) = 0.$$

We get:

$$\dot{\xi}_t = 0, \qquad \dot{\xi}_\varphi = 0, \tag{3.27}$$

$$\dot{t} = 2\left(1 - \frac{C^2}{r^2}\right)\xi_t - 2\frac{C}{r^2}\xi_\varphi, \qquad \dot{\varphi} = -\frac{2}{r^2}\xi_\varphi - 2\frac{C}{r^2}\xi_t, \tag{3.28}$$

$$\dot{\xi}_r = -2\frac{C}{r^2}\left(\partial_r C - \frac{C}{r}\right), \qquad \dot{\xi}_z = -2\frac{C}{r^2}\partial_z C, \tag{3.29}$$

$$\dot{r} = -2\xi_r, \qquad \dot{z} = -2\xi_z, \tag{3.30}$$

$$(\dot{t} - C\dot{\varphi})^2 - r^2\dot{\varphi}^2 - \dot{r}^2 - \dot{z}^2 = 0. \tag{3.31}$$

We obtain:

$$\xi_t = 1, \qquad \xi_\varphi = 0, \qquad \dot{t} = 2\left(1 - \frac{C^2}{r^2}\right), \qquad \dot{\varphi} = -2\frac{C}{r^2}, \tag{3.32}$$

and since m is a null geodesic

$$\dot{t} - C\dot{\varphi} \;=\; \dot{t}(0) - C\dot{\varphi}(0) = 2, \tag{3.33}$$

$$\big(C^2 - r^2\big)\dot{\varphi} - C\dot{t} \;=\; \big(C^2(r_0, z_0) - r_0^2\big)\dot{\varphi}(0) - C\dot{t}(0) = 0, \tag{3.34}$$

$$0 \leqslant \dot{r}^2 + \dot{z}^2 = 4\left(1 - \frac{C^2}{r^2}\right). \tag{3.35}$$

We deduce that

$$\gamma(\mathbb{R}) \cap T^*(\mathbb{T}) = \emptyset, \tag{3.36}$$

and since Σ_0 is Non-Confining we have:

$$\gamma(\mathbb{R}) \cap T^*(\mathbb{T} \cup \Sigma) = \emptyset. \tag{3.37}$$

Then there exists τ such that $m(\tau) \notin \mathbb{T}_0 \cup \Sigma_0$. We get from (3.30), (3.35) that $\xi(\tau) \notin \mathbb{R}\,\mathrm{d}t$. Since $u \in H^1$ microlocally near $\gamma(\tau)$ we deduce from the theorem of propagation of singularities that $u \in H^1$ microlocally near $\gamma(0)$. We conclude that $u \in H^1_{\mathrm{loc}}(\mathcal{M})$ and

$$\partial_t u \in L^2_{\mathrm{loc}}\big(\mathbb{R}_t; L^2_{\mathrm{loc}}\big(\mathbb{R}^3_x\big)\big). \tag{3.38}$$

Let χ be in $C_0^\infty(\mathbb{R}_x^3)$ equal to 1 on a neighborhood of $\mathcal{T}_0$. Then $v := (1-\chi)u$ satisfies:

$$v \in L^2_{\mathrm{loc}}\big(\mathbb{R}_t; W^1\big(\mathbb{R}_x^3\big)\big), \qquad \partial_t^2 v - \Delta_x v \in L^2_{\mathrm{loc}}\big(\mathbb{R}_t; L^2\big(\mathbb{R}_x^3\big)\big).$$

We deduce that

$$\partial_t^2 v \in L^2_{\mathrm{loc}}\big(\mathbb{R}_t; \big[W^1\big(\mathbb{R}_x^3\big)\big]'\big),$$

hence by the theorem of intermediate derivates ([26], p. 19, Theorem 2.3):

$$\partial_t v \in L^2_{\mathrm{loc}}\big(\mathbb{R}_t; L^2\big(\mathbb{R}_x^3\big)\big). \tag{3.39}$$

The result follows from (3.38) and (3.39). □

The previous result allows define the trace of u and $\partial_t u$ on M_t. We refer to [26] for the definitions and properties of the usual Sobolev spaces H^s, H_0^s.

Proposition 3.4. *We assume that Σ_0 is Non-Confining. Let u be such that*

$$u \in L^2_{\mathrm{loc}}\big(\mathbb{R}_t; W^1\big(\mathbb{R}_x^3\big)\big), \quad Lu \in L^2_{\mathrm{loc}}\big(\mathbb{R}_t; L^2\big(\mathbb{R}_x^3\big)\big).$$

Then we have:

$$u \in C^0\big(\mathbb{R}_t; H^{\frac{1}{2}}\big(\mathbb{R}_x^3\big)\big), \tag{3.40}$$

$$\left(1-\frac{C}{r}\right)\partial_t u \in C^0\big(\mathbb{R}_t; H^{-\frac{1}{2}}\big(\mathbb{R}_x^3\big)\big), \tag{3.41}$$

$$\partial_t u \in C^0\big(\mathbb{R}_t; H^{-1}(\mathbb{T}_0)\big). \tag{3.42}$$

Proof. Since $u \in L^2_{\mathrm{loc}}(\mathbb{R}_t; W^1(\mathbb{R}_x^3))$, and Σ_0 is Non-Confining, the previous theorem implies that $\partial_t u \in L^2_{\mathrm{loc}}(\mathbb{R}_t; L^2(\mathbb{R}_x^3))$. Thus $u \in C^0(\mathbb{R}_t; H^1(\mathbb{R}_x^3))$, and the intermediate derivates theorem (Theorem 3.1 of [26], p. 23) assures that

$$u \in C^0\big(\mathbb{R}_t; \big[H^1\big(\mathbb{R}_x^3\big), L^2\big(\mathbb{R}_x^3\big)\big]_{\frac{1}{2}} = H^{\frac{1}{2}}\big(\mathbb{R}_x^3\big)\big).$$

Now if $Lu \in L^2_{\mathrm{loc}}(\mathbb{R}_t; L^2(\mathbb{R}_x^3))$, we have:

$$\left(1-\frac{C^2}{r^2}\right)\partial_t^2 u \in L^2_{\mathrm{loc}}\big(\mathbb{R}_t; H^{-1}\big(\mathbb{R}_x^3\big)\big), \tag{3.43}$$

hence the same theorem implies that

$$\left(1-\frac{C^2}{r^2}\right)\partial_t u \in C^0\big(\mathbb{R}_t; \big[H^1\big(\mathbb{R}_x^3\big), H^{-1}\big(\mathbb{R}_x^3\big)\big]_{\frac{3}{4}} = H^{-\frac{1}{2}}\big(\mathbb{R}_x^3\big)\big). \tag{3.44}$$

That proves (3.41) since $(1+\frac{C}{r})^{-1}$ is a multiplier of $H^{\frac{1}{2}}$ because $C \in C^2$. Now since Σ_0 is Non-Confining, Theorem 11.8 of [26], p. 76 assures that:

$$\phi \in H_0^2(\mathbb{T}_0) \Longrightarrow \left(1-\frac{C^2}{r^2}\right)\phi \in H_0^1(\mathbb{T}_0).$$

Thus we deduce from (3.43) that

$$\partial_t^2 u \in L^2_{\mathrm{loc}}\big(\mathbb{R}_t; H^{-2}(\mathbb{T}_0)\big). \tag{3.45}$$

Finally (3.42) follows from (3.25) and (3.45) by the intermediate derivates theorem. □

Thanks to the result of continuity stated in Proposition 3.4, we may investigate the uniqueness of a possible solution of $Lu = 0$ for data specified on M_{t_0}. First we prove that $u = 0$ on $\mathcal{M}$ when $u = (C-r)\partial_t u = 0$ on M_{t_0}. This result is neither a consequence of the uniqueness theorem for the strictly hyperbolic operators ([18], Theorem 23.2.7) because the level surfaces M_t are not non-characteristic since $P_2(m, \mathrm{d}t) = 0$ on Σ, nor a direct application of the conservation of the energy since $E(u)$ is not definite positive.

Moreover, when $\mathcal{M}$ is totally vicious, i.e. $\mathbb{T} \neq \emptyset$, and the Non-Confining condition is fullfiled, we would like that $u = 0$ on $\mathcal{M}$ when $u = 0$ on $\mathbb{T}$. Unfortunately, although Σ is non-characteristic, we cannot use the classical results of unique continuation:

on the one hand, 0 is a double real root of $P_2(m, \mathrm{d}t + \tau N) = 0$ for $m \in \Sigma$, $N = (\partial_r C(m) - 1)\,\mathrm{d}r + \partial_z C(m)\,\mathrm{d}z$, hence we cannot apply the Calderon theorem ([18], Theorem 28.1.8). On the other hand, we have for $m \in \Sigma$:

$$\big\{P_2, \{P_2, C - r\}\big\}(m, \mathrm{d}t) = -4\big(\big|\partial_r C(m) - 1\big|^2 + \big|\partial_z C(m)\big|^2\big) < 0,$$

hence Σ is nowhere strongly pseudo-convex, and we can no more use the uniqueness theorems for second-order operators of real principal type due to N. Lerner and L. Robbiano (see [18], Theorem 28.4.3) to deduce that $u = 0$ on $\mathcal{M}$, from $u = 0$ on $\mathbb{T}$. This leads to make some assumption of analyticity on C and V near Σ_0, in order to apply the Holmgren theorem.

Theorem 3.5. *We assume that Σ_0 is Non-Confining and $\mathbb{T}_0 \neq \emptyset$. Let u be satisfying* (3.10) *and one of the following conditions for some $t_0 \in \mathbb{R}$*:

(1) $u = (1 - \frac{C}{r})\,\partial_t u = 0$ *on* M_{t_0}.
(2) $u = \partial_t u = 0$ *on* $\{t_0\} \times \mathbb{T}_0$ *and V and C are real analytic in a neighborhood of Σ_0.*

Then

$$u = 0 \quad \text{on } \mathcal{M}. \tag{3.46}$$

We shall see in Section 5 another uniqueness result for the incoming solutions.

Proof. A key ingredient is the following

Lemma 3.6. *We assume that Σ_0 is Non-Confining. Let u satisfying* (3.10) *and such that for some $t_0 \in \mathbb{R}$*:

$$u = \partial_t u = 0 \quad \text{on } \{t_0\} \times \mathbb{T}_0. \tag{3.47}$$

Then

$$u = 0 \quad \text{on } \mathbb{T}. \tag{3.48}$$

Proof. For $v \in C_0^\infty(\mathbb{R}^4)$, $m \in \mathbb{Z}$ we put:

$$P_m v(t, r, z) := \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}m\varphi} v(t, \varphi, r, z)\,\mathrm{d}\varphi.$$

The Fubini's theorem and the Parseval equality assure that P_m has a unique extension from $L^2_{\mathrm{loc}}(\mathcal{M})$ to

$$L^2_{\mathrm{loc}}(\mathbb{R}_t \times [0, \infty[_r \times \mathbb{R}_z, r\,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}z)$$

satisfying for $0 < r_0 < r_1$, $z_0 < z_1$:

$$\int_{-T}^{T}\int_0^{2\pi}\int_{r_0}^{r_1}\int_{z_0}^{z_1} \big|v(t, \varphi, r, z)\big|^2 r\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}\varphi\,\mathrm{d}r\,\mathrm{d}z = \sum_{m \in \mathbb{Z}} \int_{-T}^{T}\int_{r_0}^{r_1}\int_{z_0}^{z_1} \big|P_m v(t, r, z)\big|^2 r\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}r\,\mathrm{d}z. \tag{3.49}$$

If u satisfies (3.10), $u_m(t, \varphi, r, z) := \mathrm{e}^{\mathrm{i}m\varphi} P_m u(t, r, z)$ is solution of:

$$\left(1 - \frac{C^2}{r^2}\right)\partial_t^2 u_m - \Delta_x u_m - 2\mathrm{i}m\frac{C}{r^2}\partial_t u_m + V u_m = 0, \quad (t, x) \in \mathbb{T}, \tag{3.50}$$

$$u_m \in H^1_{\mathrm{loc}}(\mathbb{T}), \qquad u_m \in C^0\big(\mathbb{R}_t; H^{\frac{1}{2}}(\mathbb{T}_0)\big), \qquad \partial_t u_m \in C^0\big(\mathbb{R}_t; H^{-1}(\mathbb{T}_0)\big), \tag{3.51}$$

$$u_m = \partial_t u_m = 0 \quad \text{on } \{t_0\} \times \mathbb{T}_0. \tag{3.52}$$

(3.50) shows that u_m is solution of an elliptic equation in $\mathbb{T}$, therefore (3.52) and the Aronszajn–Cordes uniqueness theorem (see, e.g., [18], Theorem 17.2) imply that $u_m = 0$ on $\mathbb{T}$. Then (3.48) follows from (3.49). □

We now consider condition (1). It is sufficient to prove that $u = 0$ for $t > 0$. We define:

$$t > 0 \Longrightarrow v(t, x) = u(t, x), \qquad t < 0 \Longrightarrow v(t, x) = 0.$$

v satisfies (3.10). We use the approximation procedure (3.21) by putting:

$$v_j(t) = j \int_{-\infty}^{\infty} \theta(js)v(t-s)\,\mathrm{d}s.$$

v_j is a smooth solution and $v_j(t) = 0$ for $t < -1$ if θ is supported in $[-1, 1]$. Hence $v_j = 0$ in $\mathbb{T}$ by Lemma 3.6. Since Σ_0 is Non-Confining, $\partial\mathbb{T}_0 = \Sigma_0$, and the trace of $v_j(t)$ is zero on Σ_0. We use Theorem 11.3 of Lions and Magenes ([26], p. 65) to get

$$\int_{|x|\leqslant R} \left|1 - \frac{C^2}{r^2}\right|^{-1} \left|v_j(t,x)\right|^2 \mathrm{d}x \leqslant c_R \int_{\mathbb{R}^3} \left|\nabla_x v_j(t,x)\right|^2 \mathrm{d}x.$$

Now we evaluate

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^3} \left|1 - \frac{C^2}{r^2}\right| \left|\partial_t v_j(t,x)\right|^2 + \left|\nabla_x v_j(t,x)\right|^2 \mathrm{d}x\right) \leqslant 2\int_{\mathbb{R}^3} \left|V v_j \overline{\partial_t v_j}\right| \mathrm{d}x$$
$$\leqslant c'\left(\int_{\mathbb{R}^3} \left|1 - \frac{C^2}{r^2}\right| \left|\partial_t v_j(t,x)\right|^2\right)^{1/2} \left(\int_{\mathbb{R}^3} \left|\nabla_x v_j(t,x)\right|^2 \mathrm{d}x\right)^{1/2}.$$

We obtain by the Gronwall lemma:

$$\int_{\mathbb{R}^3} \left|1 - \frac{C^2}{r^2}\right| \left|\partial_t v_j(t,x)\right|^2 + \left|\nabla_x v_j(t,x)\right|^2 \mathrm{d}x \leqslant \mathrm{e}^{\beta|t+1|} \int_{\mathbb{R}^3} \left|1 - \frac{C^2}{r^2}\right| \left|\partial_t v_j(-1,x)\right|^2 + \left|\nabla_x v_j(-1,x)\right|^2 \mathrm{d}x. \tag{3.53}$$

Therefore $v_j = 0$ since $v_j(-1) = \partial_t v_j(-1) = 0$, and condition (1) implies (3.46).

For the second condition, we consider u_j given by (3.21). Since $u = 0$ in $\mathbb{T}$ by Lemma 3.6, we get that u_j is a smooth solution that is null in $\mathbb{T}$. As for (3.53), we obtain:

$$\int_{\mathbb{R}^3} \left|1 - \frac{C^2}{r^2}\right| \left|\partial_t u_j(t,x)\right|^2 + \left|\nabla_x u_j(t,x)\right|^2 \mathrm{d}x \leqslant \mathrm{e}^{\beta|t|} \int_{\mathbb{R}^3} \left|1 - \frac{C^2}{r^2}\right| \left|\partial_t u_j(0,x)\right|^2 + \left|\nabla_x u_j(0,x)\right|^2 \mathrm{d}x. \tag{3.54}$$

We put

$$w_j(t,x) := \mathrm{e}^{-t^2} u_j(t,x). \tag{3.55}$$

(3.54) assures that $w_j \in L^1(\mathbb{R}_t; W^1(\mathbb{R}_x^3))$, hence we can define the partial Fourier transform with respect to t of w_j:

$$\hat{w}_j(k,x) := \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kt} w_j(t,x)\,\mathrm{d}t \in C^0\big(\mathbb{R}_k; W^1\big(\mathbb{R}_x^3\big)\big). \tag{3.56}$$

$\hat{w}_j$ satisfies

$$(k,x) \in \mathbb{R} \times \mathbb{T}_0 \Longrightarrow \hat{w}_j(k,x) = 0, \tag{3.57}$$

$$\hat{A}\hat{w}_j = 0 \quad \text{on } \mathbb{R}_k \times \mathbb{R}_x^3, \tag{3.58}$$

where

$$\hat{A} := 4\left(1 - \frac{C^2}{r^2}\right)\partial_k^2 + \Delta_x + 4\mathrm{i}\frac{C}{r^2}\partial_{k,\varphi}^2 + 4k\left(1 - \frac{C^2}{r^2}\right)\partial_k + 2\mathrm{i}k\frac{C}{r^2}\partial_\varphi + \left(1 - \frac{C^2}{r^2}\right)\big(k^2 + 2 - V\big). \tag{3.59}$$

We remark that $\hat{A}$ is elliptic on $\mathbb{R}_k \times (\mathbb{R}_x^3 \setminus (\mathbb{T}_0 \cup \Sigma_0))$. Moreover, the Non-Confining condition assures that $\mathbb{R}_k \times \Sigma_0$ is a non-characteristic hypersurface with respect to $\hat{A}$. Since $\hat{w}_j(k,x) = 0$ in $\mathbb{R}_k \times \mathbb{T}_0$, the Holmgren theorem implies $\hat{w}_j(k,x) = 0$ on a neighborhood of $\mathbb{R}_k \times \Sigma_0$. We now conclude by the Aronszajn–Cordes theorem that $\hat{w}_j = 0$ everywhere. Finally $u_j = 0$ and condition (2) implies (3.46). □

We have studied the uniqueness of the sufficiently smooth solutions. The sequel of this work deals with the problem of the existence of such solutions, that is not obvious when the manifold is not causal. We introduce the vector space:

$$\mathcal{E} := \left\{u \in C^0\left(\mathbb{R}_t; W^1\left(\mathbb{R}^3_x\right)\right);\ Lu = 0,\ \partial_t u \in C^0\left(\mathbb{R}_t; L^2_C\left(\mathbb{R}^3_x\right)\right)\right\}, \tag{3.60}$$

endowed with the indefinite form $E(u)$ given by (3.20) and the space of the admissible Cauchy data:

$$\mathcal{H} := \left\{(f,g) \in W^1\left(\mathbb{R}^3_x\right) \times L^2_C\left(\mathbb{R}^3_x\right);\ \exists u \in \mathcal{E},\ \mathbf{u}(0) = (f,g)\right\}, \tag{3.61}$$

where for $v \in C^1(\mathbb{R}_t; \mathcal{D}'(\mathbb{R}^3_x))$, we put:

$$\mathbf{v} := \begin{pmatrix} v \\ \partial_t v \end{pmatrix}. \tag{3.62}$$

A priori, when $\mathbb{T} \neq \emptyset$, $\mathcal{H}$ is not an Hilbert space for the norm of $W^1 \times L^2_C$. The previous theorem assures that the family of maps

$$U(t) : \mathbf{u}(0) \in \mathcal{H} \mapsto \mathbf{u}(t) \in \mathcal{H} \tag{3.63}$$

is a strongly continuous group of linear operators on $\mathcal{H}$. In the following parts we construct global solutions u with $E(u) = 0$ or $E(u) > 0$. We let open the problem of the existence of global solution with negative energy.

4. The resonant states

In this section, we investigate the global solutions $u \in H^1_{\mathrm{loc}}(\mathcal{M})$ by separation of the variable t:

$$u(t,x) = \mathrm{e}^{\lambda t} v(x), \tag{4.1}$$

with $\lambda \in \mathbb{C}$ and v is a distribution on $\mathbb{R}^3_x$. Then u is solution of

$$Lu = 0 \quad \text{in } \mathcal{M}, \tag{4.2}$$

iff $v \in L^2_{\mathrm{loc}}(\mathbb{R}^3_x)$ is solution of the homogeneous reduced wave equation:

$$\Delta v + \frac{2C\lambda}{r^2}\partial_\varphi v - \lambda^2\left(1 - \frac{C^2}{r^2}\right)v - Vv = 0 \quad \text{on } \mathbb{R}^3. \tag{4.3}$$

By the standard results of elliptic regularity, $v \in H^2_{\mathrm{loc}}(\mathbb{R}^3)$ and $v \in C^\infty$ for $|x|$ large enough, since C and V are continuous and compactly supported. (4.3) is similar to the acoustic wave equation in an inhomogeneous medium (see, e.g., [6,20,30,34]); the crucial difference is that $1 - r^{-2}C^2$ that plaies the role of the refractive index, is null on Σ_0 and negative in $\mathbb{T}_0$.

We start by proving a result of Rellich type, stating that there exists no t-periodic, non-constant, solution of $Lu = 0$ satisfying some natural constraint at the space infinity.

Lemma 4.1. *Let v be a solution of* (4.3) *for $\lambda \in \mathrm{i}\mathbb{R}^*$, satisfying one of the following conditions*:

$$v \in L^2\left(\mathbb{R}^3\right) \cup W^1\left(\mathbb{R}^3\right); \tag{4.4}$$

$$\frac{x}{|x|} \cdot \nabla v + \lambda v = \mathrm{O}\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty; \tag{4.5}$$

$$\frac{x}{|x|} \cdot \nabla v - \lambda v = \mathrm{O}\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty. \tag{4.6}$$

Then $v = 0$.

For $\lambda = 0$ the result is well known: for non-negative potential V, the conclusion of the lemma is valid; for general potential V, since the form $v \mapsto \int V|v|^2$ is compact on $H^1_{\mathrm{loc}}(\mathbb{R}^3)$, the space of solutions of (4.3) with $\lambda = 0$ is of finite dimension.

Proof of Lemma 4.1. Let $\lambda = \mathrm{i}k$, $k \in]0,\infty[$. We have $V(x) = C(x) = 0$ for $|x| \geqslant R_0$. Thanks to the Aronszajn–Cordes theorem, it is sufficient to prove

$$|x| \geqslant R_0 \Longrightarrow v(x) = 0. \tag{4.7}$$

Since v is solution of the homogeneous Helmholtz equation for large x, v has the following expansion with respect to the spherical harmonics Y_n^m:

$$|x| \geqslant R \Longrightarrow v(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{m=n} a_n^m(\rho) Y_n^m(\omega), \quad \rho = |x|,\ \omega = \rho^{-1} x,$$

with

$$a_n^m(\rho) = \alpha_n^m h_n^{(1)}(k\rho) + \beta_n^m h_n^{(2)}(k\rho), \quad \alpha_n^m, \beta_n^m \in \mathbb{C},$$

where $h_n^{(1,2)}$ are the spherical Hankel functions, which satisfy as $\rho \to \infty$ (see, e.g., [6], p. 30):

$$\begin{aligned} &h_n^{(1,2)}(\rho) = \rho^{-1} \mathrm{e}^{\pm \mathrm{i}(\rho - \frac{n\pi}{2} - \frac{\pi}{2})}\big[1 + \mathrm{O}\big(\rho^{-1}\big)\big], \\ &\frac{\mathrm{d}}{\mathrm{d}\rho} h_n^{(1,2)}(\rho) = \rho^{-1} \mathrm{e}^{\pm \mathrm{i}(\rho - \frac{n\pi}{2})}\big[1 + \mathrm{O}\big(\rho^{-1}\big)\big]. \end{aligned} \tag{4.8}$$

From the Parseval equality

$$\int_{S^2} \big|v(\rho\omega)\big|^2 \mathrm{d}\omega = \sum_{n=0}^{\infty} \sum_{m=-n}^{m=n} \big|a_n^m(\rho)\big|^2, \qquad \int_{S^2} \big|\omega \cdot \nabla v(\rho\omega)\big|^2 \mathrm{d}\omega = \sum_{n=0}^{\infty} \sum_{m=-n}^{m=n} \left|\frac{\mathrm{d}}{\mathrm{d}\rho} a_n^m(\rho)\right|^2$$

we deduce with the asymptotic behaviours (4.8), that (4.4) implies (4.7).

We now assume that (4.5) or (4.6) is satisfied. We multiply (4.3) by $\bar{v}$ and integrate on $|x| \leqslant R$. Since $\partial_\varphi C = 0$, we get by the Green formula:

$$\Im \int_{S^2} \partial_\rho v(R\omega) \overline{v(R\omega)}\, \mathrm{d}\omega = 2kR^{-2} \Im \mathrm{i} \int_{|x| \leqslant R} \frac{C}{r^2} \bar{v} \partial_\varphi v \, \mathrm{d}x = 0.$$

Then the Rellich theorem (e.g., [6], Theorem 2.12) assures (4.7). □

Lemma 4.1 shows that we have to look for the non-trivial solutions of the homogeneous reduced wave equation, for $\lambda \in \mathbb{C} \setminus \mathrm{i}\mathbb{R}$. We adapt at our problem the concept of outgoing (resp. incoming) solution by Lax and Phillips [24]. Given $\lambda \in \mathbb{C}$, $f \in \mathcal{E}'$, the space of the compactly supported distributions, a solution $v_\lambda^{+(-)}$ of

$$\Delta v + \frac{2C\lambda}{r^2} \partial_\varphi v - \lambda^2 \left(1 - \frac{C^2}{r^2}\right) v - V v = f \quad \text{on } \mathbb{R}^3, \tag{4.9}$$

is said to be λ-*outgoing* (resp. λ-*incoming*) if

$$v_\lambda^{+(-)} = \gamma_\lambda^{+(-)} * \left[f - \frac{2C\lambda}{r^2} \partial_\varphi v_\lambda^{+(-)} - \lambda^2 \frac{C^2}{r^2} v_\lambda^{+(-)} + V v_\lambda^{+(-)} \right], \tag{4.10}$$

where

$$\gamma_\lambda^{+(-)}(x) := -\frac{\mathrm{e}^{-(+)\lambda|x|}}{4\pi|x|}. \tag{4.11}$$

It is well known that in the case $\lambda \in \mathrm{i}\mathbb{R}$, the λ-outgoing (resp. λ-incoming) condition is equivalent to the Sommerfeld radiation condition (4.5) (resp. (4.6)). A complex number λ is an *outgoing resonance* (resp. *incoming resonance*), if there exists a non-null λ-outgoing (resp. λ-incoming) solution $v_\lambda^{+(-)}$ of (4.3), called *resonant state*. We remark that when a resonant state v_λ has a finite energy, i.e. $v_\lambda \in H^1(\mathbb{R}^3)$, the total energy (3.20) of the time dependant solution $u_\lambda(t,x) = \mathrm{e}^{\lambda t} v_\lambda(x)$ is zero:

$$E(u_\lambda) = \frac{1}{2} \mathrm{e}^{2\Re(\lambda)t} \int_{\mathbb{R}^3} |\lambda|^2 \left(1 - \frac{C^2}{r^2}\right) |v_\lambda|^2 + |\nabla v_\lambda|^2 + V|v_\lambda|^2 \, \mathrm{d}x = 0. \tag{4.12}$$

We denote $\mathcal{R}^{+(-)}$ the set of the outgoing (incoming) resonances. Because C and V are real axisymmetric, and since we may take $v_\lambda^+(x^1, -x^2, z) = v_{-\lambda}^-(x^1, x^2, z)$, it is easy to see that:

$$\lambda \in \mathcal{R}^+ \iff \bar{\lambda} \in \mathcal{R}^+, \tag{4.13}$$

$$\lambda \in \mathcal{R}^+ \iff -\lambda \in \mathcal{R}^-. \tag{4.14}$$

Hence we shall consider only the set of the outgoing resonances, simply called "resonances", and we omit the superscript $+$: $\mathcal{R} := \mathcal{R}^+$, $v_\lambda := v_\lambda^+$.

We summarize the properties of the set of the resonances:

Theorem 4.2. *$\mathcal{R}$ is a discrete subset of $\mathbb{C}$, and we have*:

$$\mathcal{R} \cap \mathrm{i}\mathbb{R}^* = \emptyset; \tag{4.15}$$

$$\lambda \in \mathcal{R},\, 0 < \Re(\lambda) \Longrightarrow v_\lambda \in H^2(\mathbb{R}^3); \tag{4.16}$$

$$\mathbb{T}_0 = \emptyset \Longrightarrow \mathrm{Card}\{\lambda \in \mathcal{R};\, 0 \leqslant \Re(\lambda)\} < \infty; \tag{4.17}$$

$$\mathbb{T}_0 = \emptyset,\, 0 \leqslant V \Longrightarrow \{\lambda \in \mathcal{R};\, 0 \leqslant \Re(\lambda)\} = \emptyset; \tag{4.18}$$

$$\mathcal{T}_0 \neq \emptyset,\, \lambda \in \mathcal{R} \cap]0, \infty[\Longrightarrow \partial_\varphi v_\lambda = 0; \tag{4.19}$$

$$\mathbb{T}_0 \neq \emptyset \Longrightarrow \mathrm{Card}(\mathcal{R} \cap]0, \infty[) = \infty. \tag{4.20}$$

We know that for the scattering by obstacle there exists no real resonance, and for the scattering by non-positive potential, or metric perturbation, or Schwarzschild black-hole, there exists only a finite set of real resonances with finite energy (see, e.g., [1,24]). (4.17) and (4.18) show that this remains true even if there is a closed null geodesic ($\Sigma_0 \neq \emptyset$) but no closed timelike curve ($\mathbb{T}_0 = \emptyset$). The main novelty, (4.20), due to the existence of a closed timelike curve, is that this set is *infinite*. This last result can be physically interpreted as follows: in the framework of the studies of the stability of the manifolds of the General Relativity, the existence of an infinite set of resonant states with finite energy means that we cannot prove the possible stability of the metric (2.2) by a method of perturbation (see, e.g., the works of Y. Choquet-Bruhat, A. Fischer, J. Marsden); hence we can suspect that the manifold is actually nonlinearly instable in a suitable set of solutions of inhomogeneous Einstein equations. This agrees with the "conjecture of chronological protection" by S. Hawking [16], that states that any universe with closed timelike curve is instable.

Proof of Theorem 4.2. Let B an open ball of $\mathbb{R}^3$, that contains the supports of C and V. We introduce the operator $K(\lambda)$ on $H^1(B)$ by putting:

$$K(\lambda)v(x) := \frac{1}{4\pi} \int_B \frac{\mathrm{e}^{-\lambda|x-y|}}{|x-y|} \left[\lambda \frac{2C}{r^2} \partial_\varphi v + \lambda^2 \frac{C^2}{r^2} v - V v\right](y)\, \mathrm{d}y. \tag{4.21}$$

If $v_\lambda \neq 0$ is a resonant state, then $K(\lambda)(v_{\lambda|B}) = v_{\lambda|B}$. Conversely, if $v \in H^1(B) \setminus \{0\}$ satisfies $K(\lambda)v = v$, then v_λ defined by v in B, and by the right hand of (4.21) for $x \in \mathbb{R}^3 \setminus B$, is a resonant state. Therefore the problem is reduced to investigating the solutions of:

$$K(\lambda)v = v, \quad v \in H^1(B) \setminus \{0\},\ \lambda \in \mathbb{C}, \tag{4.22}$$

and $\mathcal{R}$ is the set of complex numbers λ such that 1 is eigenvalue of $K(\lambda)$. By the classical results on the potential γ_λ^+ (e.g., [6], Theorem 8.2), $K(\lambda)$ is a bounded operator from $H^1(B)$ to $H^2(B)$. Hence the Sobolev theorem assures that $K(\lambda)$ is an analytic family on $\mathbb{C}_\lambda$, of compact operators on $H^1(B)$. Then the Atkinson theorem (see [21], Theorem 1.9, p. 370) assures that $\mathcal{R} = \mathbb{C}$ or $\mathcal{R}$ is discrete. The first alternative is excluded by (4.15) that is stated in Lemma 4.1.

(4.16) is an obvious consequence of the asymptotic behaviour of $\gamma_\lambda^+(x)$ as $|x| \to \infty$, and we have:

$$\int_{\mathbb{R}^3} |\lambda|^2 \left(1 - \frac{C^2}{r^2}\right) |v_\lambda|^2 + |\nabla v_\lambda|^2 + V|v_\lambda|^2\, \mathrm{d}x = 0.$$

Since $C \leqslant r$ and the form $f \mapsto \int V|f|^2\, \mathrm{d}x$ is compact on H^1, we get (4.17) and (4.18).

Let $v_\lambda \in H^2$ be a resonant state for $\lambda > 0$. We use the Fourier expansion of v_λ:

$$v_\lambda = \sum_{m \in \mathbb{Z}} v_{\lambda,m}, \quad v_{\lambda,m}\big(x = (r\cos\varphi, r\sin\varphi, z)\big) := \mathrm{e}^{\mathrm{i}m\varphi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}m\theta} v_\lambda(r\cos\theta, r\sin\theta, z)\, \mathrm{d}\theta.$$

$v_{\lambda,m}$ is solution of (4.3) and an integration by parts gives:

$$2\mathrm{i}m\lambda \int_{\mathbb{R}^3} \frac{C}{r^2} |v_{\lambda,m}|^2\, \mathrm{d}x = \int_{\mathbb{R}^3} |\nabla v_{\lambda,m}|^2 + \lambda^2 \left(1 - \frac{C^2}{r^2}\right) |v_{\lambda,m}|^2 + V|v_{\lambda,m}|^2\, \mathrm{d}x = 0.$$

We deduce that $v_{\lambda,m}=0$ on the non-empty support of C for $m\neq 0$. Since $v_{\lambda,m}$ is solution of (4.3), the Aronszajn–Cordes theorem assures that $v_{\lambda,m}=0$ everywhere and (4.19) is proved.

Given an axy-symmetric domain $\Omega\subset\mathbb{R}^3$, we introduce

$$L^2_0(\Omega):=\bigl\{v\in L^2(\Omega,\mathrm{d}x);\ \partial_\varphi v=0\bigr\}. \tag{4.23}$$

To establish (4.20) we show that given $\lambda_0>0$, there exists $\lambda\geqslant\lambda_0$ such that $-\lambda^2$ is an eigenvalue of the densely defined self-adjoint operator on $L^2_0(\mathbb{R}^3)$,

$$A(\lambda):=-\Delta-\lambda^2\frac{C^2}{r^2}+V, \tag{4.24}$$

with domain

$$D\bigl(A(\lambda)\bigr)=H^2\bigl(\mathbb{R}^3\bigr)\cap L^2_0\bigl(\mathbb{R}^3\bigr). \tag{4.25}$$

Since Cr^{-1} and V are continuous, and compactly supported, the Weyl theorem assures that

$$\sigma_{\mathrm{ess}}\bigl(A(\lambda)\bigr)=[0,\infty[,\qquad \sigma\bigl(A(\lambda)\bigr)\cap]-\infty,0[=\sigma_{pp}\bigl(A(\lambda)\bigr)\cap\left[-\lambda^2\left\|\frac{C^2}{r^2}\right\|_{L^\infty}-\|V\|_{L^\infty},0\right[,$$

hence for $\lambda>0$ we have:

$$\dim P_{]-\infty,-\lambda^2]}\bigl(A(\lambda)\bigr)<\infty, \tag{4.26}$$

where $(P_I(T))_{I\subset\mathbb{R}}$ is the family of spectral projections of a self-adjoint operator T. We choose $0<r_0<r_1$, $z_0<z_1$ such that

$$(r,z,\varphi)\in\mathbb{T}_1:=]r_0,r_1[\times]z_0,z_1[\times S^1\Longrightarrow\frac{C^2}{r^2}\geqslant 1+\varepsilon>1. \tag{4.27}$$

We introduce the self-adjoint operators:

$$B_1(\lambda)\ :=\ -\Delta-\lambda^2(1+\varepsilon)+\|V\|_{L^\infty}, \tag{4.28}$$

$$D\bigl(B_1(\lambda)\bigr)\ =\ \bigl\{v_1\in L^2_0(\mathbb{T}_1);\ \Delta v_1\in L^2(\mathbb{T}_1),\ v_1=0\text{ on }\partial\mathbb{T}_1\bigr\}, \tag{4.29}$$

$$B_2(\lambda)\ :=\ -\Delta-\lambda^2\frac{C^2}{r^2}+V, \tag{4.30}$$

$$D\bigl(B_2(\lambda)\bigr)\ =\ \bigl\{v_2\in L^2_0\bigl(\mathbb{R}^3\setminus\mathbb{T}_1\bigr);\ \Delta v_2\in L^2\bigl(\mathbb{R}^3\setminus\mathbb{T}_1\bigr),\ v_2=0\text{ on }\partial\mathbb{T}_1\bigr\}, \tag{4.31}$$

$$A_D(\lambda)\ :=\ B_1(\lambda)\oplus B_2(\lambda). \tag{4.32}$$

By Proposition 4 of [31], tome 4, p. 270, we have:

$$-\lambda^2\left\|\frac{C^2}{r^2}\right\|_{L^\infty}-\|V\|_{L^\infty}\leqslant A(\lambda)\leqslant A_D(\lambda),$$

hence the min-max principle implies that

$$\dim P_{]-\infty,-\lambda^2]}\bigl(A(\lambda)\bigr)\geqslant\dim P_{]-\infty,-\lambda^2]}\bigl(A_D(\lambda)\bigr)\geqslant\dim P_{]-\infty,-\lambda^2]}\bigl(B_1(\lambda)\bigr); \tag{4.33}$$

$B_1(0)$ is a positive self-adjoint operator on $L^2_0(\mathbb{T}_1)$, and its resolvant is compact by the Sobolev theorem. Let $(\alpha_n)_{n\in\mathbb{N}}$ be the sequence of its eigenvalues. We have

$$\sigma\bigl(B_1(\lambda)\bigr)\cap\ \bigl]-\infty,-\lambda^2\bigr]=\bigl\{\alpha_n-(1+\varepsilon)\lambda^2;\ \alpha_n\leqslant\varepsilon\lambda^2\bigr\}.$$

Since $\alpha_n\to\infty$ as $n\to\infty$, we deduce that:

$$\dim P_{]-\infty,-\lambda^2]}\bigl(B_1(\lambda)\bigr)\to\infty,\quad \lambda\to\infty. \tag{4.34}$$

We assume there exists $\lambda_0>0$ such that

$$\lambda\geqslant\lambda_0\Longrightarrow-\lambda^2\notin\sigma\bigl(A(\lambda)\bigr). \tag{4.35}$$

In this case, since $\lambda\mapsto A(\lambda)$ is an analytic family of operators in the sense of Kato, its resolvant $(A(\lambda)-z)^{-1}$ is an analytic function of two variables on $\{(\lambda,z);\ \lambda\in\mathbb{R},z\notin\sigma(A(\lambda))\}$, and we have:

$$P_{]-\infty,-\lambda^2]}\bigl(A(\lambda)\bigr)=\frac{1}{2\mathrm{i}\pi}\oint_{\partial D(\lambda)}\bigl(A(\lambda)-z\bigr)^{-1}\,\mathrm{d}z,$$

with

$$D(\lambda):=\left\{z=a+\mathrm{i}b;\ -\lambda^2\left\|\frac{C^2}{r^2}\right\|_{L^\infty}-\|V\|_{L^\infty}-1\leqslant a\leqslant 0,\ -1\leqslant b\leqslant 1\right\}.$$

We deduce that $\lambda\geqslant\lambda_0\mapsto P_{]-\infty,-\lambda^2]}(A(\lambda))\in\mathcal{L}(L^2_0(\mathbb{R}^3))$ is continuous, therefore

$$\lambda\geqslant\lambda_0\Longrightarrow \dim P_{]-\infty,-\lambda^2]}\big(A(\lambda)\big)=\dim P_{]-\infty,-\lambda_0^2]}\big(A(\lambda_0)\big)<\infty.$$

This a contradiction with (4.33) and (4.34). □

5. Scattering states

When $\mathbb{T}$ is not empty, the manifold is totally vicious, hence there exists no Cauchy hypersurface. Nevertheless we shall prove that the global Cauchy problem is well posed for regular data specified at the past null infinity, and these solutions are asymptotically free at the future null infinity (*Scattering States*). Furthermore, the Scattering Operator S is well defined for any free wave with finite energy, but, unlike the usual situations, the wave operators are *not* causal. As regards the mathematical tools, we keep the features of the scattering theory, that involve neither the positivity of the energy, nor the existence of a unitary group: we use the generalised eigenfunctions method.

We start with a uniqueness result for the solutions with some given asymptotic behaviour. We recall some basic notations for the wave equation on the Minkowski space-time:

$$L_0u_0:=\partial_t^2u_0-\Delta_x u_0=0,\quad (t,x)\in\mathbb{R}\times\mathbb{R}^3. \tag{5.1}$$

The Cauchy problem is solved in $\mathcal{D}'(\mathbb{R}^3_x)$ by the group $U_0(t)$:

$$U_0(t)\mathbf{u}_0(0)=\mathbf{u}_0(t). \tag{5.2}$$

We introduce the spaces associated with the finite energy waves:

$$\mathcal{E}_0:=\left\{u_0\in C^0\big(\mathbb{R}_t;W^1\big(\mathbb{R}^3_x\big)\big);\ L_0u_0=0,\partial_t u_0\in C^0\big(\mathbb{R}_t;L^2\big(\mathbb{R}^3_x\big)\big)\right\},\qquad \mathcal{H}_0:=W^1\big(\mathbb{R}^3_x\big)\times L^2\big(\mathbb{R}^3_x\big), \tag{5.3}$$

which are Hilbert spaces for the energy norm

$$\|u_0\|^2_{\mathcal{E}_0}=\big\|\mathbf{u}_0(t)\big\|^2_{\mathcal{H}_0}:=\frac{1}{2}\int_{\mathbb{R}^3}\big|\partial_t u_0(t,x)\big|^2+\big|\nabla u_0(t,x)\big|^2\,\mathrm{d}x, \tag{5.4}$$

and $U_0(t)$ is a strongly continuous unitary group on $\mathcal{H}_0$.

Theorem 5.1. *Let u be in $\mathcal{E}$. We assume that one of the two following conditions is fullfiled*:

(1) $$u\in L^1\big(\mathbb{R}_t;L^2_{\mathrm{loc}}\big(\mathbb{R}^3_x\big)\big), \tag{5.5}$$

$$\big\|\mathbf{u}(t)\big\|_{W^1\times L^2_C}\to 0,\quad t\to-\infty. \tag{5.6}$$

(2) $\mathbb{T}\neq\emptyset$, *and there exist $a,c,R\geqslant 0$, such that*

$$\big\|u(t)\big\|_{W^1}\leqslant c\mathrm{e}^{a|t|}, \tag{5.7}$$

$$|x|\leqslant -t-R\Longrightarrow u(t,x)=0. \tag{5.8}$$

Then

$$u=0\quad \textit{on } \mathcal{M}. \tag{5.9}$$

We make some remarks. (1) The global constraint (5.5) is usefull when $\mathbb{T}\neq\emptyset$: the outgoing resonant states with finite energy satisfy (5.5) but are exponentially increasing as $t\to+\infty$. (2) It is known that when $\mathbb{T}=\emptyset$ and $0\leqslant V$ there exists non-null solutions satisfying (5.7) and (5.8). (3) Lemma 3.6 and (3.53) show that (5.7) is a consequence of (5.8) when Σ_0 is Non-Confining.

Proof of Theorem 5.1. We assume the first condition is satisfied. Let u_j be defined by (3.21). Then $u_j\in\mathcal{E}$ satisfies (5.5) and for any $k\in\mathbb{N}$, $\partial_t^k u_j\in L^1(\mathbb{R}_t;L^2_{\mathrm{loc}}(\mathbb{R}^3_x))$. Therefore

$$\big\|\mathbf{u}_j(t)\big\|_{\mathcal{H}_0}\to 0,\quad t\to-\infty,$$

$$\frac{C^2}{r^2}\partial_t^2 u_j,\ \frac{C}{r^2}\partial_{t,\varphi}^2 u_j,\ Vu_j \in L^1\big(\mathbb{R}_t; L^2\big(\mathbb{R}_x^3\big)\big).$$

We deduce that

$$\mathbf{u}_j(t) = \int_{-\infty}^{t} U_0(t-s)\begin{pmatrix}0\\ q_j(s)\end{pmatrix}\mathrm{d}s, \quad q_j := \frac{C^2}{r^2}\partial_t^2 u_j + 2\frac{C}{r^2}\partial_{t,\varphi}^2 u_j - Vu_j. \tag{5.10}$$

Since u_j satisfies (5.5), we may consider the Fourier transform

$$\hat{u}_j(k) := \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}kt} u_j(t)\,\mathrm{d}t \in C^0\big(\mathbb{R}_k; L^2_{\mathrm{loc}}\big(\mathbb{R}_x^3\big)\big);$$

(5.10) implies that

$$\hat{u}_j(k) = \gamma_{ik}^{+} * \left(k^2\frac{C^2}{r^2}\hat{u}_j(k) - 2\mathrm{i}k\frac{C}{r^2}\partial_\varphi \hat{u}_j(k) + V\hat{u}_j(k)\right).$$

Hence $\hat{u}_j$ is a ik-outgoing solution of the homogeneous equation reduced wave equation (4.3). Therefore Lemma 4.1 assures that $\hat{u}_j = 0$, and (5.9) is proved.

We now consider the second condition. (5.7) and (5.8) allow to define the Fourier–Laplace transform

$$\hat{u}(\lambda) := \int_{\mathbb{R}} \mathrm{e}^{\lambda t} u(t)\,\mathrm{d}t,$$

which is an $L^2_{\mathrm{loc}}(\mathbb{R}_x^3)$-valued analytic function of λ, $\Re(\lambda) < -a$. $\hat{u}(\lambda)$ is solution of the elliptic equation (4.3). Moreover (5.8) and Lemma 3.6 imply that $\hat{u}(\lambda) = 0$ on $\mathbb{T}_0$. We conclude that $\hat{u}(\lambda) = 0$ on $\mathbb{R}_x^3$, so (5.9) is established. □

We now return to the problem of global solutions by constructing Wave Operators. We denote $\mathcal{E}_0^\infty$ the space of the *regular wave packets* that are the smooth solutions u_0 of (5.1) such that

$$\hat{u}_0(0,\xi) := \int \mathrm{e}^{-\mathrm{i}x\cdot\xi} u_0(0,x)\,\mathrm{d}x,\ \partial_t\hat{u}_0(0,\xi) := \int \mathrm{e}^{-\mathrm{i}x\cdot\xi}\partial_t u_0(0,x)\,\mathrm{d}x \in C_0^\infty\big(\mathbb{R}_\xi^3 \setminus \{0\}\big). \tag{5.11}$$

Theorem 5.2. *Given $u_0^- \in \mathcal{E}_0^\infty$, there exists a unique $u \in \mathcal{E}$ such that $\partial_t u \in C^0(\mathbb{R}_t; L^2(\mathbb{R}_x^3))$ and satisfying* (5.5) *such that*

$$\big\|\mathbf{u}(t) - \mathbf{u}_0^-(t)\big\|_{\mathcal{H}_0} \to 0, \quad t \to -\infty. \tag{5.12}$$

Moreover there exists a unique $u_0^+ \in \mathcal{E}_0$ such that:

$$\big\|\mathbf{u}(t) - \mathbf{u}_0^+(t)\big\|_{\mathcal{H}_0} \to 0, \quad t \to +\infty, \tag{5.13}$$

and we have:

$$\big\|u_0^-\big\|^2_{\mathcal{E}_0} = E(u) = \big\|u_0^+\big\|^2_{\mathcal{E}_0}, \tag{5.14}$$

$$u_0^+ \in \mathcal{E}_0^\infty. \tag{5.15}$$

This theorem allows to introduce the Wave Operators:

$$W^- : u_0^- \mapsto u, \qquad W^+ : u_0^+ \mapsto u. \tag{5.16}$$

To make the link between these both operators, we use the time reverse operator:

$$R : u\big(t, x^1, x^2, z\big) \mapsto (Ru)\big(t, x^1, x^2, z\big) = u\big(-t, x^1, -x^2, z\big). \tag{5.17}$$

Since $R(Lu) = L(Ru)$, we have

$$W^+ = RW^-R. \tag{5.18}$$

These wave operators are defined on $\mathcal{E}_0^\infty$, but when the chronology is violated, $\mathbb{T} \neq \emptyset$, we do know to characterize neither their ranges, nor the possible continuity property. Furthermore, they are no causal in the usual sense, since Theorem 5.1 shows that if $u = W^- u_0^-$ exists for some free wave $u_0^- \in \mathcal{E}_0$ satisfying the initially incoming condition (5.8), and $u = u_0^-$ for $t \ll 0$, then $u = u_0^- = 0$.

We now consider the Scattering Operator:

$$S : u_0^- \mapsto u_0^+. \tag{5.19}$$

The previous theorem assures that S is an isometry from $\mathcal{E}_0^\infty$ onto $\mathcal{E}_0^\infty$, and by (5.18) we have

$$S^{-1} = RSR. \tag{5.20}$$

Therefore S can be extended by continuity and density, into an unitary operator on $\mathcal{E}_0$, denoted S again. To investigate this operator, we recall two important tools (see [6,24,29]): the translation representation for the free wave equation is the map:

$$u_0 \in \mathcal{E}_0 \mapsto f^\sharp \in L^2\big(\mathbb{R}_s \times S^2_\omega, \mathrm{d}s\,\mathrm{d}\omega\big), \quad f^\sharp(s,\omega) = -\lim_{|t|\to\infty} t\partial_t u_0\big(t, x = (t+s)\omega\big) \quad \text{in } L^2_{\mathrm{loc}}\big(\mathbb{R}_s \times S^2_\omega, \mathrm{d}s\,\mathrm{d}\omega\big), \tag{5.21}$$

that is an isometry from $\mathcal{E}_0$ onto $L^2(\mathbb{R}_s \times S^2_\omega, \mathrm{d}s\,\mathrm{d}\omega)$; the spectral representation is the isometry $u_0 \mapsto \tilde f$ from $\mathcal{E}_0$ onto $L^2(\mathbb{R}_k \times S^2_\omega, \mathrm{d}k\,\mathrm{d}\omega)$ defined by (5.40). The link between these both representations is the Fourier transform with respect to s:

$$\tilde f(k,\omega) = \frac{1}{\sqrt{2\pi}} \int \mathrm{e}^{\mathrm{i}ks} f^\sharp(s,\omega)\,\mathrm{d}s. \tag{5.22}$$

We put

$$\mathbf{S} : \mathbf{u}_0^-(0) \mapsto \mathbf{u}_0^+(0). \tag{5.23}$$

Then $\mathbf{S}$ is an isometry from $\mathcal{H}_0$ onto $\mathcal{H}_0$, and because of the invariance of the wave equation $Lu = 0$ by the time translation, we have for any $t \in \mathbb{R}$:

$$U_0(t)\mathbf{S} = \mathbf{S}U_0(t). \tag{5.24}$$

With obvious notations, we can also represent the scattering operator by putting:

$$S^\sharp f_-^\sharp = f_+^\sharp, \qquad \tilde S \tilde f_- = \tilde f_+. \tag{5.25}$$

Since $\mathbf{S}$ commutes with the free group $U_0(t)$, $S^\sharp$ commutes with the s-translation. Then $\tilde S$ is represented as a multiplicative operator-valued function $\tilde S(k)$ on $L^2(S^2_\omega)$. We shall state in Proposition 5.5 that we can represent $\tilde S(k)$ by using the distorded plane waves as well as for the usual globally hyperbolic case.

Proof of Theorem 5.2. We start by constructing global solutions of type *distorded plane waves* $\Phi(t,x;k,\omega)$:

Lemma 5.3. *For all $k \in \mathbb{C}$, $ik \notin \mathcal{R}$, $\omega \in S^2$, there exists a unique* ik*-outgoing function $\Psi(x;k,\omega)$ that is a $H^2_{\mathrm{loc}}(\mathbb{R}^3_x)$-valued analytic function on $(\mathbb{C}_k \setminus \mathrm{i}\mathcal{R}) \times S^2_\omega$, such that*

$$\Phi(t,x;k,\omega) := \mathrm{e}^{\mathrm{i}k(t - x\cdot\omega)} + \mathrm{e}^{\mathrm{i}kt}\Psi(x;k,\omega) \tag{5.26}$$

is solution of $L\Phi = 0$.

Proof. We use the notations of the proof of Theorem 4.2. We remark that $L\Phi = 0$ iff Ψ is an ik-outgoing solution of:

$$k^2\left(1 - \frac{C^2}{r^2}\right)\Psi + \Delta\Psi + 2\mathrm{i}k\frac{C}{r^2}\partial_\varphi\Psi - V\Psi = \left(k^2\frac{C^2}{r^2} - 2k^2\frac{C}{r^2}\big(x^1\omega^2 - x^2\omega^1\big) + V\right)\mathrm{e}^{-\mathrm{i}kx\cdot\omega}. \tag{5.27}$$

Hence Ψ exists and is unique, iff the equation

$$\big(K(\mathrm{i}k) - \mathrm{Id}\big)\Psi(\cdot;k,\omega) = \gamma^+_{\mathrm{i}k} * \left(k^2\frac{C^2}{r^2} - 2k^2\frac{C}{r^2}\big(x^1\omega^2 - x^2\omega^1\big) + V\right)\mathrm{e}^{-\mathrm{i}kx\cdot\omega} \tag{5.28}$$

has a unique solution in $H^2(B)$. We know that $K(\mathrm{i}k)$ is an analytic family of bounded operators from $H^1(B)$ to $H^2(B)$, hence of compact operators on $H^1(B)$. Therefore the Fredholm theorem assures that this equation has a unique solution when ik is not a resonance, and by the Steinberg theorem, this solution depends analytically of k and ω. □

In the previous lemma, we can take $k \in \mathbb{R}^*$ since Theorem 4.2 states that there is no resonance in $\mathrm{i}\mathbb{R}^*$, and by using these distorded plane waves, we can get global solutions with finite energy:

Lemma 5.4. *For any $\tilde{f}_- \in C_0^\infty(\mathbb{R}_k^* \times S_\omega^2)$, the function*

$$u(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{S^2} \Phi(t,x;k,\omega) \tilde{f}_-(k,\omega) \, \mathrm{d}k \, \mathrm{d}\omega \tag{5.29}$$

satisfies

$$u \in C^0\big(\mathbb{R}_t; W^1\big(\mathbb{R}_x^3\big)\big), \qquad \partial_t u \in C^0\big(\mathbb{R}_t; L^2\big(\mathbb{R}_x^3\big)\big), \qquad Lu = 0. \tag{5.30}$$

Proof. It is clear that we have $Lu = 0$. We write $u = u_0^- + v$ with

$$u_0^-(t,x) := \frac{1}{2\pi} \int_{\mathbb{R}} \int_{S^2} \mathrm{e}^{\mathrm{i}k(t - x\cdot\omega)} \tilde{f}_-(k,\omega) \, \mathrm{d}k \, \mathrm{d}\omega, \tag{5.31}$$

$$v(t,x) := \frac{1}{2\pi} \int_{\mathbb{R}} \int_{S^2} \mathrm{e}^{\mathrm{i}kt} \Psi(x;k,\omega) \tilde{f}_-(k,\omega) \, \mathrm{d}k \, \mathrm{d}\omega. \tag{5.32}$$

We have

$$u_0^-(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}x\cdot\xi} \left(\mathrm{e}^{-\mathrm{i}t|\xi|} \tilde{f}_-\left(-|\xi|, \frac{\xi}{|\xi|}\right) + \mathrm{e}^{\mathrm{i}t|\xi|} \tilde{f}_-\left(|\xi|, -\frac{\xi}{|\xi|}\right)\right) \frac{1}{|\xi|^2} \, \mathrm{d}\xi,$$

hence $u_0^- \in C^\infty(\mathbb{R}_t; \mathcal{S}(\mathbb{R}_x^3))$ is a regular wave packet, and we have for any $\alpha \in \mathbb{N}^4$, $N \in \mathbb{N}$:

$$|x| \leqslant R_0 \Longrightarrow \big|\partial^\alpha u_0^-(t,x)\big| \leqslant c_{\alpha,N}\big(1 + |t|\big)^{-N}. \tag{5.33}$$

On the other hand we have

$$k \mapsto \int_{S^2} \Psi(x;k,\omega) \tilde{f}_-(k,\omega) \, \mathrm{d}\omega \in C_0^\infty\big(\mathbb{R}_k^*; H_{\mathrm{loc}}^2\big(\mathbb{R}_x^3\big)\big),$$

thus

$$v \in \mathcal{S}\big(\mathbb{R}_t; H_{\mathrm{loc}}^2\big(\mathbb{R}_x^3\big)\big). \tag{5.34}$$

We introduce

$$q := L_0 v. \tag{5.35}$$

We have $q = (L_0 - L)(v - u_0^-)$, thus (5.33) and (5.34) assure that

$$q \in L^1\big(\mathbb{R}_t; L^2\big(\mathbb{R}_x^3\big)\big), |x| \geqslant R_0 \Longrightarrow q(t,x) = 0. \tag{5.36}$$

Moreover since $\Psi(\cdot;k,\omega)$ is $\mathrm{i}k$*-outgoing*, (5.27) imply that

$$\Psi(x;k,\omega) = \gamma_{\mathrm{i}k}^+ * F(\cdot;k,\omega) \tag{5.37}$$

with

$$F(x;k,\omega) = \left[k^2 \frac{C^2}{r^2} - 2\mathrm{i}k \frac{C}{r^2} \partial_\varphi + V\right]\big(\mathrm{e}^{-\mathrm{i}kx\cdot\omega} + \Psi(x;k,\omega)\big). \tag{5.38}$$

A function $u \in C^1(\mathbb{R}_t; \mathcal{D}'(\mathbb{R}_x^3))$ is *outgoing* in the sense of Cooper and Strauss [7], if there exists $a \geqslant 0$ such that for all $T \in \mathbb{R}$, $U_0(t-T)\mathbf{u}(T)$ vanishes in the forward cone $|x| < t - T - a$. We put $w(t,x;k,\omega) := \mathrm{e}^{\mathrm{i}kt}\Psi(x;k,\omega)$. From the well known result

$$|x| \leqslant t - T \Longrightarrow U_0(t-T) \begin{pmatrix} \gamma_{\mathrm{i}k}^+ \\ \mathrm{i}k\gamma_{\mathrm{i}k}^+ \end{pmatrix} = 0,$$

we get

$$|x| \leqslant t - T - R_0 \Longrightarrow U_0(t-T)\mathbf{w}(T) = \mathrm{e}^{\mathrm{i}kT} F(\cdot; k, \omega) * \left[U_0(t-T) \begin{pmatrix} \gamma_{\mathrm{i}k}^+ \\ \mathrm{i}k\gamma_{\mathrm{i}k}^+ \end{pmatrix} \right] = 0.$$

We deduce that $w(t, x; k, \omega)$ is outgoing hence v is also outgoing and by Theorem 4 of [7] and (5.36), we get

$$\mathbf{v}(t) = \int_{-\infty}^{t} U_0(t-s) \begin{pmatrix} 0 \\ q(s) \end{pmatrix} \mathrm{d}s \in C^0(\mathbb{R}_t; \mathcal{H}_0). \qquad \square \tag{5.39}$$

We return to the proof of the theorem. Since the problem is linear, the uniqueness of u is assured by Theorem 5.1, and (5.13) and the conservation of the energy imply the uniqueness of u_0^+. To construct these waves, we put:

$$\tilde{f}_-(k, \omega) = \frac{1}{2(2\pi)^{3/2}} \left[k^2 \hat{u}_0^-(0, -k\omega) - \mathrm{i}k\partial_t \hat{u}_0^-(0, -k\omega) \right]. \tag{5.40}$$

Then (5.31) is satisfied, and u is given by Lemma 5.4. With q defined by (5.35) we put:

$$\mathbf{u}_0^+(t) := \mathbf{u}_0^-(t) + U_0(t) \int_{\mathbb{R}} U_0(-s) \begin{pmatrix} 0 \\ q(s) \end{pmatrix} \mathrm{d}s. \tag{5.41}$$

By (5.36) and (5.39) we get:

$$\left\| \mathbf{u}(t) - \mathbf{u}_0^-(t) \right\|_{\mathcal{H}_0} \leqslant \int_{-\infty}^{t} \left\| q(s) \right\|_{L^2} \mathrm{d}s \to 0, \quad t \to -\infty,$$

$$\left\| \mathbf{u}(t) - \mathbf{u}_0^+(t) \right\|_{\mathcal{H}_0} \leqslant \int_{t}^{\infty} \left\| q(s) \right\|_{L^2} \mathrm{d}s \to 0, \quad t \to +\infty.$$

Finally (5.14) is a consequence of (5.12), (5.13) and of the conservation of the energy (3.20). It remains to prove that u_0^+ is a regular wave packet. Since (4.15) assures that $\mathcal{R} \cap \mathrm{i}\mathbb{R}^* = \emptyset$, it is a direct consequence of the following spectral representation of the scattering kernel.

Proposition 5.5. *There exists a function $\tilde{S}(\omega', k, \omega)$ analytic on $S^2_{\omega'} \times (\mathbb{C}_k \setminus \mathrm{i}\mathcal{R}) \times S^2_\omega$ such that*

$$\Psi(x; k, \omega) = \frac{\mathrm{e}^{-\mathrm{i}k|x|}}{|x|} \tilde{S}\left(\frac{x}{|x|}, k, \omega \right) + \mathrm{O}\left(\frac{\mathrm{e}^{-\mathrm{i}k|x|}}{|x|^2} \right), \quad |x| \to \infty, \tag{5.42}$$

$$\frac{x}{|x|} \cdot \nabla_x \Psi(x; k, \omega) = -\mathrm{i}k \frac{\mathrm{e}^{-\mathrm{i}k|x|}}{|x|} \tilde{S}\left(\frac{x}{|x|}, k, \omega \right) + \mathrm{O}\left(\frac{\mathrm{e}^{-\mathrm{i}k|x|}}{|x|^2} \right), \quad |x| \to \infty. \tag{5.43}$$

For any $\tilde{f}_- \in L^2(\mathbb{R}_k \times S^2_\omega)$, we have

$$(\tilde{S}\tilde{f}_-)(k, \omega) = \tilde{f}_-(k, \omega) - \frac{\mathrm{i}k}{2\pi} \int_{S^2} \tilde{S}(\omega, k, \omega') \tilde{f}_-(k, \omega') \, \mathrm{d}\omega'. \tag{5.44}$$

Proof. From the formula (5.37),

$$\Phi(x; k, \omega) = -\frac{1}{4\pi} \int_{|y| \leqslant R_0} \frac{\mathrm{e}^{-\mathrm{i}k|x-y|}}{|x-y|} F(y; k, \omega) \, \mathrm{d}y,$$

we easily get that

$$\Phi(x; k, \omega) = -\frac{1}{4\pi} \frac{\mathrm{e}^{-\mathrm{i}k|x|}}{|x|} \int_{|y| \leqslant R_0} \mathrm{e}^{\mathrm{i}k \frac{x}{|x|} \cdot y} F(y; k, \omega) \, \mathrm{d}y + \mathrm{O}\left(\frac{\mathrm{e}^{-\mathrm{i}k|x|}}{|x|^2} \right), \quad |x| \to \infty,$$

$$\frac{x}{|x|} \cdot \nabla_x \Psi(x; k, \omega) = -\mathrm{i}k \Psi(x; k, \omega) + \mathrm{O}\left(\frac{\mathrm{e}^{-\mathrm{i}k|x|}}{|x|^2} \right), \quad |x| \to \infty.$$

Since F is given by (5.38), the function

$$\tilde{S}(\omega', k, \omega) := -\frac{1}{4\pi} \int_{|y| \leqslant R_0} e^{ik\omega' \cdot y} F(y; k, \omega)\, dy \tag{5.45}$$

is analytic on $S^2_{\omega'} \times (\mathbb{C}_k \setminus i\mathcal{R}) \times S^2_\omega$ and satisfies (5.42).

To prove the spectral representation of S we denote $f_+^\sharp$ the translation representation of u_0^+. By (5.13), (5.21) we have:

$$f_+^\sharp(s, \omega) = -\lim_{t \to \infty} t\partial_t u\big(t, x = (t+s)\omega\big) \quad \text{in } L^2_{\mathrm{loc}}\big(\mathbb{R}_s \times S^2_\omega\big).$$

We get from (5.32) that

$$t\partial_t u\big(t, x = (t+s)\omega\big) = t\partial_t u_0^-\big(t, x = (t+s)\omega\big) + \frac{1}{2\pi} \int_{\mathbb{R}} \int_{S^2} ikt e^{ikt} \Psi\big((t+s)\omega; k, \omega'\big) \tilde{f}_-(k, \omega')\, dk\, d\omega'.$$

On the one hand (5.31) gives

$$t\partial_t u_0^-\big(t, x = (t+s)\omega\big) \to -f_-^\sharp(s, \omega), \quad t \to +\infty.$$

On the other hand (5.42) assures that

$$\int_{\mathbb{R}} \int_{S^2} ikt e^{ikt} \Psi\big((t+s)\omega; k, \omega'\big) \tilde{f}_-(k, \omega')\, dk\, d\omega' \to \int_{\mathbb{R}} \int_{S^2} ik e^{-iks} \tilde{S}(\omega, k, \omega') \tilde{f}_-(k, \omega')\, dk\, d\omega', \quad t \to +\infty.$$

We deduce that

$$f_+^\sharp(s, \omega) = f_-^\sharp(s, \omega) - \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iks} \left(\int_{S^2} ik \tilde{S}(\omega, k, \omega') \tilde{f}_-(k, \omega')\, d\omega' \right) dk,$$

and by (5.22) and taking the inverse Fourier transform, we obtain (5.44). □

When the manifold is globally hyperbolic, i.e. $\mathbb{T} = \Sigma = \emptyset$, we can apply the general results of the "black box" scattering (see, e.g., [36]), that assure that $k \in \mathbb{C} \mapsto \tilde{S}(k) \in \mathcal{L}(L^2(S^2))$ defined by (5.44) is meromorphic on $\mathbb{C}$ and the poles essentially correspond to the resonances. More precisely, the multiplicity of a pole k of $\tilde{S}$ is equal to the difference between the multiplicities of the possible resonances ik and $-ik$. We state a less precise result when the metric is not causal.

Theorem 5.6. *The $\mathcal{L}(L^2(S^2))$ valued scattering matrix $\tilde{S}(k)$ is meromorphic on $\mathbb{C}_k$. If $k_0 \in \mathbb{C}$ is a pole, then $ik_0 \in \mathcal{R}$. Conversely a complex number k_0 satisfying*

$$\Re(ik_0) > 0, \qquad ik_0 \in \mathcal{R}, \qquad -ik_0 \notin \mathcal{R}, \tag{5.46}$$

is a pole of $\tilde{S}$.

When the manifold is chronological, $\mathbb{T} = \emptyset$, but non-causal, $\Sigma \neq \emptyset$, and if $0 \leqslant V$, then there exists no resonance with positive real part (Theorem 4.2, (4.18)). In this case, the Fourès–Segal theorem [9] implies that the scattering operator S is causal. When the manifold is non chronological, $\mathbb{T} \neq \emptyset$, we have stated in Theorem 4.2, (4.20), that there exists infinitely many resonances with positive part. We conjecture that some resonance satisfies (5.46) and the scattering operator is not causal.

Proof of Theorem 5.6. The analytic Fredholm theorem assures that the solution Ψ of (5.28), considered as a $H^2_{\mathrm{loc}}(\mathbb{R}^3_x \times S^2_\omega)$-valued map of k, is meromorphic on $\mathbb{C}$, and the poles k satisfy $ik \in \mathcal{R}$. Therefore the meromorphy of the map $k \in \mathbb{C} \mapsto \tilde{S}(k)$ follows from (5.44) and (5.45).

Since **S** is unitary on $\mathcal{H}_0$, $\tilde{S}(k)$ is unitary on $L^2(S^2)$ for almost every real k. We deduce from the analyticity of $\tilde{S}(\cdot)$ that

$$ik \notin \mathcal{R} \Rightarrow \tilde{S}(k) = \big[\big(\tilde{S}(\bar{k})\big)^*\big]^{-1}. \tag{5.47}$$

Therefore to prove that k_0 is a pole of $\tilde{S}$, it is sufficient to establish that

$$\mathrm{Ker}\big(\tilde{S}(\bar{k}_0)\big)^* \neq \{0\}. \tag{5.48}$$

We easily get from (5.44) that for $k \in \mathbb{C}$, $\mathrm{i}k \notin \mathcal{R}$, and $g \in L^2(S^2)$, we have:

$$\big[\big(\tilde{S}(k)\big)^* g\big](\omega) = g(\omega) + \frac{\mathrm{i}\bar{k}}{2\pi} \int_{S^2} \overline{\tilde{S}(\omega', k, \omega)} g(\omega')\, \mathrm{d}\omega'.$$

We remark that from the uniqueness of Ψ assured by Lemma 5.3 we get

$$\overline{\Psi(x; k, \omega)} = \Psi(x; -\bar{k}, \omega),$$

hence

$$\overline{\tilde{S}(\omega', k, \omega)} = \tilde{S}(\omega', -\bar{k}, \omega), \tag{5.49}$$

so we obtain

$$-\mathrm{i}k \notin \mathcal{R} \Longrightarrow \big[\big(\tilde{S}(\bar{k})\big)^* g\big](\omega) = g(\omega) + \frac{\mathrm{i}k}{2\pi} \int_{S^2} \tilde{S}(\omega', -k, \omega) g(\omega')\, \mathrm{d}\omega'. \tag{5.50}$$

To show (5.48), we use the following:

Lemma 5.7. *Let v be a $\mathrm{i}k_0$-outgoing resonant state associated with the resonance $\mathrm{i}k_0 \in \mathcal{R} \setminus \{0\}$. Then there exists $g \in C^\infty(S^2)$ such that*

$$v(x) = \frac{\mathrm{e}^{-\mathrm{i}k_0|x|}}{|x|} g\left(\frac{x}{|x|}\right)\big(1 + \mathrm{o}(1)\big), \qquad \frac{x}{|x|} \cdot \nabla v(x) = -\mathrm{i}k_0 \frac{\mathrm{e}^{-\mathrm{i}k_0|x|}}{|x|} g\left(\frac{x}{|x|}\right)\big(1 + \mathrm{o}(1)\big), \quad |x| \to \infty, \tag{5.51}$$

$$g \neq 0. \tag{5.52}$$

Proof. The existence of g satisfying (5.51) is a direct consequence of the integral representation (4.10) (see so [24], p. 127). Moreover Theorem 4.5 of [24], p. 129, assures that the translation representative $h \in \mathcal{D}'(\mathbb{R}_s \times S^2_\omega)$ of the eventually outgoing data $(v, \mathrm{i}k_0 v)$ satisfies:

$$s < -R_0 \Rightarrow h(s, \omega) = 0, \qquad s > R_0 \Rightarrow h(s, \omega) = \mathrm{e}^{-\mathrm{i}k_0 s} g(\omega).$$

Assume that $g = 0$. Then formula (3.1g) of [24], p. 111, implies that $(v, \mathrm{i}k_0 v)$ is also initially incoming. Theorem 4.2 of [24], p. 123, yields that $v(x) = 0$ for $|x| > R_0$. Since v is solution of the homogeneous elliptic equation (4.3), the unique continuation theorem shows that $v = 0$, that is a contradiction. □

We now claim that function g given by the previous lemma for k_0 satisfying (5.46) belongs to $\mathrm{Ker}(\tilde{S}(\bar{k}_0))^*$. From the equations

$$\Delta v + 2\mathrm{i}k_0 \frac{C}{r^2} \partial_\varphi v + k_0^2 \left(1 - \frac{C^2}{r^2}\right) v - V v = 0,$$

$$\left[\Delta - 2\mathrm{i}k_0 \frac{C}{r^2} \partial_\varphi + k_0^2 \left(1 - \frac{C^2}{r^2}\right) - V\right]\big(\mathrm{e}^{\mathrm{i}k_0 x \cdot \omega} + \Psi(x; -k_0, \omega)\big) = 0,$$

get by the Green formula:

$$\int_{|x|=R} \frac{x}{|x|} \cdot \nabla v(x)\big(\mathrm{e}^{\mathrm{i}k_0 x \cdot \omega} + \Psi(x; -k_0, \omega)\big) - v(x) \frac{x}{|x|} \cdot \nabla\big(\mathrm{e}^{\mathrm{i}k_0 x \cdot \omega} + \Psi(x; -k_0, \omega)\big)\, \mathrm{d}S(x) = 0.$$

On the one hand we have:

$$\begin{aligned}
&\int_{|x|=R} \frac{x}{|x|} \cdot \nabla v(x)\big(\mathrm{e}^{\mathrm{i}k_0 x \cdot \omega}\big) - v(x) \frac{x}{|x|} \cdot \nabla\big(\mathrm{e}^{\mathrm{i}k_0 x \cdot \omega}\big)\, \mathrm{d}S(x) \\
&\quad = -\mathrm{i}k_0 R \int_{S^2} \mathrm{e}^{-\mathrm{i}k_0 R(1 - \omega' \cdot \omega)} g(\omega')(1 + \omega' \cdot \omega)\, \mathrm{d}\omega' + \mathrm{o}(1) \\
&\quad = -4\pi g(\omega) + \mathrm{o}(1), \quad R \to \infty.
\end{aligned}$$

On the other hand:

$$\int_{|x|=R} \frac{x}{|x|}\cdot\nabla v(x)\Psi(x;-k_0,\omega) - v(x)\frac{x}{|x|}\cdot\nabla\Psi(x;-k_0,\omega)\,\mathrm{d}S(x)$$
$$= -2\mathrm{i}k_0 \int_{S^2} g(\omega')\tilde{S}(\omega',-k_0,\omega)\,\mathrm{d}\omega' + \mathrm{o}(1), \quad R\to\infty.$$

We conclude that

$$g(\omega) + \frac{2\mathrm{i}k_0}{2\pi}\int_{S^2}\tilde{S}(\omega',-k_0,\omega)g(\omega')\,\mathrm{d}\omega' = 0. \qquad \square$$

6. Scattering by a causality violation in a chronological space-time

In this part we prove the completeness of the wave operators in the case where the manifold is chronological but non-globally hyperbolic:

$$\mathbb{T} = \emptyset, \tag{6.1}$$
$$\Sigma \neq \emptyset. \tag{6.2}$$

The case of the globally hyperbolic space-time, $\mathbb{T}=\emptyset$, $\Sigma=\emptyset$, has been treated by D. Häfner [15]. Thus we assume that:

$$\sup\frac{C}{r} = 1. \tag{6.3}$$

In order to use some energy estimates, we impose the positivity of the total energy,

$$0\leqslant V. \tag{6.4}$$

First we consider the Cauchy problem with data on M_{t_0}. We show that this problem is well posed despite the existence of closed null geodesics. That is not entirely surprising since M_{t_0} is *weakly spacelike* according to the terminology of L. Hörmander who has studied the characteristic Cauchy problem on a globally hyperbolic manifold [19]. Nevertheless, because of the violation of the causality, we have to be carefull to define the set of the possible initial data.

We observe that Σ_0 is necessarily confining, hence we cannot invoke Theorem 3.5 to assure the uniqueness. But since the conserved energy $E(u)$ is now positive, $\mathcal{E}$, $\mathcal{H}$ defined by (3.60), (3.61), are Hilbert spaces, and $u\mapsto\mathbf{u}(0)$ is an isometry from $\mathcal{E}$ onto $\mathcal{H}$, for the norms:

$$\|u\|^2_{\mathcal{E}} := E_\infty(u,t) = \big\|\mathbf{u}(0)\big\|^2_{\mathcal{H}} := \frac{1}{2}\big\|\partial_t u(0)\big\|^2_{L^2_C} + \frac{1}{2}\big\|u(0)\big\|^2_1. \tag{6.5}$$

We have used the equivalent norm on $W^1(\mathbb{R}^3_x)$:

$$\|f\|^2_1 := \int_{\mathbb{R}^3}\big|\nabla f(x)\big|^2 + V(x)\big|f(x)\big|^2\mathrm{d}x.$$

Since $U(t)$ given by (3.63) is a strongly continuous unitary group $U(t)$ on $\mathcal{H}$, the Stone theorem assures that there exists a self-adjoint operator A on $\mathcal{H}$, with dense domain $D(A)$, such that

$$U(t) = \mathrm{e}^{\mathrm{i}tA}.$$

It is easy to characterize $D(A)$ in terms of more regular solutions:

$$D(A) = \big\{\mathbf{u}(0);\ u\in\mathcal{E}^1\big\}, \quad \mathcal{E}^1 := \{u\in\mathcal{E};\ \partial_t u\in\mathcal{E}\}. \tag{6.6}$$

To state that the space of the admissible Cauchy data is large, we introduce the set:

$$\mathcal{D} := \left\{(f,g)\in W^1\big(\mathbb{R}^3_x\big)\times H^1\big(\mathbb{R}^3_x\big);\ \Delta f\in L^2\big(\mathbb{R}^3_x\big), \Delta f + 2\frac{C}{r^2}\partial_\varphi g - Vf = 0 \text{ on a neighborhood } \mathcal{V}_{(f,g)} \text{ of } \Sigma_0\right\}, \tag{6.7}$$

and the Beppo–Levi space $W^1_0(\mathbb{R}^3_x\setminus\Sigma_0)$ as completion of $C^\infty_0(\mathbb{R}^3_x\setminus\Sigma_0)$ for the norm (3.11).

Theorem 6.1. *We assume that* (6.1) *and* (6.4) *are fullfiled. Then we have*:

$$C_0^\infty\big(\mathbb{R}_x^3 \setminus \Sigma_0\big) \times C_0^\infty\big(\mathbb{R}_x^3 \setminus \Sigma_0\big) \subset D(A), \tag{6.8}$$

$$W_0^1\big(\mathbb{R}_x^3 \setminus \Sigma_0\big) \times L_C^2\big(\mathbb{R}_x^3\big) \subset \mathcal{H}, \tag{6.9}$$

$$\mathcal{D} \subset \mathcal{H}. \tag{6.10}$$

Moreover if the Lebesgue measure of Σ_0 is zero, then

$$\overline{\mathcal{D}} = \mathcal{H} = W^1\big(\mathbb{R}_x^3\big) \times L_C^2\big(\mathbb{R}_x^3\big). \tag{6.11}$$

Proof. Given $\varepsilon \in]0, 1]$, we put:

$$C_\varepsilon := (1-\varepsilon)C.$$

We have:

$$0 \leqslant \sup \frac{C_\varepsilon}{r} < 1.$$

We define metric g_ε by (2.2) where we replace C by C_ε. Then the manifold $(\mathcal{M}, g_\varepsilon)$ is globally hyperbolic and M_{t_0} is a Cauchy hypersurface for the operator:

$$L_\varepsilon := \left(1 - \frac{C_\varepsilon^2}{r^2}\right)\partial_t^2 - \Delta_x - 2\frac{C_\varepsilon}{r^2}\partial_t\partial_\varphi + V.$$

Hence, given $(f, g) \in W^1(\mathbb{R}_x^3) \times L^2(\mathbb{R}_x^3)$, the Cauchy problem:

$$\begin{aligned} &u_\varepsilon \in C^0\big(\mathbb{R}_t; W^1\big(\mathbb{R}_x^3\big)\big), \qquad \partial_t u_\varepsilon \in C^0\big(\mathbb{R}_t; L_{C_\varepsilon}^2\big(\mathbb{R}_x^3\big)\big),\\ &L_\varepsilon u_\varepsilon = 0, \qquad u_\varepsilon(0) = f, \qquad \partial_t u_\varepsilon(0) = g, \end{aligned} \tag{6.12}$$

is solved by the usual way thanks to a unitary group on $W^1(\mathbb{R}_x^3) \times L_{C_\varepsilon}^2(\mathbb{R}_x^3)$, and we have the energy estimate:

$$\begin{aligned} &\int_{\mathbb{R}^3}\left(1 - \frac{C_\varepsilon^2}{r^2}\right)\big|\partial_t u_\varepsilon(t,x)\big|^2 + \big|\nabla u_\varepsilon(t,x)\big|^2 + V(x)\big|u_\varepsilon(t,x)\big|^2 \mathrm{d}x\\ &\quad = \int_{\mathbb{R}^3}\left(1 - \frac{C_\varepsilon^2}{r^2}\right)\big|g(x)\big|^2 + \big|\nabla f(x)\big|^2 + V(x)\big|f(x)\big|^2 \mathrm{d}x. \end{aligned} \tag{6.13}$$

We deduce that the family $(u_\varepsilon)_{0<\varepsilon<1}$ satisfies:

$$\sup_{0<\varepsilon<1}\sup_{t\in\mathbb{R}}\big\|u_\varepsilon(t)\big\|_1 < \infty, \tag{6.14}$$

$$\sup_{0<\varepsilon<1}\sup_{t\in\mathbb{R}}\big\|\partial_t u_\varepsilon(t)\big\|_{L_C^2} < \infty. \tag{6.15}$$

When (f, g) belongs to $\mathcal{D}$, $\partial_t u_\varepsilon$ is a finite energy solution of

$$L_\varepsilon(\partial_t u_\varepsilon) = 0, \qquad \partial_t u_\varepsilon(0) = g, \qquad \partial_t^2 u_\varepsilon(0) = \left(1 - \frac{C_\varepsilon^2}{r^2}\right)^{-1}\left[\Delta f + 2\frac{C_\varepsilon}{r^2}\partial_\varphi g - Vf\right],$$

hence, we get a second estimate:

$$\begin{aligned} &\big\|\Delta u_\varepsilon(t)\big\|_{L^2}^2 + \big\|\partial_t u_\varepsilon(t)\big\|_1^2\\ &\quad \leqslant Cst.\int_{\mathbb{R}^3}\left(1 - \frac{C_\varepsilon^2}{r^2}\right)^{-1}\left|\Delta u_\varepsilon(t,x) + 2\frac{C_\varepsilon}{r^2}\partial_t\partial_\varphi u_\varepsilon(t,x) - V(x)u_\varepsilon(t,x)\right|^2\\ &\qquad + \big|\nabla\partial_t u_\varepsilon(t,x)\big|^2 + V(x)\big|\partial_t u_\varepsilon(t,x)\big|^2 \mathrm{d}x\\ &\quad = Cst.\int_{\mathbb{R}^3}\left(1 - \frac{C_\varepsilon^2}{r^2}\right)^{-1}\left|\Delta f(x) + 2\frac{C_\varepsilon}{r^2}\partial_\varphi g(x) - V(x)f(x)\right|^2 + \big|\nabla g(x)\big|^2 + V(x)\big|g(x)\big|^2 \mathrm{d}x\\ &\quad \leqslant Cst'.\sup_{x\notin\mathcal{V}_{(f,g)}(\Sigma_0)}(r-C)^{-1}\big[\|\Delta f\|_{L^2}^2 + \|g\|_1^2\big]. \end{aligned} \tag{6.16}$$

We deduce that the family $(u_\varepsilon)_{0<\varepsilon<1}$ satisfies:

$$\sup_{0<\varepsilon<1}\sup_{t\in\mathbb{R}}\left\|\Delta u_\varepsilon(t)\right\|_{L^2}<\infty, \tag{6.17}$$

$$\sup_{0<\varepsilon<1}\sup_{t\in\mathbb{R}}\left\|\partial_t u_\varepsilon(t)\right\|_1<\infty. \tag{6.18}$$

On the one hand (6.14) and (6.18) assure that $(u_\varepsilon)_{0<\varepsilon<1}$ is equicontinuous in $C^0(\mathbb{R}_t;W^1(\mathbb{R}^3_x))$. On the other hand, (6.15) and (6.14) show that $(\partial_t u_\varepsilon)_{0<\varepsilon<1}$ is equicontinuous in $C^0(\mathbb{R}_t;L^2_C(\mathbb{R}^3_x))$. Now if f and g are compactly supported, since $L_\varepsilon=L_0$ for large $|x|$, we get that $u_\varepsilon(t,\cdot)$ and $\partial_t u_\varepsilon(t,\cdot)$ are supported in a compact that is independent of ε. Then (6.17), 6.18) and the Sobolev embedding assure that $(\mathbf{u}_\varepsilon(t,\cdot))_{0<\varepsilon<1}$ is relatively compact in $W^1(\mathbb{R}^3_x)\times L^2_C(\mathbb{R}^3_x)$. We conclude with the Ascoli theorem that there exists $(u,v)\in C^0(\mathbb{R}_t;W^1(\mathbb{R}^3_x)\times L^2_C(\mathbb{R}^3_x))$, and a sequence $\varepsilon_n\to 0^+$ such that:

$$\mathbf{u}_{\varepsilon_n}\to(u,v)\quad\text{in } C^0\big(\mathbb{R}_t;W^1\big(\mathbb{R}^3_x\big)\times L^2_C\big(\mathbb{R}^3_x\big)\big),\ n\to\infty.$$

We get that $\partial_t u|_{\mathcal{M}\setminus\Sigma}=v$. Moreover we have:

$$\partial_t^2 u_{\varepsilon_n}\to\partial_t^2 u,\qquad \partial_t\partial_\varphi u_{\varepsilon_n}\to\partial_t\partial_\varphi u\quad\text{in } H^{-2}_{\text{loc}}\big(\mathbb{R}^4_{(t,x)}\big),\ n\to\infty,$$

thus $Lu=0$ since $C_{\varepsilon_n}\to C$ in $H^2(\mathbb{R}^3_x)$ as $n\to\infty$. Therefore we have proved that the subset of the elements of $\mathcal{D}$ which are compactly supported, is included in $\mathcal{H}$. Since this subspace is dense in $\mathcal{D}$, (6.10) is established. Moreover we have $C_0^\infty(\mathbb{R}^3_x\setminus\Sigma_0)\times C_0^\infty(\mathbb{R}^3_x\setminus\Sigma_0)\subset\mathcal{D}$, thus we have (6.9).

To prove (6.8), we consider the solution $u\in\mathcal{E}$ with initial data $(f,g)\in C_0^\infty(\mathbb{R}^3_x\setminus\Sigma_0)\times C_0^\infty(\mathbb{R}^3_x\setminus\Sigma_0)$. Let $u'\in\mathcal{E}$ the solution with initial data $(f',g')\in C_0^\infty(\mathbb{R}^3_x\setminus\Sigma_0)\times C_0^\infty(\mathbb{R}^3_x\setminus\Sigma_0)$, where $f'=g$, $g'=(1-C^2r^{-2})[\Delta f+2Cr^{-2}\partial_\varphi g-Vf]$. We put:

$$v:=u-f-\int_0^t u'(s)\,\mathrm{d}s.$$

We easily check that $v\in\mathcal{E}$ and $\mathbf{v}(0)=0$, hence $v=0$ and $\partial_t u=u'\in\mathcal{E}$. Therefore we get (6.8).

To prove (6.11), since $C_0^\infty(\mathbb{R}^3_x\setminus\Sigma_0)$ is dense in $L^2_C(\mathbb{R}^3_x)$, it is sufficient to establish that

$$\mathcal{D}_1:=\big\{f\in W^1\big(\mathbb{R}^3_x\big);\ \Delta f\in L^2\big(\mathbb{R}^3_x\big),\ \Delta f-Vf=0\text{ on a neighborhood }\mathcal{V}_f\text{ of }\Sigma_0\big\},$$

is dense in $W^1(\mathbb{R}^3_x)$. We introduce:

$$\mathcal{D}_0:=\big\{f\in W^1\big(\mathbb{R}^3_x\big);\ \Delta f-Vf=0\text{ on a neighborhood }\mathcal{V}_f\text{ of }\Sigma_0\big\}.$$

Given $f\in\mathcal{D}_0$, we choose $\chi\in C_0^\infty(\mathcal{V}_f)$ such that $\chi=1$ on a neighborhood $\mathcal{V}'_f$ of Σ_0. Then $\chi f\in\mathcal{D}_1$ and $(1-\chi)f\in W^1(\mathbb{R}^3_x)$ equals to zero on $\mathcal{V}'_f$. Given $\varepsilon>0$ there exists $g\in C_0^\infty(\mathbb{R}^3_x\setminus\Sigma_0)$ such that $\|(1-\chi)f-g\|_1\leqslant\varepsilon$. Therefore $f_1:=\chi f+g\in\mathcal{D}_1$ and $\|f-f_1\|_1\leqslant\varepsilon$. We conclude that

$$\overline{\mathcal{D}_1}=\overline{\mathcal{D}_0}.$$

Let $F\in\mathcal{D}_0^\perp$. Let $\mathcal{V}$ be an open neighborhood of Σ_0 and we assume that its boundary $\partial\mathcal{V}$ is sufficiently smooth to that the Dirichlet problem for the Laplacian is well posed. We put $f=F$ on $\mathbb{R}^3_x\setminus\mathcal{V}$, and $f=u$ in $\mathcal{V}$, where u is the unique solution of

$$-\Delta u+Vu=0,\quad u\in H^1(\mathcal{V}),\qquad u=F\quad\text{on }\partial\mathcal{V}.$$

Then $f\in\mathcal{D}_0$, and we have:

$$\begin{aligned}0&=\int_{\mathbb{R}^3}\nabla F\cdot\overline{\nabla f}+VF\bar f\,\mathrm{d}x=\int_{\mathbb{R}^3\setminus\mathcal{V}}|\nabla F|^2+V|F|^2\,\mathrm{d}x+\langle u,\overline{\partial_\nu u}\rangle_{H^{1/2}(\partial\mathcal{V}),H^{-1/2}(\partial\mathcal{V})}\\&=\int_{\mathbb{R}^3\setminus\mathcal{V}}|\nabla F|^2+V|F|^2\,\mathrm{d}x+\int_{\mathcal{V}}|\nabla u|^2+V|u|^2\,\mathrm{d}x.\end{aligned}$$

We conclude that $F=0$ on $\mathbb{R}^3_x\setminus\mathcal{V}$. Now, given $n\in\mathbb{N}^*$, there exists $x(n,j)\in\Sigma_0$, $1\leqslant j\leqslant N_n$, such that

$$\Sigma_0\subset\mathcal{V}_n:=\bigcup_{j=1}^{N_n}B\left(x(n,j),\frac{1}{n}\right).$$

Since $\bigcap_{n=1}^{\infty}\mathcal{V}_n=\Sigma_0$, we get that $F=0$ on $\mathbb{R}^3_x\setminus\Sigma_0$. When Σ_0 is negligible, we conclude that $F=0$ and $\mathcal{D}_0$ is dense in $W^1(\mathbb{R}^3_x)$. □

We now return to the scattering theory. We have seen that the scattering operator S is an isometry from $\mathcal{E}_0$ onto $\mathcal{E}_0$. Nevertheless, when the space time is totally vicious ($\mathbb{T}\neq\emptyset$), we can define the wave operators $W^{+(-)}$ only on the dense set of the regular wave packets, $\mathcal{E}_0^{\infty}$, and the range of these operators is not known. Taking advantage of the fact that the conserved energy is positive when $\mathbb{T}=\emptyset$, we could extend by continuity the wave operators (5.16) previously defined on $\mathcal{E}_0^{\infty}$, but in order to be more concrete, we prefer to directely construct them, by replacing $W^1\times L^2$ by $W^1\times L^2_C$ in the control of the asymptotic behaviour, and using a time-dependent method. Despite the violation of the causality ($\Sigma_0\neq\emptyset$), we are able to develop a strategy *à la* Lax and Phillips [24] because the chronology is respected, and we get

$$\operatorname{Ran}W^+=\operatorname{Ran}W^-=\mathcal{E}. \tag{6.19}$$

We need the R-outgoing (R-incoming) subspaces:

$$D_R^{+(-)}:=\{F=(f,g)\in\mathcal{H}_0;\ |x|\leqslant +(-)t+R\Rightarrow U_0(t)F=0\},\quad 0\leqslant R. \tag{6.20}$$

Proposition 6.2. *We assume that* (6.1) *and* (6.4) *are fullfiled. Given* $u_0^{+(-)}\in\mathcal{E}_0$*, there exists a unique* $u^{+(-)}\in\mathcal{E}$ *such that*:

$$\left\|\mathbf{u}^{+(-)}(t)-\mathbf{u}_0^{+(-)}(t)\right\|_{W^1\times L^2_C}\to 0,\quad t\to +(-)\infty. \tag{6.21}$$

Moreover, we have:

$$\left\|u^{+(-)}\right\|_{\mathcal{E}}=\left\|u_0^{+(-)}\right\|_{\mathcal{E}_0}. \tag{6.22}$$

Proof. It is sufficient to study the past wave operator. Since

$$\left\|V^{\frac12}u_0^-(t)\right\|_{L^2}\to 0,\qquad 0\leqslant\left\|\partial_t u_0^-(t)\right\|_{L^2}-\left\|\partial_t u_0^-(t)\right\|_{L^2_C}\to 0,\quad t\to-\infty,$$

(6.22) is a consequence of (6.21); that assures the uniqueness. To establish the existence of u^-, we first assume that u_0^- is a free incoming wave, i.e. $(u_0(0),\partial_t u_0(0))\in D_R^-$ for some $R>0$. Let R_0 be given by (3.5) and $t_0<-R-R_0$. Thanks to Theorem 6.1 there exists a unique solution u of $Lu=0$ equal to u_0 for $t\leqslant t_0$. Hence (6.21) is satisfied. Now given $u_0^-\in\mathcal{E}_0$, we choose a sequence of free incoming waves, $u_{0,n}^-\in\mathcal{E}_0$, such that

$$\left\|u_{0,n}^- - u_0^-\right\|_{\mathcal{E}_0}\to 0,\quad n\to\infty.$$

(6.22) implies that $W^-(u_{0,n}^-)$ is a Cauchy sequence in $\mathcal{E}$. Let $u^-:=\lim_{n\to\infty}W^-(u_{0,n}^-)\in\mathcal{E}$. We evaluate:

$$\begin{aligned}\left\|\mathbf{u}^-(t)-\mathbf{u}_0^-(t)\right\|_{W^1\times L^2_C}\leqslant{}& c\left\|u^- - W^-(u_{0,n}^-)\right\|_{\mathcal{E}}^{\frac12}+c\left\|u_{0,n}^- - u_0^-\right\|_{\mathcal{E}_0}^{\frac12}\\ &+\left\|\nabla_x W^-(u_{0,n}^-)(t)-\nabla_x u_{0,n}^-(t)\right\|_{L^2}+\left\|\partial_t W^-(u_{0,n}^-)(t)-\partial_t u_{0,n}^-(t)\right\|_{L^2_C}.\end{aligned}$$

That concludes the proof. □

Therefore we have proved that the Wave Operators

$$W^{+(-)}:u_0^{+(-)}\mapsto u^{+(-)} \tag{6.23}$$

extend the wave operators (5.16) defined only on $\mathcal{E}_0^{\infty}$, and are isometries from $\mathcal{E}_0$ to $\mathcal{E}$. The main result of this part states these operators are onto.

Theorem 6.3. *We assume that* (6.1) *and* (6.4) *are fullfiled. Then for all* $u\in\mathcal{E}$*, there exists a unique* $u_0^{+(-)}\in\mathcal{E}_0$ *such that*:

$$\left\|\mathbf{u}(t)-\mathbf{u}_0^{+(-)}(t)\right\|_{W^1\times L^2_C}\to 0,\quad t\to +(-)\infty. \tag{6.24}$$

Moreover, we have:

$$\|u\|_{\mathcal{E}}=\left\|u_0^{+(-)}\right\|_{\mathcal{E}_0}. \tag{6.25}$$

The crucial point is the decay of the local energy that we establish by using the RAGE theorem.

Lemma 6.4. *Let $u \in \mathcal{E}$. Then for all $R > 0$ we have*:

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T \sqrt{E_R(u,t)}\,\mathrm{d}t = 0. \tag{6.26}$$

Proof. It is sufficient to consider the case where $(f,g) \in D(A)$. Then the solution $u \in \mathcal{E}$ satisfying $u(0) = f$, $\partial_t u(0) = g$, belongs to $\mathcal{E}_1$. Thus

$$\partial_t u \in C^0\big(\mathbb{R}_t; L^2_C\big(\mathbb{R}^3_x\big) \cap W^1\big(\mathbb{R}^3_x\big) = H^1\big(\mathbb{R}^3_x\big)\big), \qquad \partial_t^2 u \in C^0\big(\mathbb{R}_t; L^2_C\big(\mathbb{R}^3_x\big)\big).$$

By using equation $Lu = 0$ we get:

$$\Delta_x u \in C^0\big(\mathbb{R}_t; L^2\big(\mathbb{R}^3_x\big)\big),$$

hence we deduce that

$$D(A) \subset \big\{(f,g) \in W^1\big(\mathbb{R}^3_x\big) \times H^1\big(\mathbb{R}^3_x\big);\ \Delta f \in L^2\big(\mathbb{R}^3_x\big)\big\}, \tag{6.27}$$

$$\|f\|_{W^1} + \|\Delta f\|_{L^2} + \|g\|_{H^1} \leqslant \mathrm{const}\big\{\big\|(f,g)\big\|_{\mathcal{H}} + \big\|A(f,g)\big\|_{\mathcal{H}}\big\}. \tag{6.28}$$

Given $\chi \in C_0^\infty(\mathbb{R}^3_x)$ we define the cut-off operator

$$\boldsymbol{\chi} : (f,g) \mapsto (\chi f, \chi g);$$

(6.28) and the Rellich's compactness theorem imply that $\boldsymbol{\chi}(A + \mathrm{i})^{-1}$ is a compact operator from $\mathcal{H}$ to $\mathcal{H}_0$. On the other hand, Lemma 4.1 and the remark that follows it, show that A has no point spectrum. Then the RAGE theorem (see, e.g., [29], Theorem 1.2.1) assures that

$$\forall F \in \mathcal{H}, \quad \lim_{T\to\infty} \frac{1}{T}\int_0^T \big\|\boldsymbol{\chi} U(t)F\big\|_{\mathcal{H}}\,\mathrm{d}t = 0.$$

The result of decay of the local energy immediately follows. $\square$

Lemma 6.5. *For all $R \geqslant R_0$ we have*:

$$\overline{\bigcup_{t\in\mathbb{R}} U(t)D_R^+} = \overline{\bigcup_{t\in\mathbb{R}} U(t)D_R^-} = \mathcal{H}. \tag{6.29}$$

Proof. Let F be in $\mathcal{H}$, orthogonal to $U(t)D_R^+$ for all t. This condition is equivalent to:

$$\forall G \in D_R^+, \forall t \in \mathbb{R}, \quad \big\langle U(t)F, G\big\rangle_{\mathcal{H}} = 0.$$

We prove that $F = 0$. We choose $\theta \in C_0^\infty(\mathbb{R}_t)$ such that $\int \theta(t)\,\mathrm{d}t = 1$, and we put for $j \in \mathbb{N}$:

$$F_j = j\int \theta(jt)U(t)F\,\mathrm{d}t.$$

We easily check that $F_j \to F$ in $\mathcal{H}$, as $j \to \infty$. Moreover

$$\frac{U(t)F_j - F_j}{t} \to -j^2\int \theta'(js)U(s)F\,\mathrm{d}s \in \mathcal{H}, \quad t \to 0^+,$$

hence $F_j \in D(A)$. We have also:

$$\forall G \in D_R^+, \quad \big\langle U(t)F_j, G\big\rangle_{\mathcal{H}} = j\int \theta(js)\big\langle U(t+s)F, G\big\rangle_{\mathcal{H}}\,\mathrm{d}s = 0.$$

Therefore it is sufficient to consider the case:

$$F \in D(A), \forall G \in D_R^+, \forall t \in \mathbb{R}, \quad \big\langle U(t)F, G\big\rangle_{\mathcal{H}} = 0. \tag{6.30}$$

We remark that $U(t)F \in D(A) \subset \mathcal{H}_0$ and for $G \in D_R^+$,

$$\big\langle U(t)F, G\big\rangle_{\mathcal{H}_0} = \big\langle U(t)F, G\big\rangle_{\mathcal{H}} = 0.$$

Since $U_0(-R)D_R^+ = D_0^+$ we have:

$$\forall G_0 \in D_0^+, \quad \langle U_0(-R)U(t)F, G_0\rangle_{\mathcal{H}_0} = 0,$$

that to say $U_0(-R)U(t)F \in D_0^-$. We deduce that

$$s \leqslant 0, |x| \leqslant -s + R \Longrightarrow U_0(s-2R)U(t)F(x) = 0. \tag{6.31}$$

By uniqueness of the solution we conclude that

$$\forall s \leqslant 0, \forall t \in \mathbb{R}, \quad U_0(s-2R)U(t)F = U(s)U_0(-2R)U(t)F. \tag{6.32}$$

We need the local norms:

$$\begin{aligned} \|(f,g)\|_{R,\mathcal{H}}^2 &:= \frac{1}{2}\int_{|x|\leqslant R} \left(1-\frac{C^2}{r^2}\right)|g(x)|^2 + |\nabla_x f(x)|^2 + V(x)|f(x)|^2 \mathrm{d}x, \\ \|(f,g)\|_{R,\mathcal{H}_0}^2 &:= \frac{1}{2}\int_{|x|\leqslant R} |g(x)|^2 + |\nabla_x f(x)|^2 \mathrm{d}x. \end{aligned} \tag{6.33}$$

Thanks to (3.12) and (6.28) we can compare these norms:

$$\begin{aligned} \|(f,g)\|_{R,\mathcal{H}} &\leqslant c\|(f,g)\|_{R,\mathcal{H}_0}, \\ \|(f,g)\|_{R,\mathcal{H}_0} &\leqslant c'\{\|(f,g)\|_{R,\mathcal{H}} + \|A(f,g)\|_{R,\mathcal{H}}\}. \end{aligned} \tag{6.34}$$

Since $F \in D(A)$, Lemma 6.4 implies that

$$\frac{1}{T}\int_0^T \|U(t)F\|_{R,\mathcal{H}} + \|AU(t)F\|_{R,\mathcal{H}} \mathrm{d}t \to 0, \quad T \to \infty.$$

Therefore (6.34) assures that given $\varepsilon > 0$, $k \in \mathbb{N}^+$, there exists $T > (k+1)R$ such that

$$\|U(T)F\|_{5R,\mathcal{H}} + \|U(T)F\|_{5R,\mathcal{H}_0} \leqslant \varepsilon. \tag{6.35}$$

Applying (3.17) and (6.34) we have:

$$\|U_0(-2R)U(T)F\|_{3R,\mathcal{H}} \leqslant c\|U_0(-2R)U(T)F\|_{3R,\mathcal{H}_0} \leqslant c\|U(T)F\|_{5R,\mathcal{H}_0} \leqslant c\varepsilon, \tag{6.36}$$

$$\|U(T-2R)F\|_{3R,\mathcal{H}} \leqslant \|U(T)F\|_{5R,\mathcal{H}} \leqslant \varepsilon. \tag{6.37}$$

Since $L = L_0$ for $|x| \geqslant R_0$, we have $U_0(-2R)U(T)F = U(-2R+T)F$ for $|x| \geqslant 3R$, therefore with (6.36), (6.37), we get:

$$\|U_0(-2R)U(T)F - U(-2R)U(T)F\|_{\mathcal{H}} = \|U_0(-2R)U(T)F - U(-2R)U(T)F\|_{3R,\mathcal{H}} \leqslant (c+1)\varepsilon.$$

We apply $U(2R-T)$ to find

$$\|U(2R-T)U_0(-2R)U(T)F - F\|_{\mathcal{H}} \leqslant (c+1)\varepsilon.$$

By (6.32) with $s = 2R - T$, we have $U_0(-T)U(T)F = U(2R-T)U_0(-2R)U(T)F$, hence

$$\|U_0(-T)U(T)F - F\|_{\mathcal{H}} \leqslant (c+1)\varepsilon.$$

Finally thanks to (6.31), $U_0(-T)U(T)F = 0$ for $|x| \leqslant T - R$, and since $T > (k+1)R$ we conclude that

$$\forall k \in \mathbb{N}^+, \quad \|F\|_{kR,\mathcal{H}} \leqslant (c+1)\varepsilon. \qquad \square$$

Proof of Theorem 6.3. To prove the uniqueness of the symptotic waves, we consider $u_0^+, u_1^+ \in \mathcal{E}_0$ satisfying (6.24). Then $\|\mathbf{u}_0^+(t) - \mathbf{u}_1^+(t)\|_{W^1\times L_C^2} \to 0$ as $t \to +\infty$. Since the local energy of the free waves decaies, we get that $\|u_0^+ - u_1^+\|_{\mathcal{E}_0} = 0$.

To establish (6.25), we deduce from Lemma 6.4 that there exists $t_n \to \infty$ such that $E_{R_0}(u, t_n) \to 0$, as $n \to \infty$. Then $\sqrt{2}\|u\|_{\mathcal{E}} - \|\mathbf{u}(t_n)\|_{W^1\times L_C^2} \to 0$. Since $\|\mathbf{u}_0^+(t_n)\|_{W^1\times L_C^2} - \sqrt{2}\|u_0^+\|_{\mathcal{E}_0} \to 0$, (6.25) is a consequence of (6.24).

Let u be in $\mathcal{E}$. Lemma 6.5 assures that there exists $t_n \in \mathbb{R}$, $F_n \in D_{R_0}^+$, such that

$$\|U(t_n)F_n - \mathbf{u}(0)\|_{\mathcal{H}} \to 0, \quad n \to \infty.$$

We note that $U_0(t+t_n)F_n = U(t+t_n)F_n$ when $t+t_n \geqslant 0$. We put $F_n^+ := U_0(t_n)F_n$, and for $t_p \geqslant t_n$ we evaluate

$$\|F_n^+ - F_p^+\|_{\mathcal{H}_0} = \|F_n - U_0(t_p - t_n)F_p\|_{\mathcal{H}_0} = \|F_n - U_0(t_p - t_n)F_p\|_{\mathcal{H}} = \|U(t_n)F_n - U(t_p)F_p\|_{\mathcal{H}}.$$

We deduce that F_n^+ is a Cauchy sequence in $\mathcal{H}_0$. We denote $F^+ := \lim_{n\to\infty} F_n^+$, $\mathbf{u}_0^+(t) := U_0(t)F^+$. We estimate for $t+t_n \geqslant 0$,

$$\begin{aligned}\|\mathbf{u}_0^+(t) - \mathbf{u}(t)\|_{W^1\times L^2_C} &\leqslant \|U_0(t)\big(F^+ - F_n^+\big)\|_{W^1\times L^2_C} + \|U_0(t+t_n)F_n - \mathbf{u}(t)\|_{W^1\times L^2_C}\\ &\leqslant \sqrt{2}\|F^+ - F_n^+\|_{\mathcal{H}_0} + \sqrt{2}\|U(t_n)F_n - \mathbf{u}(0)\|_{\mathcal{H}}.\end{aligned}$$

(6.24) immediately follows. □

We achieve this study by some remarks on the Scattering Operator S. We have shown that even if the chronology is violated ($\mathbb{T} \neq \emptyset$), the scattering operator is a well defined isometry on $\mathcal{E}_0$, but in this case, its meaning is somewhat mysterious since we can construct the Wave Operators only on $\mathcal{E}_0^\infty$. When the chronology is not violated, we deduce from the previous theorem that $(W^+)^{-1}$ is well defined from $\mathcal{E}$ to $\mathcal{E}_0$, and with Proposition 6.2 we conclude that the Scattering Operator is actually defined by:

$$S := \big(W^+\big)^{-1} W^-. \tag{6.38}$$

Moreover since D_R^+ and D_R^- are orthogonal, the scattering operator S is causal in the usual sense (e.g., [24]), i.e.

$$\big(|x| \leqslant -t \Rightarrow u_0^-(t,x) = 0\big) \Longrightarrow \big(|x| \leqslant -t \Rightarrow u_0^+(t,x) = 0\big),$$

although the manifold $\mathcal{M}$ is non-causal (it would be preferable to say S is chronological, since this is this property of $\mathcal{M}$ that assures the so-called causality of S). This is also a consequence of the theorem of Fourès and Segal [9], and of the spectral representation of S, Proposition 5.5, since we have stated in Theorem 4.2 (4.18) that there exists no resonance with positive real part.

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