The Dirac Sea for the Non-Separable Hilbert Spaces

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We give a rigorous construction of the Dirac Sea for the fermionic quantization in the non-separable Hilbert spaces. These CAR-representations depend on the Axiom of Choice, hence are not unique, nevertheless they are unitarily equivalent to the classic Fock representation.

I. INTRODUCTION

The old hole theory of Dirac, where the vacuum is replaced by a *sea* filled of all the negative energy states, continues to arouse the interest of physicists and mathematicians despite the severe criticism by Weinberg. For instance the Dirac sea plays an important role in the theory of the fermionic projector of Finster [9], and recently it appears also in quantum cosmology [4]. Of the mathematical point of view, a rigorous definition of such a sea is an interesting issue. In the case of the usual framework of the separable Hilbert spaces, an elegant construction is provided by a semi-infinite wedge product in the work of Dimock [6] for the free Dirac equation, and for the external field problem by Deckert and co-authors [5]. At the first glance, this approach heavely depends of the countability of the Hilbert basis and its generalization to the *non*-separable Hilbert spaces is not obvious. The purpose of this paper consists in showing that this goal is achievable with a little work of set theory. This one consists in defining in a coherent way a suitable notion describing the "parity" of the size of all the subsets of an *infinite* set X. In short we prove the existence of a homomorphism π from the Abelian group of the power set of X endowed with the symetric difference, to $\mathbb{Z}/2\mathbb{Z}$, such that $\pi(A) = 0$ (resp. 1) if A is a finite part of X with an even (resp. odd) cardinal. The proof that is based on the technics of the ultrapowers, is strongly inspired by the theory of the numerosities of Benci *et alii* [2], [3] (a shorter proof uses powerful arguments from the Boolean Algebras Theory). As regards the axiomatic framework, the price to pay to be able to treat the case of the non-separable Hilbert spaces, is a triple recourse to some consequences of the Axiom of Choice (see *e.g* [12]): 1) for the existence of a Hilbert basis $(e_x)_{x \in X}$ on a non-separable Hilbert space; 2) for the existence of a linear order on the set X that indexes this basis; 3) in the proofs of the existence of π, we use one of the following three consequences of Zorn's Lemma: *i)* the existence of a suitable ultrafilter on the set of the finite parts of X, *ii)* Sikorski's extension theorem, *iii)* the equivalence between "completeness" and "injectivity" for the Boolean Algebras. As a consequence, our construction leads to a lot of different quantizations. Fortunately, they are all unitarily equivalent to the classic Fock quantization. Concerning the role of the non-separable Hilbert spaces in Physics, the negative opinion of Streater and Wightman in *PCT, Spin and Statistics, and All That*, is well known. Nevertheless this issue is always matter to debate, see Earman [8] for a highlighting discussion. We also remark that the non-separable Hilbert spaces naturally arise in loop quantum gravity (see *e.g.* [1]).

We now introduce our strategy. We first fix the notations by recalling the well known basics of the fermionic quantization (see *e.g.* [7], [13], [15]). We consider a complex Hilbert space $(\mathfrak{h}, <; >_{\mathfrak{h}})$ where $<; >_{\mathfrak{h}}$ is the inner product linear with respect to the second argument, and we look for a Hilbert space $\mathfrak{H}$ and an antilinear map Ψ from $\mathfrak{h}$ to the space of the linear maps on $\mathfrak{H}$ satisfying the canonical anticommutation relations (CAR): for any $u, v \in \mathfrak{h}$ we have

$$\{\Psi(u), [\Psi(v)]^*\} = < u; v >_{\mathfrak{h}} Id_{\mathfrak{H}}, \tag{1}$$

$$\{\Psi(u), \Psi(v)\} = 0, \tag{2}$$

where $\{A, B\} := AB + BA$ is the anticommutator of two operators A and B. Taking the adjoint of (2) we also have

$$\{[\Psi(u)]^*, [\Psi(v)]^*\} = 0. \tag{3}$$

Another important consequence of the CAR is that $\Psi(u)$ belongs to the space $\mathcal{L}(\mathfrak{H})$ of the bounded linear maps on $\mathfrak{H}$ and we have

$$\|\Psi(u)\|_{\mathcal{L}(\mathfrak{H})} = \|[\Psi(u)]^*\|_{\mathcal{L}(\mathfrak{H})} = \|u\|_{\mathfrak{h}}. \tag{4}$$

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The classic representation of the CAR on $\mathfrak{h}$ is given by choosing the antisymetric Fock space

$$\mathfrak{H} = \mathcal{F}^\wedge(\mathfrak{h}) := \bigoplus_{n=0}^{\infty} \mathfrak{h}^{\wedge n}, \tag{5}$$

and $\Psi = a$, the usual annihilation operator that is the adjoint of the creation operator a^*, which is nicely expressed by using the wedge product:

$$a^*(u)(v_1 \wedge v_2 \wedge ... \wedge v_n) = u \wedge v_1 \wedge v_2 \wedge ... \wedge v_n. \tag{6}$$

Up to an unitary transform, the Fock quantization is the unique irreducible representation of the CAR on $\mathfrak{h}$.

We are now ready to introduce the issue of the Dirac sea in this framework. The idea is that, unlike the classic Dirac quantum field, which annihilates a particle but *creates* an anti-particle, the fermionic quantum field should just be an annihilation operator in a suitable sense: the creation of an antiparticle in the Fock quantization should be understood as the creation of a hole in the Dirac sea, *i.e.* the annihilation of a state of negative energy, the Dirac sea being filled with all these states. This idea has been rigorously implemented by Dimock in [6] when $\mathfrak{h} = \mathfrak{h}_- \oplus \mathfrak{h}_+$ is *separable*. In order to point out the role of the separability, we briefly describe his approach based on the semi-infinite wedge products. In the sequel, if X is a set of integers or ordinals, we put $X^* := X \setminus \{0\}$. Given a Hilbert basis $(e_{\pm j})_{j \in \mathbb{N}^*}$ of $\mathfrak{h}_\pm$, we consider the Hilbert completion $\mathfrak{H}$ of the free vector space spanned by the formal symbols

$$e_I := e_{i_1} \wedge e_{i_2} \wedge ... \tag{7}$$

where $I = \{i_s, \ s \in \mathbb{N}^*\}$ and $(i_s)_{s \in \mathbb{N}^*} \in (\mathbb{Z}^*)^{\mathbb{N}^*}$ is strictly decreasing sequence (for the usual order on $\mathbb{Z}$) satisfying for s large enough

$$i_{s+1} = i_s - 1. \tag{8}$$

Then the Dirac sea is the vector

$$\Omega_D := e_{-1} \wedge e_{-2} \wedge e_{-3} \wedge ... \tag{9}$$

and the quantum field is the antilinear map Ψ from $\mathfrak{h}$ to $\mathcal{L}(\mathfrak{H})$ defined by:

$$\Psi(e_j)e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge ... = \begin{cases} 0 \ \ if \ \ \forall s \in \mathbb{N}, \ \ j \neq i_s, \\ (-1)^{s+1} e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_{s-1}} \wedge e_{i_{s+1}} \wedge ... \ \ if \ \ \exists s \in \mathbb{N}, \ \ j = i_s. \end{cases} \tag{10}$$

The key point in definition (10) is the factor $(-1)^s$ that is well defined thanks to the obvious fact that $s = s(j, I)$ defined by

$$j = i_s \tag{11}$$

is a *finite* ordinal that is just the cardinal of the subset

$$X(j, I) := \{ i \in I, \ \ i \geq j \}. \tag{12}$$

A crucial property to obtain the CAR is that

$$k > j, \ k \in I \Rightarrow (-1)^{s(j,I)} = -(-1)^{s(j,I \setminus \{k\})}, \tag{13}$$

since

$$k > j, \ k \in I \Rightarrow s(j, I) = s(j, I \setminus \{k\}) + 1. \tag{14}$$

The situation drastically changes when $\mathfrak{h}$ is a *non*-separable Hilbert space. In this case we have to consider a Hilbert basis $(e_j)_{j \in X}$ of $\mathfrak{h}$, where X is a totally ordered set of uncountable cardinal $| X |= \aleph > \aleph_0$, and, instead of the *countable* wedge products, the straight generalization of (7) and (8) would consist in considering the formal symbols

$$e_I = \bigwedge_{j \in I} e_j \tag{15}$$

where $I = \{i_s, \ s \in \aleph^*\}$ is *uncountable*, and $(i_s)_{s \in \aleph^*} \in X^{\aleph^*}$ is a strictly decreasing generalized sequence satisfying for some $a \in X$

$$x < a \Rightarrow x \in I. \tag{16}$$

Now given I and $j \in X$, $s = s(j, I)$ solution of (11) is an ordinal that belongs to $\aleph$ and we have to define $(-1)^s$ satisfying (13). Of course the difficulty arises when s is an *infinite* ordinal. We could try to use the Cantor's normal form theorem that assures that s can be uniquely expressed as

$$s = \lambda + N,$$

with $N \in \mathbb{N}$ and λ is a limit ordinal. Deciding classically that the limit ordinals are even, we could define

$$(-1)^s := (-1)^N. \tag{17}$$

Unfortunately this definition does not assure the key point (13) since (14) can be wrong due to the absorbing property of the ordinal addition

$$1 + \lambda = \lambda.$$

For an elementary example, we can take $I = \{-n; \ n \in \mathbb{N}^*\} \cup \{-\omega\}$ where $-\omega < -n$ for any integer n. Then with $j = -\omega$ and $k = -n$ for some integer n, we have $s(-\omega, I) = \aleph_0$ and also $s(-\omega, I \setminus \{-n\}) = \aleph_0$. Therefore with definition (17) we have

$$(-1)^{s(-\omega, I)} = 1 = (-1)^{s(-\omega, I \setminus \{-n\})}$$

that contradicts (13). We conclude that the ordinal calculus is not sufficient to associate 1 or -1 to the set $X(j, I)$ in such a way that (13) is satisfied even if $X(j, I)$ is infinite. To overcome this difficulty, we simply remark that in the separable case for which $X(j, I)$ is finite, $(-1)^{s(j,I)} = 1$ if the cardinal of $X(j, I)$ is even and $(-1)^{s(j,I)} = -1$ if the cardinal of $X(j, I)$ is odd. Finally we are led to ask a somewhat weird question: what is the parity of the size of an infinite set? Clearly, in the infinite case, the cardinal is not a tool sufficiently subtle to distinguish the size of $X(j, I)$ from the size of $X(j, I \setminus \{k\})$. In fact a refined concept of size of an infinite set has been introduced by Benci and co-authors [2], [3]. It is a hypernatural s and we could define $(-1)^s$ in the framework of this theory of the numerosities, from the parity of s, but we prefer to give a direct construction in the next section. We introduce the quantum fields in an abstract setting in the third part. We present the application to the Dirac theory in the last section.

II. PARITY OF AN INFINITE SET

We first introduce some notations. The cardinal of a set X is denoted $|X|$, $\mathcal{P}(X)$ is its power set, and $\mathcal{P}_F(X)$ is the set of the finite parts of X. The symetric difference Δ is defined by

$$A, B \in \mathcal{P}(X), \ \ A \Delta B := (A \cup B) \setminus (A \cap B).$$

Given an infinite set X we look for a homomorphism π from $(\mathcal{P}(X), \Delta)$ to $(\mathbb{Z}/2\mathbb{Z}, +)$ such that

$$\forall a \in X, \ \pi(\{a\}) = 1. \tag{18}$$

The requirement

$$\pi(A \Delta B) = \pi(A) + \pi(B) \tag{19}$$

is equivalent to the couple of properties

$$\forall A, B \in \mathcal{P}(X), \ \ A \cap B = \emptyset \Rightarrow \pi(A \cup B) = \pi(A) + \pi(B) \ \ in \ \ \mathbb{Z}/2\mathbb{Z}, \tag{20}$$

$$\forall A, B \in \mathcal{P}(X), \ \ B \subset A \Rightarrow \pi(A \setminus B) = \pi(A) - \pi(B) \ \ in \ \ \mathbb{Z}/2\mathbb{Z}. \tag{21}$$

If π exists, then (18) and (20) imply that for $A \in \mathcal{P}_F(X)$, $\pi(A)$ is the usual parity of the cardinal of A, defined as 0 if it is even and 1 if it is odd. The extension of π from $\mathcal{P}_F(X)$ to the whole $\mathcal{P}(X)$ is not at all obvious: its existence depends on the Axiom of Choice, and then such a π is not unique since we may impose $\pi(X) = 0$ or $\pi(X) = 1$ as well.

Theorem II.1 (Parity function) *We consider an infinite set X and $p \in \mathbb{Z}/2\mathbb{Z}$. Then there exists a homomorphism $\pi : (\mathcal{P}(X), \Delta) \to (\mathbb{Z}/2\mathbb{Z}, +)$ satisfying (18) and*

$$\pi(X) = p. \tag{22}$$

We present two proofs of this result. The first one, rather long and pedestrian, uses a suitable Ultrafilter. The second one, short and elegant, has been suggested by the anonymous referee; it is based on a powerful tool of the Boolean Algebras Theory: either Sikorski's extension theorem, or the injectivity of $\{0, 1\}$.

First proof. We identify the additive group $\mathbb{Z}/2\mathbb{Z}$ and $\{0, 1\}$ and we denote $\mathcal{I} := \mathcal{P}_F(X) \setminus \{\emptyset\}$. Given $p \in \{0, 1\}$, we define for any $i \in \mathcal{I}$

$$C_p(i) := \{j \in \mathcal{I}; \ i \subset j, \ |j| \in 2\mathbb{N} + p\} \in \mathcal{P}(\mathcal{I}) \setminus \{\emptyset\}. \tag{23}$$

We have for any $i, j \in \mathcal{I}$

$$C_p(i) \cap C_p(j) = C_p(i \cup j), \tag{24}$$

hence the family $\mathcal{B}_p := (C_p(i))_{i \in \mathcal{I}}$ is a filter basis on $\mathcal{I}$. Applying the Axiom of Choice, we consider an ultrafilter $\mathcal{U}_p$ containing $\mathcal{B}_p$. $\{0, 1\}^{\mathcal{I}}$ being the additive group of the maps from $\mathcal{I}$ to $\mathbb{Z}/2\mathbb{Z}$, we consider its ultrapower

$$\{0, 1\}^{\mathcal{I}}/\mathcal{U}_p := \{\dot{\varphi}, \ \varphi \in \{0, 1\}^{\mathcal{I}}\}, \quad \dot{\varphi} := \{\psi \in \{0, 1\}^{\mathcal{I}}; \ \{i; \ \varphi(i) = \psi(i)\} \in \mathcal{U}_p\}. \tag{25}$$

In fact this ultrapower is just a pair set: since $\mathcal{U}_p$ is an ultrafilter, either $\{i \in \mathcal{I}; \ \varphi(i) = 0\}$ or $\{i \in \mathcal{I}; \ \varphi(i) = 1\}$ belongs to $\mathcal{U}_p$. We denote p_φ the element of $\{0, 1\}$ such that $\{i \in \mathcal{I}; \ \varphi(i) = p_\varphi\} \in \mathcal{U}_p$. Now for $\psi \in \dot{\varphi}$ we have

$$\{i \in \mathcal{I}; \ \psi(i) = p_\varphi\} \supset \{i \in \mathcal{I}; \ \psi(i) = \varphi(i)\} \cap \{i \in \mathcal{I}; \ \varphi(i) = p_\varphi\} \in \mathcal{U}_p,$$

hence $\{i \in \mathcal{I}; \ \psi(i) = p_\varphi\} \in \mathcal{U}_p$ and thus $p_\varphi = p_\psi$, and we may introduce the map p

$$p : \dot{\varphi} \in \{0, 1\}^{\mathcal{I}}/\mathcal{U}_p \longmapsto p(\dot{\varphi}) := p_\varphi \in \{0, 1\}. \tag{26}$$

p is a group isomorphism. It is obviously surjective, it is a homomorphism since

$$\{i; \ \varphi(i) + \psi(i) = p_\varphi + p_\psi\} \supset \{i; \ \varphi(i) = p_\varphi\} \cap \{i; \ \psi(i) = p_\psi\} \in \mathcal{U}_p,$$

hence $\{i; \ \varphi(i) + \psi(i) = p_\varphi + p_\psi\} \in \mathcal{U}_p$ and thus $p(\dot{\varphi} + \dot{\psi}) = p(\dot{\varphi}) + p(\dot{\psi})$, and finally it is injective since

$$p(\dot{\varphi}) = 0 \Leftrightarrow p_\varphi = 0 \Leftrightarrow \{i; \ \varphi(i) = 0\} \in \mathcal{U}_p \Leftrightarrow \dot{\varphi} = \dot{0}.$$

To any $A \in \mathcal{P}(X)$ we associate $\varphi_A \in \{0, 1\}^{\mathcal{I}}$ by

$$\forall i \in \mathcal{I}, \ \varphi_A(i) = 0 \Leftrightarrow |A \cap i| \in 2\mathbb{N}, \ \varphi_A(i) = 1 \Leftrightarrow |A \cap i| \in 2\mathbb{N} + 1,$$

and we define

$$\pi(A) := p(\dot{\varphi}_A). \tag{27}$$

We remark that for any $a \in X$, we have

$$\{i \in \mathcal{I}; \ a \in i\} \in \mathcal{U}_p \tag{28}$$

since

$$\{i \in \mathcal{I}; \ a \in i\} \supset C_p(\{a\}) \in \mathcal{U}_p.$$

We have

$$\varphi_{\{a\}}(i) = 1 \Leftrightarrow i \in \{j \in \mathcal{I}; \ a \in j\},$$

hence $p_{\varphi_{\{a\}}} = 1$ and we deduce that (18) is satisfied. Now for $A, B \in \mathcal{P}(X)$ with $A \cap B = \emptyset$, we have

$$\{i; \ \varphi_{A \cup B}(i) = p_{\varphi_A} + p_{\varphi_B}\} \supset \{i; \ \varphi(A)(i) = p_{\varphi_A}\} \cap \{i; \ \varphi(B)(i) = p_{\varphi_B}\} \in \mathcal{U}_p$$

hence $p_{\varphi_{A \cup B}} = p_{\varphi_A} + p_{\varphi_B}$ and (20) is established. Moreover (21) is a consequence of (18) and (20) since if $B \subset A$ we have $\pi(A) = \pi(A \setminus B) + \pi(B)$. Finally we have

$$\{i; \ \varphi_X(i) = p\} = \{i; \ |i| \in 2\mathbb{N} + p\} \supset C_p(\{a\}) \in \mathcal{U}_p$$

hence $\{i; \ \varphi_X(i) = p\} \in \mathcal{U}_p$, and thus $\pi(X) = p$.

Q.E.D.

Second proof. We consider the collection $\mathcal{A}$ of the subsets of X that are either finite or co-finite: $\mathcal{A}$ is a Boolean algebra. We define a map $\pi_0 : \mathcal{A} \to \{0, 1\}$ as follows: If A is a finite subset of X, then $\pi_0(A) = 0$ if $| A |$ is even, otherwise $\pi_0(A) = 1$. If A is a co-finite subset of X, then $\pi_0(A) = p$ if $| X \setminus A |$ is even, otherwise $\pi_0(A) = 1 - p$. π_0 is clearly a Boolean homomorphism from $\mathcal{A}$ to $\{0, 1\}$ considered as a Boolean Algebra. Since $\{0, 1\}$ is trivially a complete algebra, π_0 can be extended into to a Boolean homomorphism π from $\mathcal{P}(X)$ to $\{0, 1\}$. The existence of π is assured either by Sikorski's extension theorem ([14], p.141, Theorem 33.1) or by the injectivity of $\{0, 1\}$ due to the theorem on the equivalence between the completeness and the injectivity ([11], p.141, Theorem 19).

Q.E.D.

We now are ready to deduce the main tool for the quantization, the "ϵ-functions" that replace $(-1)^s$ in the definition of the quantum fields.

Corollary II.2 ("ϵ-function") *Given a partially ordered set $(X, \leq)$, there exists a map*

$$\epsilon : \ I \in \mathcal{P}(X) \longmapsto \epsilon_I \in \{-1, +1\}^I \tag{29}$$

satisfying

$$\forall x \in X, \ \ \epsilon_{\{x\}}(x) = 1, \tag{30}$$

$$j, k \in I, \ j < k \Rightarrow \epsilon_I(k) = -\epsilon_{I \setminus \{j\}}(k), \tag{31}$$

$$j \in I, \ k \in X, \ \ j < k \Rightarrow \epsilon_I(j) = \epsilon_{I \cup \{k\}}(j). \tag{32}$$

Proof. We choose a parity function π given by the previous theorem. For $I \in \mathcal{P}(X)$, $x \in X$, we put $I_x := I \cap \{y \in X; \ y \leq x\}$. We obviously have: for any $x \in X$,

$$\{x\}_x = \{x\}, \ \ \pi(\{x\}_x) = 1,$$

for any $j, k \in I$ with $j < k$,

$$I_k = (I \setminus \{j\})_k \cup \{j\}, \ \ \pi(I_k) = \pi((I \setminus \{j\})_k) + 1,$$

for any $j \in I$ and $k > j$

$$I_j = (I \cup \{k\})_j.$$

Therefore it is sufficient to define for $I \in \mathcal{P}(X)$, $j \in I$

$$\epsilon_I(j) := (-1)^{\pi(I_j)+1}.$$

Q.E.D.

To make the link with the countable case, we consider $X = \mathbb{Z}^*$ and $I = \{i_s, \ s \in \mathbb{N}^*\}$ where $(i_s)_{s \in \mathbb{N}^*} \in (\mathbb{Z}^*)^{\mathbb{N}^*}$ is a strictly decreasing sequence (for the usual order on $\mathbb{Z}$) satisfying (8) for s large enough. We now endow $\mathbb{Z}^*$ with the *reverse* order of the usual order, then $\epsilon_I(j) := (-1)^{s+1}$ where $j = i_s$, satisfies (30), (31) and (32).

III. ABSTRACT QUANTIZATION

We now consider an infinite ordered set $(X, \leq)$ and a subset $\mathcal{I} \subset \mathcal{P}(X)$ such that

$$X \text{ is totally ordered}, \tag{33}$$

$$\forall I \in \mathcal{I}, \;\; \forall A \in \mathcal{P}_F(X), \;\; I \cup A \in \mathcal{I}, \;\; I \setminus A \in \mathcal{I}. \tag{34}$$

We also take an ϵ-function given by Corollary II.2.

For the quantization process, X enumerates a Hilbert basis $(e_x)_{x \in X}$ of the Hilbert space $\mathfrak{h}$ of the classical fields, and $\mathcal{I}$ indexes a Hilbert basis of the Hilbert space $\mathfrak{H}$ on which the quantum fields act. The existence of a linear order on X is assured by the Ordering Principle which is strictly weaker than the Zermelo Axiom. The ϵ-function will play the role of $(-1)^s$ in the definition of the quantum fields.

We introduce the equivalence relation $\mathcal{R}$ on the abelian group $(\mathcal{P}(X), \Delta)$ associated to the subgroup $\mathcal{P}_F(X)$. $\mathcal{R}$ is defined by

$$\forall A, B \in \mathcal{P}(X), \;\; A \mathcal{R} B \Leftrightarrow A \Delta B \in \mathcal{P}_F(X). \tag{35}$$

Since $A = (A \cap B) \sqcup (A \cap (A \Delta B))$ we also have

$$A \mathcal{R} B \Leftrightarrow \exists A', B', \in \mathcal{P}_F(X), \;\; C \in \mathcal{P}(X), \;\; A = A' \sqcup C, \;\; B = B' \sqcup C. \tag{36}$$

$\mathcal{R}$ is an equivalence relation on $\mathcal{P}(X)$ for which we denote $\mathcal{P}(X)/\mathcal{P}_F(X)$ its quotient set and $[A]_X$ the equivalence class of $A \in \mathcal{P}(X)$. The equivalence class $[A]_X$ is also described as

$$I \in [A]_X \Leftrightarrow \exists I_F \in \mathcal{P}_F(X), \;\; I = A \Delta I_F, \tag{37}$$

and the map

$$I_F \in \mathcal{P}_F(X) \longmapsto I = A \Delta I_F \in [A]_X \tag{38}$$

is a bijection from $\mathcal{P}_F(X)$ onto $[A]_X$. We note that $[\emptyset]_X = \mathcal{P}_F(X)$ and $[X]_X$ is just the set of the co-finite subsets of X. Putting $I_- := I_F \cap A$ and $I_+ := I_F \cap (X \setminus A)$, we also have

$$\forall A \in \mathcal{P}(X), \;\; \forall I \in [A]_X, \;\; \exists I_- \in \mathcal{P}_F(A), \;\; \exists I_+ \in \mathcal{P}_F(X \setminus A), \;\; I = (A \setminus I_-) \sqcup I_+. \tag{39}$$

We obviously have for any $A \subset X$:

$$\mid [A]_X \mid = \mid X \mid, \tag{40}$$

and since

$$\mathcal{P}(X) = \bigsqcup_{[A]_X \in \mathcal{P}(X)/\mathcal{P}_F(X)} [A]_X, \;\; \mid \mathcal{P}(X) \mid = 2^{|X|},$$

we have

$$\mid \mathcal{P}(X)/\mathcal{P}_F(X) \mid = 2^{|X|}. \tag{41}$$

Moreover $\mathcal{I} \subset \mathcal{P}(X)$ satisfies (34) iff

$$\mathcal{I} = \bigcup_{A \in \mathcal{I}} [A]_X. \tag{42}$$

Now we introduce the free vector space $\mathcal{V}(\mathcal{I})$ spanned by $\mathcal{I}$, *i.e.* we consider the elements I of $\mathcal{I}$ as vectors denoted e_I, and $\mathcal{V}(\mathcal{I})$ becomes a pre-Hilbert space if we decide $\mathcal{I}$ is an orthonormal basis.

The fundamental example arising for the quantization in the separable case is given by:

$$X = \mathbb{Z}^* = \mathbb{Z}^- \sqcup \mathbb{Z}^+, \;\; \mathbb{Z}^\pm := \{n \in \mathbb{Z}; \;\; \pm n \geq 1\}, \tag{43}$$

and $<$ is just the *reverse* of the usual strict order on $\mathbb{Z}$. We choose $\mathcal{I}$ to be the set of the "maya diagrams" (see *e.g.* [10]):

$$\mathcal{I} = \{I = (\mathbb{Z}^- \setminus I^-) \sqcup I^+; \;\; I^\pm \in \mathcal{P}_F(\mathbb{Z}^\pm)\} = [\mathbb{Z}^-]_{\mathbb{Z}^*}. \tag{44}$$

The *Dirac sea* is the vector

$$\Omega_D := e_{\mathbb{Z}^-} \in \mathcal{V}([\mathbb{Z}^-]_{\mathbb{Z}^*}). \tag{45}$$

We easily generalize this example to take into account the non-separable Hilbert spaces for which X is uncountable. Given two infinite ordinals $\Lambda_\pm$ we introduce

$$X = \Lambda_-^* \sqcup \Lambda_+^* \tag{46}$$

endowed with the total strict order $<$ defined by

$$\forall \lambda_\pm, \lambda'_\pm \in \Lambda_\pm^*, \ \ \lambda_+ < \lambda_-, \ \ \lambda_- < \lambda'_- \Leftrightarrow \lambda_- \prec \lambda'_-, \ \ \lambda_+ < \lambda'_+ \Leftrightarrow \lambda'_+ \prec \lambda_+, \tag{47}$$

where $\prec$ is the usual strict well order on the ordinals. To generalize (44) and (45), we can take

$$\mathcal{I} = [\Lambda_-^*]_{\Lambda_-^* \sqcup \Lambda_+^*} = \{I : (\Lambda_-^* \setminus I_-) \sqcup I_+; \ I_\pm \in \mathcal{P}_F(\Lambda_\pm^*)\}, \tag{48}$$

$$\Omega_D := e_{\Lambda_-^*} \in \mathcal{V}([\Lambda_-^*]_{\Lambda_-^* \sqcup \Lambda_+^*}). \tag{49}$$

We now choose the Hilbert framework. Given a set E, we introduce the Hilbert space of the complex-valued square integrable functions on E with respect to the counting measure, that is also the Hilbert closure of the free vector space $\mathcal{V}(E)$ spanned by E,

$$l^2(E) := \left\{ u \in \mathbb{C}^E; \ \|u\| := \left(\sum_{x \in E} |u(x)|^2 \right)^{\frac{1}{2}} < \infty \right\}. \tag{50}$$

We use the canonical Hilbert basis

$$\mathcal{B}(E) := \left\{ e_x \in \mathbb{C}^E; \ x \in E \right\}, \ \forall x, y \in E, \ \ e_x(y) = \delta_{x,y}, \tag{51}$$

here $\delta_{x,y}$ is the Kronecker symbol ($\delta_{x,x} = 1$, $\delta_{x,y} = 0$ if $x \neq y$). Then $u \in l^2(E)$ can be written as

$$u = \sum_{x \in E} c_x e_x, \ \ c_x \in \mathbb{C}, \tag{52}$$

where $\{x; \ c_x \neq 0\}$ is countable and

$$\|u\|^2 = \sum_{x \in E} |c_x|^2. \tag{53}$$

For $F \subset E$ we denote P_F the orthogonal projector on $l^2(F)$

$$P_F(u) = \sum_{x \in F} <e_x; u> e_x. \tag{54}$$

For any infinite set X and $A \subset X$, we have

$$l^2(X) = l^2(A) \oplus l^2(X \setminus A), \ \ l^2(\mathcal{P}(X)) = \bigoplus_{[A]_X \in \mathcal{P}(X)/\mathcal{P}_F(X)} l^2([A]_X), \tag{55}$$

and since the elements I of $[A]_X$ can be indexed by $I_F \in \mathcal{P}_F(X)$ or $(I_-, I_+) \in \mathcal{P}_F(A) \times \mathcal{P}_F(X \setminus A)$, we can identify the following spaces with natural isometries:

$$l^2([A]_X), \ l^2(\mathcal{P}_F(X)), \ l^2(\mathcal{P}_F(A) \times \mathcal{P}_F(X \setminus A)), \ l^2(\mathcal{P}_F(A)) \otimes l^2(\mathcal{P}_F(X \setminus A)), \ l^2([\emptyset]_A) \otimes l^2([\emptyset]_{X \setminus A}).$$

We now construct two maps, ψ, the "annihilation operator", and ψ^*, the "creation operator", from $l^2(X)$ to the space $\mathcal{L}(l^2(\mathcal{P}(X)))$ of the bounded linear maps on $l^2(\mathcal{P}(X))$. Given $j \in X$ and $I \in \mathcal{P}(X)$, we put:

$$\psi(e_j) e_I := \begin{cases} 0 & \text{if } j \notin I, \\ \epsilon_I(j) e_{I \setminus \{j\}} & \text{if } j \in I, \end{cases} \tag{56}$$

$$\psi^*(e_j)e_I := \begin{cases} 0 & \text{if } j \in I, \\ \epsilon_{I \cup \{j\}}(j)e_{I \cup \{j\}} & \text{if } j \notin I. \end{cases} \tag{57}$$

Since $\psi(e_j)$ and $\psi^*(e_j)$ map $\mathcal{B}(\mathcal{P}(X))$ into $\mathcal{B}(\mathcal{P}(X)) \cup \{0\}$, we can extend $\psi(e_j)$ and $\psi^*(e_j)$ as bounded linear operators on $l^2(\mathcal{P}(X))$ with

$$\|\psi(e_j)\|_{\mathcal{L}(l^2(\mathcal{I}))} = 1, \quad \|\psi^*(e_j)\|_{\mathcal{L}(l^2(\mathcal{I}))} = 1. \tag{58}$$

We note that (30) implies that

$$\psi(e_j)e_{\{j\}} = e_\emptyset, \quad \psi^*(e_j)e_\emptyset = e_{\{j\}}. \tag{59}$$

The physical interpretation of these operators is classic: $\psi(e_j)$ annihilates the state $e_{\{j\}}$ and $\psi^*(e_j)$ creates the state $e_{\{j\}}$. We also have with (30), (31) and (32)

$$j < k \Rightarrow \psi^*(e_k)\psi^*(e_j)e_\emptyset = -e_{\{j,k\}}, \quad \psi^*(e_j)\psi^*(e_k)e_\emptyset = e_{\{j,k\}} \tag{60}$$

More generally, we arrive to the fundamental algebraic properties: $\psi(e_j)$ and $\psi^*(e_j)$ are adjoint to each other and satisfy the canonical anticommutation relations (CAR).

Lemma III.1 *For any $j, k \in X$, we have*

$$\psi^*(e_j) = [\psi(e_j)]^*, \tag{61}$$

$$\{\psi(e_j), \psi^*(e_k)\} = \delta_{jk} Id, \tag{62}$$

$$\{\psi(e_j), \psi(e_k)\} = 0, \tag{63}$$

$$\{\psi^*(e_j), \psi^*(e_k)\} = 0. \tag{64}$$

For any $A \subset X$, $\psi(e_j)$ and $\psi^*(e_k)$ leave invariant $l^2([A]_X)$.

Proof. To prove (61) it is sufficient to establish for any $I, J \in \mathcal{I}$ that:

$$\langle \psi(e_j)e_I; e_J \rangle = \langle e_I; \psi^*(e_j)e_J \rangle. \tag{65}$$

We have

$$\langle \psi(e_j)e_I; e_J \rangle = \begin{cases} 0 & \text{if } j \notin I, \\ \epsilon_I(j)\delta_{I \setminus \{j\}, J} & \text{if } j \in I, \end{cases} \tag{66}$$

$$\langle e_I; \psi^*(e_j)e_J \rangle = \begin{cases} 0 & \text{if } j \in J, \\ \epsilon_{J \cup \{j\}}(j)\delta_{J \cup \{j\}, I} & \text{if } j \notin J. \end{cases} \tag{67}$$

Therefore $\langle \psi(e_j)e_I; e_J \rangle$ and $\langle e_I; \psi^*(e_j)e_J \rangle$ are non zero iff $j \notin J$ and $I = J \cup \{j\}$. Moreover in this case they are equal to $\epsilon_I(j)$. That proves (65).

Now we consider $j \in I$ and we evaluate

$$\psi(e_j)\psi^*(e_j)e_I = 0, \quad \psi^*(e_j)\psi(e_j)e_I = \epsilon_I(j)\psi^*(e_j)e_{I \setminus \{j\}} = \epsilon_I(j)\epsilon_I(j)e_I = e_I$$

hence we have

$$j \in I \Rightarrow \{\psi(e_j), \psi^*(e_j)\}e_I = e_I. \tag{68}$$

Then we consider $j \notin I$ and we evaluate

$$\psi^*(e_j)\psi(e_j)e_I = 0, \quad \psi(e_j)\psi^*(e_j)e_I = \epsilon_{I \cup \{j\}}(j)\psi(e_j)e_{I \cup \{j\}} = \epsilon_{I \cup \{j\}}(j)\epsilon_{I \cup \{j\}}(j)e_I = e_I$$

that implies

$$j \notin I \Rightarrow \{\psi(e_j), \psi^*(e_j)\}e_I = e_I. \tag{69}$$

Now we take $j \neq k$. 1) We first assume $j \in I$ and $k \notin I$. We compute:

$$\psi(e_j)\psi^*(e_k)e_I = \epsilon_{I\cup\{k\}}(j)\epsilon_{I\cup\{k\}}(k)e_{(I\setminus\{j\})\cup\{k\}}, \tag{70}$$

$$\psi^*(e_k)\psi(e_j)e_I = \epsilon_I(j)\epsilon_{(I\setminus\{j\})\cup\{k\}}(k)e_{(I\setminus\{j\})\cup\{k\}}. \tag{71}$$

If $j < k$, then $\epsilon_{I\cup\{k\}}(j) = \epsilon_I(j)$ by (32), $\epsilon_{(I\setminus\{j\})\cup\{k\}}(k) = -\epsilon_{I\cup\{k\}}(k)$ by (31) and thus

$$\{\psi(e_j), \psi^*(e_k)\}e_I = 0. \tag{72}$$

If $k < j$, then $\epsilon_{I\cup\{k\}}(j) = -\epsilon_I(j)$ by (31), $\epsilon_{(I\setminus\{j\})\cup\{k\}}(k) = \epsilon_{I\cup\{k\}}(k)$ by (32) and thus (72) is satisfied again. 2) We assume $j \notin I$ and $k \in I$. Then we have:

$$\psi^*(e_k)e_I = 0 = \psi(e_j)e_I, \tag{73}$$

hence (72) is trivially satisfied. 3) We assume $j \in I$ and $k \in I$. Then $k \in I \setminus \{j\}$ and so

$$\{\psi(e_j), \psi^*(e_k)\}e_I = \psi^*(e_k)\psi(e_j)e_I = \epsilon_I(j)\psi^*(e_k)e_{I\setminus\{j\}} = 0. \tag{74}$$

4) Finally we assume $j \notin I$ and $k \notin I$. Then $\psi(e_j)e_I = 0$ and $j \notin I \cup \{k\}$, hence

$$\{\psi(e_j), \psi^*(e_k)\}e_I = \psi(e_j)\psi^*(e_k)e_I = \epsilon_{I\cup\{k\}}(k)\psi(e_j)e_{I\cup\{k\}} = 0, \tag{75}$$

and the proof of (62) is complete.

To prove (63), we distinguish the different cases again. 1) If $j \notin I$, $k \notin I$ then $\psi(e_j)e_I = \psi(e_k)e_I = 0$ and obviously we have

$$\{\psi(e_j), \psi(e_k)\}e_I = 0. \tag{76}$$

2) If $j \notin I$, $k \in I$, we have $\psi(e_j)e_I = 0$ and

$$\{\psi(e_j), \psi(e_k)\}e_I = \epsilon_I(k)\psi(e_j)e_{I\setminus\{k\}} = 0 \tag{77}$$

since $j \notin I \setminus \{k\}$. 3) The case $j \in I$, $k \notin I$ is analogous. 4) Now if $j \in I$, $k \in I$, $j \neq k$, we compute

$$\{\psi(e_j), \psi(e_k)\}e_I = \left(\epsilon_I(k)\epsilon_{I\setminus\{k\}}(j) + \epsilon_I(j)\epsilon_{I\setminus\{j\}}(k)\right)e_{I\setminus\{j,k\}}. \tag{78}$$

If $j < k$, then (32) implies that $\epsilon_I(j) = \epsilon_{I\setminus\{k\}}(j)$ and (31) assures that $\epsilon_I(k) = -\epsilon_{I\setminus\{j\}}(k)$, and we conclude that (76) follows from (78). 5) Finally if $j = k \in I$, $\psi(e_j)\psi(e_j)e_I = \epsilon_I(j)\psi(e_j)e_{I\setminus\{j\}} = 0$. The proof of (63) is achieved. Moreover (64) follows from (63) by taking the adjoint.

To end, since the quantum fields map δ_I to zero, or $\pm\delta_{I\setminus\{j\}}$, or $\pm\delta_{I\cup\{j\}}$, they leave invariant $l^2([A]_X)$.

Q.E.D.

We now define ψ and ψ^* on the whole space $l^2(X)$ by the usual way used in the separable case [6]. If $u \in l^2(X)$ is expressed as

$$u = \sum_{k\in\mathbb{N}} c_k e_{j_k}, \quad c_k \in \mathbb{C}, \quad \sum_{k\in\mathbb{N}} |c_k|^2 < \infty, \tag{79}$$

we introduce for any $N \in \mathbb{N}$

$$\psi_N(u) := \sum_{k\leq N} c_k^* \psi(e_{j_k}), \quad \psi_N^*(u) := \sum_{k\leq N} c_k \psi^*(e_{j_k}) = [\psi_N(u)]^*. \tag{80}$$

Given two integers N, M with $M \geq N + 1$, we deduce from (62) that for any $V \in l^2(\mathcal{P}(X))$ we have

$$< V; \{\psi_M(u) - \psi_N(u), \psi_M^*(u) - \psi_N^*(u)\}V >= \left(\sum_{k=N+1}^{M} |c_k|^2 \right) \|V\|^2.$$

Therefore

$$\|\psi_M(u) - \psi_N(u)\|^2_{\mathcal{L}(l^2(\mathcal{P}(X)))} + \|\psi_M^*(u) - \psi_N^*(u)\|^2_{\mathcal{L}(l^2(\mathcal{P}(X)))} \leq \sum_{k=N+1}^{M} |c_k|^2.$$

We conclude that we may define the quantum fields

$$\psi(u) := \lim_{N \to \infty} \psi_N(u), \quad \psi^*(u) := \lim_{N \to \infty} \psi_N^*(u) \ \ in \ \ \mathcal{L}(l^2(\mathcal{P}(X))). \tag{81}$$

The previous lemma directly implies the main properties of the quantum fields:

Theorem III.2 ψ is an anti-linear map from $l^2(X)$ to $\mathcal{L}(l^2(\mathcal{P}(X)))$ and we have for any $u, v \in l^2(X)$

$$[\psi(u)]^* = \psi^*(u), \tag{82}$$

$$\|\psi(u)\|_{\mathcal{L}(l^2(\mathcal{P}(X)))} = \|u\|_{l^2(X)}, \tag{83}$$

$$\{\psi(u), \psi^*(v)\} =< u, v > Id_{l^2(\mathcal{P}(X))}, \tag{84}$$

$$\{\psi(u), \psi(v)\} = 0, \tag{85}$$

$$\{\psi^*(u), \psi^*(v)\} = 0, \tag{86}$$

$$\forall A \subset X, \ \forall u \in l^2(X), \ \ \psi(u)l^2([A]_X) \subset l^2([A]_X), \tag{87}$$

$$\forall A \subset X, \ \forall u \in l^2(X), \ \ P_A(u) = 0 \Rightarrow \psi(u)e_A = 0, \tag{88}$$

$$\forall A \subset X, \ \forall u \in l^2(X), \ \ P_{X \setminus A}(u) = 0 \Rightarrow \psi^*(u)e_A = 0. \tag{89}$$

The quantum field provided by the previous theorem is not unique since it depends on the choices of the linear order on X, and the function ϵ. Nevertheless, the representation of the CAR on $l^2(X)$ given by $(\psi, l^2([A]_X))$ is actually unique up to an unitary transform and it depends only on A and $X \setminus A$. We denote

$$\aleph_- :=| A |, \ \ \aleph_+ :=| X \setminus A |. \tag{90}$$

We take a bijection θ_- from A onto $\aleph_-$ and a bijection θ_+ from $X \setminus A$ onto $\aleph_+$. We denote θ the bijection from X onto $\aleph_- \sqcup \aleph_+$ defined by

$$j \in A \Rightarrow \theta(j) := \theta_-(j), \ \ j \in X \setminus A \Rightarrow \theta(j) := \theta_+(j). \tag{91}$$

Therefore the map

$$u \in l^2(X) \longmapsto u \circ \theta^{-1} \in l^2(\aleph_- \sqcup \aleph_+) = l^2(\aleph_-) \oplus l^2(\aleph_+) \tag{92}$$

is an isometry.

We consider an anti-unitary operator C on $l^2(\aleph_-) \oplus l^2(\aleph_+)$ and we introduce the fermionic Fock space

$$\mathfrak{H}_{\mathcal{F}} := \mathcal{F}^\wedge(Cl^2(\aleph_-)) \otimes \mathcal{F}^\wedge(l^2(\aleph_+)). \tag{93}$$

The Fock quantization $(\mathfrak{H}_\mathcal{F}, \Psi_\mathcal{F})$ on $l^2(\aleph_-) \oplus l^2(\aleph_+)$ is defined by choosing the antilinear map, $\Psi_\mathcal{F}$, from $l^2(\aleph_-) \oplus l^2(\aleph_+)$ to $\mathcal{L}(\mathfrak{H}_\mathcal{F})$ defined by

$$\Psi_\mathcal{F}(u_- \oplus u_+) := a_-^*(Cu_-) \otimes (-1)^{N_+} + Id \otimes a_+(u_+), \quad u_\pm \in l^2(\aleph_\pm), \tag{94}$$

where a_-^* (resp. a_+) is the creation (resp. annihilation) operator on $\mathcal{F}^\wedge\left(Cl^2(\aleph_-)\right)$ (resp. $\mathcal{F}^\wedge\left(l^2(\aleph_+)\right)$), and N_+ is the number operator on $\mathcal{F}^\wedge(l^2(\aleph_+))$. The *Fock vacuum* is the vector

$$\Omega_\mathcal{F} := (1, 0, 0, ...) \in [Cl^2(\aleph_-)]^{\wedge 0} \otimes [l^2(\aleph_-)]^{\wedge 0} \tag{95}$$

that obviously satisfies

$$\forall u_\pm \in l^2(\aleph_\pm), \quad \Psi_\mathcal{F}(u_+)\Omega_\mathcal{F} = 0, \quad [\Psi_\mathcal{F}(u_-)]^*\Omega_\mathcal{F} = 0. \tag{96}$$

We know that this representation is irreducible.

Theorem III.3 *For any $A \subset X$, the quantization $(\psi, l^2([A]_X)$ is an irreducible representation of the CAR on $l^2(X)$, moreover there exists a unitary transformation $\mathbb{U}_A$ from $\mathfrak{H}_\mathcal{F}$ onto $l^2([A]_X)$ such that for any $u \in l^2(X)$ we have*

$$\psi(u) = \mathbb{U}_A \Psi_\mathcal{F}\left(u \circ \theta^{-1}\right) \mathbb{U}_A^{-1} \quad on \; l^2([A]_X). \tag{97}$$

Proof. The irreducibility of the representation $(\psi, l^2([A]_X)$ follows from (97) since the Fock representation is irreducible. We express any subset $I_F \in \mathcal{P}_F(X)$ as a finite decreasing sequence

$$I_F = \{j_n \; 1 \le n \le N, \; j_N < j_{N-1} < ... < j_1\},$$

and if $\varphi = \psi, \; \psi^*$ we denote

$$\prod_{j \in I_F} \varphi(e_j) := \varphi(e_{j_N})\varphi(e_{j_{N-1}})...\varphi(e_{j_1}),$$

and for $\varphi = \Psi_\mathcal{F}, \; \Psi_\mathcal{F}^*$ we denote

$$\prod_{j \in I_F} \varphi(e_{\theta(j)}) := \varphi(e_{\theta(j_N)})\varphi(e_{\theta(j_{N-1})})...\varphi(e_{\theta(j_1)}).$$

We note that (39) assures that any $I \in [A]_X$ can be uniquely written as $I = (A \setminus I_-) \sqcup I_+$, and we have

$$\prod_{j \in I_+} \psi^*(e_j) \prod_{k \in I_-} \psi(e_k)e_A = \varepsilon e_I, \quad \varepsilon \in \{-1, 1\}.$$

We conclude that the set

$$\left\{\prod_{j \in I_+} \psi^*(e_j) \prod_{k \in I_-} \psi(e_k)e_A; \; I_- \in \mathcal{P}_F(A), \; I_+ \in \mathcal{P}_F(X \setminus A)\right\}, \tag{98}$$

is a Hilbert basis of $l^2([A]_X)$.

Since $\{e_{\theta_-(j)}; \; j \in A\}$ is a Hilbert basis of $l^2(\aleph_-)$ and $\{e_{\theta_+(j)}; \; j \in X \setminus A\}$ is a Hilbert basis of $l^2(\aleph_+)$, we know that

$$\left\{\prod_{j \in I_+} \Psi_\mathcal{F}^*(e_{\theta_+(j)}) \prod_{k \in I_-} \Psi_\mathcal{F}(e_{\theta_-(k)})\Omega_\mathcal{F}; \; I_- \in \mathcal{P}_F(A), \; I_+ \in \mathcal{P}_F(X \setminus A)\right\} \tag{99}$$

is a Hilbert basis of $\mathfrak{H}_\mathcal{F}$. We define the unitary transformation $\mathbb{U}_A$ from $\mathfrak{H}_\mathcal{F}$ onto $l^2([A]_X)$ by putting

$$\mathbb{U}_A\left(\prod_{j \in I_+} \Psi_\mathcal{F}^*(e_{\theta_+(j)}) \prod_{k \in I_-} \Psi_\mathcal{F}(e_{\theta_-(k)})\Omega_\mathcal{F}\right) = \prod_{j \in I_+} \psi^*(e_j) \prod_{k \in I_-} \psi(e_k)e_A. \tag{100}$$

Now to prove (97) it is sufficient to show that for anay $l \in X$, $I_- \in \mathcal{P}_F(A)$, $I_+ \in \mathcal{P}_F(X \setminus A)$ we have

$$\psi(e_l) \prod_{j \in I_+} \psi^*(e_j) \prod_{k \in I_-} \psi(e_k) e_A = \mathbb{U}_A \Psi_\mathcal{F} \left(e_{\theta(l)} \right) \prod_{j \in I_+} \Psi_\mathcal{F}^*(e_{\theta_+(j)}) \prod_{k \in I_-} \Psi_\mathcal{F}(e_{\theta_-(k)}) \Omega_\mathcal{F}. \tag{101}$$

The CAR assure that:

1) If $l \notin I_+$ there exists $\varepsilon \in \{-1, +1\}$ such that

$$\psi(e_l) \prod_{j \in I_+} \psi^*(e_j) \prod_{k \in I_-} \psi(e_k) e_A = \begin{cases} 0 \ if \ l \notin A, \\ \varepsilon \prod_{j \in I_+} \psi^*(e_j) \prod_{k \in I_- \cup \{l\}} \psi(e_k) e_A \ if \ l \in A, \end{cases}$$

$$\Psi_\mathcal{F} \left(e_{\theta(l)} \right) \prod_{j \in I_+} \Psi_\mathcal{F}^*(e_{\theta_+(j)}) \prod_{k \in I_-} \Psi_\mathcal{F}(e_{\theta_-(k)}) \Omega_\mathcal{F} = \begin{cases} 0 \ if \ l \notin A, \\ \varepsilon \prod_{j \in I_+} \Psi_\mathcal{F}^*(e_{\theta_+(j)}) \prod_{k \in I_- \cup \{l\}} \Psi_\mathcal{F}(e_{\theta_-(k)}) \Omega_\mathcal{F} \ if \ l \in A, \end{cases}$$

and we conclude that (101) is satisfied;

2) If $l \in I_+$ there exists $\varepsilon \in \{-1, +1\}$ such that

$$\psi(e_l) \prod_{j \in I_+} \psi^*(e_j) \prod_{k \in I_-} \psi(e_k) e_A = \varepsilon \prod_{j \in I_+ \setminus \{l\}} \psi^*(e_j) \prod_{k \in I_-} \psi(e_k) e_A,$$

$$\Psi_\mathcal{F} \left(e_{\theta(l)} \right) \prod_{j \in I_+} \Psi_\mathcal{F}^*(e_{\theta_+(j)}) \prod_{k \in I_-} \Psi_\mathcal{F}(e_{\theta_-(k)}) \Omega_\mathcal{F} = \varepsilon \prod_{j \in I_+ \setminus \{l\}} \Psi_\mathcal{F}^*(e_{\theta_+(j)}) \prod_{k \in I_-} \Psi_\mathcal{F}(e_{\theta_-(k)}) \Omega_\mathcal{F},$$

hence (101) is satisfied again. The proof is complete.

Q.E.D.

We deduce a form of the Shale-Stinespring criterion of implementation of unitary operators, adapted to our quantization.

Corollary III.4 *Given $A \subset X$, let U be an unitary operator on $l^2(X) = l^2(A) \oplus l^2(X \setminus A)$. Then there exists a unitary operator $\mathbb{U}$ on $l^2([A]_X)$ satisfying for any $u \in l^2(X)$*

$$\psi(Uu) = \mathbb{U}\psi(u)\mathbb{U}^{-1} \tag{102}$$

if and only if $P_A U P_{X \setminus A}$ and $P_{X \setminus A} U P_A$ are Hilbert-Schmidt operators.

Proof. With the previous notations we have

$$\psi(Uu) = \mathbb{U}_A \Psi_\mathcal{F}((Uu) \circ \theta^{-1}) \mathbb{U}_A^{-1} = \mathbb{U}_A \Psi_\mathcal{F}(\mathcal{U}(u \circ \theta^{-1})) \mathbb{U}_A^{-1}$$

where $\mathcal{U}$ is the unitary operator on $l^2(\aleph_- \sqcup \aleph_+)$ defined by

$$v \in l^2(\aleph_- \sqcup \aleph_+) \longmapsto \mathcal{U}v := [U(v \circ \theta)] \circ \theta^{-1}.$$

Therefore $\mathbb{U}$ exists and satisfies (102) iff $\mathcal{U}$ is implementable in the Fock representation of $l^2(\aleph_-) \oplus l^2(\aleph_+)$. The famous theorem of Shale and Stinespring states that a necessary and sufficient condition is that $P_{\aleph_-} \mathcal{U} P_{\aleph_+}$ and $P_{\aleph_+} \mathcal{U} P_{\aleph_-}$ are Hilbert-Schmidt operators. We have

$$P_{\aleph_-} \mathcal{U} P_{\aleph_+} v = [P_A U P_{X \setminus A}(v \circ \theta)] \circ \theta^{-1}$$

hence we deduce that

$$\sum_{y, y' \in \aleph_- \sqcup \aleph_+} |< P_{\aleph_-} \mathcal{U} P_{\aleph_+} e_y; e_{y'} >|^2 = \sum_{y_- \in \aleph_-} \sum_{y_+ \in \aleph_+} |< \mathcal{U} e_{y_+}; e_{y_-} |^2$$

$$= \sum_{j_- \in A} \sum_{j_+ \in X \setminus A} |< U e_{j_+}; e_{j_-} |^2 \tag{103}$$

$$= \sum_{j, j' \in X} |< P_A U P_{X \setminus A} e_y; e_{y'} >|^2 .$$

We conclude that $P_{\aleph_-} \mathcal{U} P_{\aleph_+}$ is Hilbert-Schmidt on $l^2(\aleph_- \sqcup \aleph_+)$ iff $P_A U P_{X \setminus A}$ is Hilbert-Schmidt on $l^2(X)$. And the same holds for $P_{\aleph_+} \mathcal{U} P_{\aleph_-}$ and $P_{X \setminus A} U P_A$. The proof is complete.

<div align="right">*Q.E.D.*</div>

Finally we prove that, up to a unitary transform, our irreducible representations depend only on $\mid A \mid$ and $\mid X \setminus A \mid$. We consider two totally ordered infinite sets X_i, $i = 1, 2$ and two epsilon functions ϵ_i given by Corollary IV.2, and the quantum fields ψ_i defined on $l^2(X_i)$ as previously. Let A_i be in $\mathcal{P}(X_i)$ such that

$$\mid A_1 \mid = \mid A_2 \mid, \quad \mid X_1 \setminus A_1 \mid = \mid X_2 \setminus A_2 \mid, \tag{104}$$

and we choose a bijection Θ from X_1 onto X_2 satisfying

$$\Theta(A_1) = A_2. \tag{105}$$

Theorem III.5 *There exists a unitary transform $\mathbb{U}$ from $l^2([A_2]_{X_2})$ onto $l^2([A_1]_{X_1})$ such that for any $u_2 \in l^2(X_2)$ we have*

$$\psi_2(u_2) = \mathbb{U}^{-1}\psi_1(u_2 \circ \Theta)\mathbb{U} \quad on \quad l^2([A_2]_{X_2}). \tag{106}$$

Proof. We denote $\aleph_- := \mid A_i \mid$, $\aleph_+ := \mid X_i \setminus A_i \mid$. We introduce a bijection θ_i from X_i onto $\aleph_- \sqcup \aleph_+$ such that $\theta_i(A_i) = \aleph_-$, $\theta_i(X_i \setminus A_i) = \aleph_+$. The previous Theorem assures there exists a unitary transform U_i from $\mathcal{F}^\wedge(\mathrm{Cl}^2(\aleph_-)) \otimes \mathcal{F}^\wedge(l^2(\aleph_+))$ onto $l^2([A_i]_{X_i})$ such that for any $u_i \in l^2(X_i)$ we have

$$\psi_i(u_i) = U_i \Psi_{\mathcal{F}} \left(u_i \circ \theta_i^{-1} \right) U_i^{-1}. \tag{107}$$

Now we introduce an unitary operator T on $l^2(\aleph_-) \oplus l^2(\aleph_+)$ by putting

$$T: \ v \in l^2(\aleph_-) \oplus l^2(\aleph_+) \longmapsto v \circ \theta_2 \circ \Theta \circ \theta_1^{-1} \in l^2(\aleph_-) \oplus l^2(\aleph_+). \tag{108}$$

Since T leaves invariant $l^2(\aleph_\pm)$ it is implementable in the Fock representation: there exists a unitary operator $\mathbb{T}$ on $\mathcal{F}^\wedge(\mathrm{Cl}^2(\aleph_-)) \otimes \mathcal{F}^\wedge(l^2(\aleph_+))$ such that for any $v \in l^2(\aleph_-) \oplus l^2(\aleph_+)$ we have

$$\Psi_{\mathcal{F}}(Tv) = \mathbb{T}\Psi_{\mathcal{F}}(v)\mathbb{T}^{-1}.$$

Since the previous Theorem assures that there exist unitary tranformations $\mathbb{U}_{A_i}$ from $\mathfrak{H}_{\mathcal{F}}$ onto $l^2([A_i]_{X_i})$ such that for any $u_i \in l^2(X_i)$

$$\psi_i(u_i) = \mathbb{U}_{A_i} \Psi_{\mathcal{F}} \left(u_i \circ \theta_i^{-1} \right) \mathbb{U}_{A_i}^{-1} \quad on \quad l^2([A_i]_{X_i}),$$

we compute

$$\begin{aligned}
\psi_1(u_2 \circ \Theta) &= \mathbb{U}_1 \Psi_{\mathcal{F}} \left(u_2 \circ \theta_2^{-1} \circ (\theta_2 \circ \Theta \circ \theta_1^{-1}) \right) \mathbb{U}_1^{-1} \\
&= \mathbb{U}_1 \mathbb{T} \Psi_{\mathcal{F}} \left(u_2 \circ \theta_2^{-1} \right) \mathbb{T}^{-1} \mathbb{U}_1^{-1} \\
&= \mathbb{U}_1 \mathbb{T} \mathbb{U}_2^{-1} \psi_2(u_2) \mathbb{U}_2 \mathbb{T} \mathbb{U}_1^{-1}.
\end{aligned} \tag{109}$$

Now it is sufficient to choose

$$\mathbb{U} := \mathbb{U}_1 \mathbb{T} \mathbb{U}_2^{-1}.$$

<div align="right">*Q.E.D.*</div>

Taking $X_1 = X_2$, $A_1 = A_2$, $\Theta = Id$, we immediately deduce the following

Corollary III.6 *Given an infinite set X, let ψ_1, ψ_2 be the quantizations associated to two linear orders on X and two ϵ-functions. Then for any $A \subset X$ there exists a unitary operator $\mathbb{U}$ on $l^2([A]_X)$ such that for any $u \in l^2(X)$ we have*

$$\psi_2(u) = \mathbb{U}^{-1}\psi_1(u)\mathbb{U} \quad on \quad l^2([A]_X). \tag{110}$$

IV. APPLICATION TO THE DIRAC THEORY

We consider two Hilbert spaces $\mathfrak{h}_\pm$, possibly non-separable, and we define

$$\mathfrak{h} = \mathfrak{h}_- \oplus \mathfrak{h}_+, \tag{111}$$

and we denote $P_\pm$ the orthogonal projector on $\mathfrak{h}_\pm$. Invoking the Axiom of Choice, we take two Hilbert basis $(\mathfrak{e}_j)_{j \in X_\pm}$ of $\mathfrak{h}_\pm$. We put

$$X := X_- \sqcup X_+, \tag{112}$$

then $(\mathfrak{e}_j)_{j \in X}$ is a Hilbert basis of $\mathfrak{h}$, and using the Axiom of Choice again, we endow X with a total order. Finally, we use a last time this Axiom to get an ultrafilter on $\mathcal{P}_F(X) \setminus \{\emptyset\}$ to obtain an ϵ-function given by Corollary II.2. ψ being the quantum field on $l^2(X)$ given by Theorem III.2, $(\psi, l^2([X_-]_X)$ is an irreducible representation of the CAR on $l^2(X)$. Now we introduce the canonical isometry $\mathcal{J}$ from $\mathfrak{h}$ onto $l^2(X)$,

$$\mathfrak{u} \in \mathfrak{h} \longmapsto \mathcal{J}(\mathfrak{u}) = (< \mathfrak{e}_j; \mathfrak{u} >)_{j \in X} \in l^2(X), \tag{113}$$

and we define a quantum field on $\mathfrak{h}$ by putting

$$\Psi_D(\mathfrak{u}) := \psi(\mathcal{J}(\mathfrak{u})) \in \mathcal{L}(\mathfrak{H}_D), \quad \mathfrak{H}_D := l^2([X_-]_X). \tag{114}$$

The *Dirac sea* describing the state fulfilled by $\mathfrak{h}_-$ is the vector

$$\Omega_D := e_{X_-} \in \mathfrak{H}_D. \tag{115}$$

These mathematical objects have the usual meaning of the Dirac theory: $\mathfrak{h}_{+(-)}$ is the space of the classical solutions of positive (negative) energy. The quantum field Ψ_D is an operator of annihilation: $\Psi_D(\mathfrak{e}_j)$ annihilates the state $\mathfrak{e}_j$, therefore it annihilates a particle in the state $\mathfrak{e}_j$ when $j \in X_+$, and it creates a hole in the Dirac sea when $j \in X_-$.

We shall compare with the classic Fock quantization defined by the Hilbert space

$$\mathfrak{H}_\mathcal{F} := \mathcal{F}^\wedge(\mathcal{C}\mathfrak{h}_-) \otimes \mathcal{F}^\wedge(\mathfrak{h}_+), \tag{116}$$

the Fock vacuum vector

$$\Omega_\mathcal{F} := (1, 0, 0, 0...) \in [\mathcal{C}\mathfrak{h}_-]^{\wedge 0} \otimes [\mathfrak{h}_+]^{\wedge 0}, \tag{117}$$

and the quantum field

$$\Psi_\mathcal{F}(u) := a_-^*(\mathcal{C}P_-u) \otimes (-1)^{N_+} + Id \otimes a_+(P_+u) \in \mathcal{L}(\mathfrak{H}_\mathcal{F}), \tag{118}$$

where a_-^* (resp. a_+) is the creation (resp. annihilation) operator on $\mathcal{F}^\wedge(\mathcal{C}\mathfrak{h}_-)$ (resp. $\mathcal{F}^\wedge(\mathfrak{h}_+)$), N_+ is the number operator on $\mathcal{F}^\wedge(\mathfrak{h}_+)$ and $\mathcal{C}$ is an anti-unitary operator on $\mathfrak{h}$. Now $\mathfrak{h}_+$ is the one-particle space, $\mathcal{C}\mathfrak{h}_-$ is the one-antiparticle space, and $\Psi_\mathcal{F}(\mathfrak{e}_j)$ annihilates a particle if $j \in X_+$ and creates an antiparticle if $j \in X_-$.

Theorem IV.1 $(\Psi_D, \mathfrak{H}_D)$ *defined by (114) is an irreducible representation of the CAR on $\mathfrak{h}$. All the representations obtained by changing the total order on X, or the ϵ-function, or the Hilbert basis, are unitarily equivalent. Moreover there exists a unitary transform $\mathbb{U}$ from $\mathfrak{H}_\mathcal{F}$ onto $\mathfrak{H}_D$ such that*

$$\Omega_D = \mathbb{U}\Omega_\mathcal{F}, \tag{119}$$

$$\forall u \in \mathfrak{h}, \quad \Psi_D(u) = \mathbb{U}\Psi_\mathcal{F}(u)\mathbb{U}^{-1}. \tag{120}$$

Proof. If we change the total order on X or the ϵ-function, Corollary III.6 shows that we obtain a new representation that is unitarily equivalent to the previous one. More generally, if we take another basis $(\mathfrak{e}_j')_{j \in X_\pm'}$ of $\mathfrak{h}_\pm$, Theorem III.5 assures that the quantizations built from these two choices of basis are unitarily equivalent. Moreover, if we take an anti-unitary transform $\mathcal{C}$ on $\mathfrak{h}$, Theorem III.3 implies that these quantizations are unitarily equivalent to the Fock quantization $\Psi_\mathcal{F}$ on $l^2(\aleph_-) \oplus l^2(\aleph_+)$ where $X_\pm = \aleph_\pm = dim(\mathfrak{h}_\pm)$ and $C = \mathcal{J}\mathcal{C}\mathcal{J}^{-1}$, which is obviously unitarily equivalent to the classic Fock quantization on $\mathfrak{h}_- \oplus \mathfrak{h}_+$.

Q.E.D.

We introduce a general setting for the abstract Dirac equations that have the form

$$i\frac{d\psi}{dt} = H\psi, \tag{121}$$

with the following assumptions:

1) H is a densely defined selfadjoint operator on $\mathfrak{h}$ with domain $\mathfrak{D}(H)$;

2) there exist two densely defined selfadjoint operators $H_\pm$ on $\mathfrak{h}_\pm$ with domain $\mathfrak{D}(H_\pm)$ such that

$$0 \leq H_+, \tag{122}$$

$$H = H_- \oplus H_+, \quad \mathfrak{D}(H) = \mathfrak{D}(H_-) \oplus \mathfrak{D}(H_+); \tag{123}$$

3) There exists an anti-unitary operator $\mathcal{C}$ on $\mathfrak{h}$ such that,

$$\mathcal{C}\mathfrak{h}_\pm = \mathfrak{h}_\mp, \quad \mathcal{C}P_\pm = P_\mp\mathcal{C}, \tag{124}$$

$$\mathcal{C}\mathfrak{D}(H_\pm) = \mathfrak{D}(H_\mp), \quad \mathcal{C}H = -H\mathcal{C}. \tag{125}$$

In fact these assumptions imply that

$$H_- \leq 0 \tag{126}$$

since we have:

$$0 \leq\, < P_+u; H_+P_+u > = < \mathcal{C}H_+P_+u; \mathcal{C}P_+u >$$
$$= < \mathcal{C}P_+Hu; P_-\mathcal{C}u > = < P_-\mathcal{C}Hu; P_-\mathcal{C}u >$$
$$= - < P_-H\mathcal{C}u; P_-\mathcal{C}u > = - < H_-P_-\mathcal{C}u; P_-\mathcal{C}u > .$$

The time evolution of the classical Dirac fields is given by the unitary group $e^{-itH} = e^{-itH_-} \oplus e^{-itH_+}$ on $\mathfrak{h}$, that leaves invariant $\mathfrak{h}_\pm$. The time evolution of the quantum field is defined for $t \in \mathbb{R}$, $u \in \mathfrak{h}$ by putting:

$$\Psi_D(t, u) := \Psi_D\left(e^{itH}u\right). \tag{127}$$

Theorem IV.2 *For any $t \in \mathbb{R}$, $(\Psi_D(t, .), \mathfrak{H}_D)$ is an irreducible representation of the CAR on $\mathfrak{h}$ and for any $u \in \mathfrak{D}(H)$ the map $t \mapsto \Psi_D(t, u) \in \mathcal{L}(\mathfrak{H}_D)$ is a strongly differentiable function satisfying*

$$i\frac{d}{dt}\Psi_D(t, u) = \Psi_D(t, Hu), \quad \Psi_D(0, u) = \Psi_D(u). \tag{128}$$

Moreover there exists a densely defined self-adjoint operator $\mathbb{H}_D$ on $\mathfrak{H}_D$ such that

$$0 \leq \mathbb{H}_D, \tag{129}$$

$$\forall t \in \mathbb{R}, \ \forall u \in \mathfrak{h}, \quad \Psi_D(t, u) = e^{it\mathbb{H}_D}\Psi_D(u)e^{-it\mathbb{H}_D}. \tag{130}$$

Proof. The first assertion is obvious since e^{itH} is unitary. Moreover if $u \in \mathfrak{D}(H)$, the map $t \mapsto e^{itH}u$ belongs to $C^1(\mathbb{R}_t; \mathfrak{h}) \cap C^0(\mathbb{R}_t; \mathfrak{D}(H))$, therefore, since $u \mapsto \Psi_D(u)$ is an antilinear bounded map, we can differentiate $\Psi_D(t, e^{itH}u)$ and we obtain (128)). Now since $\mathcal{C}\mathfrak{h}_- = \mathfrak{h}_+$, we consider $\mathfrak{H}_{\mathcal{F}} = \mathcal{F}^\wedge(\mathfrak{h}_+) \otimes \mathcal{F}^\wedge(\mathfrak{h}_+)$, and the positive densely defined self-adjoint operator on $\mathfrak{H}_{\mathcal{F}}$

$$\mathbb{H}_{\mathcal{F}} := d\Gamma(H_+) \otimes Id + Id \otimes d\Gamma(H_+), \tag{131}$$

where Γ is the usual second-quantization functor. Therefore, using unitary operator $\mathbb{U}$ of Theorem IV.1, the operator

$$\mathbb{H}_D := \mathbb{U}\mathbb{H}_{\mathcal{F}}\mathbb{U}^{-1} \tag{132}$$

is positive, densely defined and self-adjoint on $\mathfrak{H}_D$. We have:

$$e^{it\mathbb{H}_{\mathcal{F}}} = \Gamma\left(e^{itH_+}\right) \otimes \Gamma\left(e^{itH_+}\right), \tag{133}$$

$$e^{it\mathbb{H}_D} = \mathbb{U}e^{it\mathbb{H}_{\mathcal{F}}}\mathbb{U}^{-1}. \tag{134}$$

We achieve the proof by writting

$$
\begin{aligned}
\Psi_D(e^{itH}u) &= \mathbb{U}\Psi_{\mathcal{F}}(e^{itH}u)\mathbb{U}^{-1} \\
&= \mathbb{U}\left(a_-^*(\mathcal{C}P_-e^{itH}u) \otimes (-1)^{\mathrm{N}_+} + Id \otimes a_+(P_+e^{itH}u)\right)\mathbb{U}^{-1} \\
&= \mathbb{U}\left(a_-^*(e^{itH_+}P_+\mathcal{C}u) \otimes (-1)^{\mathrm{N}_+} + Id \otimes a_+(e^{itH_+}P_+u)\right)\mathbb{U}^{-1} \\
&= \mathbb{U}e^{it\mathbb{H}_{\mathcal{F}}}\Psi_{\mathcal{F}}(u)e^{-it\mathbb{H}_{\mathcal{F}}}\mathbb{U}^{-1} \\
&= e^{it\mathbb{H}_D}\Psi_D(u)e^{-it\mathbb{H}_D}.
\end{aligned}
\tag{135}
$$

Q.E.D.

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Conflict of Interest

The author has no conflicts of interest to disclose.

Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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