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New Dynamics in the Anti-de Sitter Universe AdS^5

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Abstract: This paper deals with the propagation of the gravitational waves in the Poincaré patch of the 5-dimensional Anti-de Sitter universe. We construct a large family of unitary dynamics with respect to some high order energies that are conserved and positive. These dynamics are associated with asymptotic conditions on the conformal time-like boundary of the universe. This result does not contradict the statement of Breitenlohner-Freedman that the hamiltonian is essentially self-adjoint in L^2 and thus accordingly the dynamics is uniquely determined. The key point is the introduction of a new Hilbert functional framework that contains the massless graviton which is not normalizable in L^2. Then the hamiltonian is not essentially self-adjoint in this new space and possesses a lot of different positive self-adjoint extensions. These dynamics satisfy a holographic principle: there exists a renormalized boundary value which completely characterizes the whole field in the bulk.

1. Introduction

The 5-dimensional Anti-de Sitter space-time AdS^5 plays a fundamental role in string cosmology and has given rise to a lot of works (see e.g. [8,14]). An important geometrical framework is the Poincaré patch $\mathcal{P}$ of AdS^5, defined by

$$\mathcal{P} := \mathbb{R}_t \times \mathbb{R}^3_{\mathbf{x}} \times]0, \infty[_z, \quad g_{\mu\nu} dx^\mu dx^\nu = \frac{1}{z^2} \left(dt^2 - d\mathbf{x}^2 - dz^2 \right).$$

$\mathcal{P}$ is a lorentzian manifold and the crucial point is that it is not globally hyperbolic : the conformal boundary $\mathbb{R}_t \times \mathbb{R}_{\mathbf{x}} \times \{z = 0\}$ is time-like and the question arises to determine the possible boundary conditions on this horizon, satisfied by the gravitational waves propagating in the bulk $\mathcal{P}$. These fields obey the D'Alembert equation

$$\Box_g u = 0, \quad \Box_g u := |g|^{-\frac{1}{2}} \partial_\mu \left(|g|^{\frac{1}{2}} g^{\mu\nu} \partial_\nu u \right). \tag{1.1}$$

If we put $\Phi =: z^{-\frac{3}{2}}u$ Eq. (1.1) in $\mathcal{P}$ takes the very simple form of the free wave equation on the 1+4-dimensional half Minkowski space-time $\mathbb{R}_t \times \mathbb{R}^3_{\mathbf{x}} \times]0,\infty[_z$, perturbed by a singular cartesian potential $\frac{15}{4z^2}$:

$$\left(\partial_t^2 - \Delta_{\mathbf{x}} - \partial_z^2 + \frac{15}{4z^2}\right)\Phi = 0, \quad \text{in } \mathbb{R}_t \times \mathbb{R}^3_{\mathbf{x}} \times]0,\infty[_z. \tag{1.2}$$

Unlike the Randal-Sundrum $RS2$-model investigated in [3], for which this equation is considered only for $z > 1$, we have to take account of the singularity at $z = 0$, and we are mainly interested in the role of the conformal boundary. More precisely, in this work, we address three questions:

(i) Since $\mathcal{P}$ is not globally hyperbolic, the dynamics is not *a priori* well defined without some boundary condition imposed on the time-like horizon $\{z = 0\}$. The usual opinion is that such a supplement constraint is not necessary because the Breitenlohner-Freedman condition is satisfied for the gravitational waves ([6,10,19] and App. of [2]), and so the hamiltonian $-\Delta_{\mathbf{x}} - \partial_z^2 + \frac{15}{4z^2}$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3_{\mathbf{x}} \times]0,\infty[_z)$ in the Hilbert space $\mathcal{H}$ chosen to be $L^2(\mathbb{R}^3_{\mathbf{x}} \times]0,\infty[_z)$. As a consequence there exists a unique dynamics in the functional framework of the fields with finite energy ([1,4,10]):

$$\mathbb{E}(\Phi) := \int_{\mathbb{R}^3}\int_0^\infty \mid \nabla_{t,\mathbf{x},z}\Phi(t,\mathbf{x},z)\mid^2 + \frac{15}{4z^2} \mid \Phi(t,\mathbf{x},z)\mid^2 d\mathbf{x}dz < \infty. \tag{1.3}$$

In fact this constraint implies an implicit Dirichlet condition on the boundary of the universe,

$$\Phi(t,\mathbf{x},0) = 0, \tag{1.4}$$

and these gravitational waves are called *Friedrichs solutions*. Nevertheless this result of uniqueness is not the end of the story because it depends deeply on the choice of the Hilbert space $\mathcal{H}$ (or the choice of the energy $\mathbb{E}(\Phi)$). In this paper we show that we can perform a rich variety of different unitary dynamics for the gravitational waves by changing the choice of the conserved energy. We construct a Hilbert space $\mathcal{H}$ such that $-\Delta_{\mathbf{x}} - \partial_z^2 + \frac{15}{4z^2}$ is *not* essentially self-adjoint on $C_0^\infty(\mathbb{R}^3_{\mathbf{x}} \times]0,\infty[_z)$ and admits many self-adjoint extensions associated with different boundary conditions at $z = 0$ of asymptotic type.

(ii) Another belief is that this cosmological model with a time-like horizon is not physically realistic since the massless graviton $\Phi_G(t,\mathbf{x},z) := z^{-\frac{3}{2}}\phi(t,\mathbf{x})$, where $\partial_t^2\phi - \Delta_{\mathbf{x}}\phi = 0$, is not normalizable (in the sense of the L^2 norm). Otherwise, the interest of such non normalizable fields has been emphasized in [15]. In this paper we prove there exists an infinity of pairwise different unitary dynamics for which this graviton is normalizable (in the sense of the new Hilbert space that contains all the fields with the same singularity). Moreover these dynamics are not trivial, *i.e* any field localized far from $z = 0$ at time $t = 0$, interacts with the massless graviton: when the field hits the boundary $z = 0$, a part of the scattered field is given by the graviton. Furthermore, many of these dynamics are stable in the sense that there is no growing mode and the conserved energy is positive.

(iii) We know that in the context of the Gauge Theory/String Theory dualities, the radial coordinate $r = \frac{1}{z}$ in AdS^5 is identified with the energy scale in the dual theory gauge which is localized at the conformal boundary $r = \infty$ (see *e.g.* [5]). The ADS/CFT conjecture deals with some one-to-one correspondence between the theory in the bulk and the dynamics on the boundary (a form of the *"Holographic Principle"*). The framework of our paper is obviously much more elementary: there is neither string nor quantum field, just a scalar field defined on the Poincaré patch, a solution of the linear hyperbolic equation (1.2). Then the issue of the Holographic Principle takes simply the following form: can we define a boundary value ϕ_2 of Φ on the conformal boundary $z = 0$, such that the operator $\Gamma : \Phi \mapsto \phi_2$ is one to one, *i.e.* ϕ_2 completely characterizes Φ ? This question would have a trivial positive answer if the partial differential equation was elliptic, but since (1.2) is hyperbolic, this question is very unusual and a positive answer is rather unexpected. For instance, it fails in the case of the Friedrichs solutions since in this case the dynamics preserves the Dirichlet condition (1.4) and so $\Gamma = 0$. An interesting feature of our new dynamics is that we can perform a renormalization on the boundary by putting $\Gamma(\Phi) := \lim_{z\to 0} z^{\frac{3}{2}}\Phi$, and this operator is one-to one. In this sense, we have constructed dynamics that satisfy the Holographic Principle.

The main result of this paper is Theorem 4.1 which provides answers to these issues. Now we describe the very simple idea of the construction of these new dynamics. We can see that Φ is a solution of (1.2) iff $\Psi(t, \mathbf{x}, Z) :=\mid Z\mid^{-\frac{5}{2}} \Phi(t, \mathbf{x}, \mid Z\mid)$ is a solution of

$$\left(\partial_t^2 - \Delta_{\mathbf{x}} - \Delta_Z\right)\Psi = 0, \quad \text{in } \mathbb{R}_t \times \mathbb{R}^3_{\mathbf{x}} \times \left(\mathbb{R}^6_Z \setminus \{Z = 0\}\right), \tag{1.5}$$

and we have proved in [4] that Φ satisfies (1.3) iff Ψ is a solution of the free wave equation in the whole Minkowski space-time $\mathbb{R}_t \times \mathbb{R}^9_{\mathbf{x},Z}$. As a consequence, to obtain new dynamics for (1.2), it is sufficient to construct solutions of (1.5) that are not free waves in $\mathbb{R}_t \times \mathbb{R}^9_{\mathbf{x},Z}$. Therefore we look for some self-adjoint extensions of the Laplace operator $\Delta_{\mathbf{x}} + \Delta_Z$ defined on $C_0^\infty\left(\mathbb{R}^3_{\mathbf{x}} \times \left(\mathbb{R}^6_Z \setminus \{Z = 0\}\right)\right)$. Since this operator is essentially self-adjoint in $L^2(\mathbb{R}^9)$, we must consider another Hilbert space and try to give sense to a perturbation localized on $\mathbb{R}^3_{\mathbf{x}} \times \{Z = 0\}$. It turns out that there has been recent progress on this question, in particular P. Kurasov in 2009 has studied the super-singular perturbations of the Laplacian [11]. Taking advantage of these novel advances in spectral analysis, we construct some new dynamics for (1.2) by considering the formal equation

$$\left(\partial_t^2 - \Delta_{\mathbf{x}} - \Delta_Z + c\delta_0(Z)\right)\Psi = 0, \quad \text{in } \mathbb{R}_t \times \mathbb{R}^3_{\mathbf{x}} \times \mathbb{R}^6_Z. \tag{1.6}$$

If Φ is the sum of a field Φ_0 satisfying (1.3), and of a graviton-like singular field $z^{-\frac{3}{2}}\phi(t, \mathbf{x})$, then $\Psi(t, \mathbf{x}, Z) =|Z|^{-\frac{5}{2}} \Phi_0(t, \mathbf{x}, |Z|) + \phi(t, \mathbf{x})\ |Z|^{-4}$ and the meaning of the super singular perturbation $c\delta_0(Z)$ is

$$c\delta_0(Z)\Psi := -4\pi^3\phi(t, \mathbf{x})\delta_0(Z).$$

A partial Fourier transform with respect to $\mathbf{x}$ allows to reduce the study of (1.6) to the investigation of the super-singular perturbations of the Klein-Gordon equation

$$\left(\partial_t^2 - \Delta_Z + m^2 + c\delta_0(Z)\right)u = 0, \quad \text{in } \mathbb{R}_t \times \mathbb{R}^6_Z,$$

that we perform in the next section.

Finally we summarize our main result stated in Theorem 4.1. We look for the gravitational waves solutions of (1.2) that have an expansion of the following form:

$$\Phi(t,\mathbf{x},z)=\Phi_r(t,\mathbf{x},z)z^{\frac{5}{2}}+\phi_{-1}(t,\mathbf{x})\chi(z)z^{\frac{5}{2}}+\phi_0(t,\mathbf{x})\chi(z)z^{\frac{5}{2}}\log z$$
$$+\phi_1(t,\mathbf{x})\chi(z)z^{\frac{1}{2}}+\phi_2(t,\mathbf{x})z^{-\frac{3}{2}}, \tag{1.7}$$

where $\chi\in C_0^\infty(\mathbb{R})$, $\chi(z)=1$ in a neighborhood of 0 and $\Phi_r(t,\mathbf{x},0)=0$. The term $\phi_2(t,\mathbf{x})z^{-\frac{3}{2}}$ is the part of the wave in the sector of the massless graviton. The behaviour of the field on the boundary of the universe is assumed to be for some $(\alpha_0,\alpha_1,\alpha_2)\in\mathbb{R}^3$:

$$\phi_{-1}(t,\mathbf{x})+\alpha_0\phi_0(t,\mathbf{x})+\alpha_1\phi_1(t,\mathbf{x})+\alpha_2\phi_2(t,\mathbf{x})=0,\quad t\in\mathbb{R},\quad \mathbf{x}\in\mathbb{R}^3. \tag{1.8}$$

For a large family of α_j, we are able to construct a Hilbert functional framework for which the Cauchy problem associated with (1.2) is well-posed. At each time, the boundary constraint (1.8) is satisfied and the graviton part ϕ_2 is non zero even if the initial data are compactly supported far from the boundary of the universe: hence these waves are not Friedrichs solutions. In fact we establish a kind of holographic principle: the operator Γ_α associating ϕ_2 in (1.7), to any solution Φ of (1.2) and (1.8),

$$\Gamma_\alpha:\quad \Phi\longmapsto\phi_2(t,\mathbf{x}):=\lim_{z\to 0}z^{\frac{3}{2}}\Phi(t,\mathbf{x},z),$$

is *one-to-one*. Therefore the renormalized field on the conformal boundary completely characterizes the whole field in the bulk. Graphically, ϕ_2 is the hologram of Φ.

Moreover there exists a conserved energy. This complicated energy involves the derivatives of third order of the fields. An interesting fact is that this energy is positive for a continuous set of α_j, more precisely when

$$\alpha_2=0,\quad 0<\alpha_1,\quad -\frac{1}{2}-\frac{3}{2}\log 2<\alpha_0+\frac{1}{2}\log\alpha_1<\frac{1}{4}-\frac{1}{2}\log 2-\gamma,$$

where γ is the Euler's constant. In this important case, the massless graviton $\Phi_G(t,\mathbf{x},z):=z^{-\frac{3}{2}}\phi(t,\mathbf{x})$ satisfies the constraint (1.8). To see that, we note that Φ_G has the form (1.7) if one sets $\Phi_r,\phi_{-1},\phi_0,\phi_1$ to zero, and $\phi_2=z^{\frac{3}{2}}\Phi_G$. Then (1.8) is trivially satisfied with $\alpha_2=0$. Moreover the energy of the massless graviton is just the usual energy

$$\mathbb{E}(\Phi_G)=c\int_{\mathbb{R}^3_{\mathbf{x}}}|\nabla_{t,\mathbf{x}}\phi(t,\mathbf{x})|^2d\mathbf{x}.$$

Furthermore, the positivity of the conserved energy assures that there is no growing mode: we can consider that these new possible dynamics of the gravitational fluctuations are stable.

As a final remark, we want to emphazise that this paper deals only with the exact Anti-de Sitter metric that allows to perform explicit computations involving special functions, useful to our spectral method. It would be very interesting to extend these results to the more realistic case of the asymptotically Anti-de Sitter spaces studied by A. Vasy in [19]. The situation should be much more complicated, and we can think that the delicate techniques developed in [19] are necessary to the investigation of these universes.

2. Super-singular Perturbation of The Wave Equation on $\mathbb{R}^{1+6}$

Since the theory of the super-singular perturbation plays a crucial role in our work, it is convenient to begin this section by a brief presentation of the elegant approach exposed by P. Kurasov in [11]. Let A be a non negative selfadjoint operator on a Hilbert space $H = \mathcal{H}_0$. We introduce the scale of the Hilbert spaces $\mathcal{H}_m, m \in \mathbb{Z}$. For $m \in \mathbb{N}, \mathcal{H}_m = Dom\left((A+1)^{\frac{m}{2}}\right)$ endowed with the norm $\|u\|_{\mathcal{H}_m} := \|(A+1)^{\frac{m}{2}}u\|_H$, and for negative m, $\mathcal{H}_m$ is the completion of H for the norm $\|u\|_{\mathcal{H}_m} := \|(A+1)^{\frac{m}{2}}u\|_H$. A super-singular perturbation is an element $\varphi \in (\mathcal{H}_n)' \simeq \mathcal{H}_{-n}$ for some $n \geq 3$, such that $\varphi \in \mathcal{H}_{-n} \setminus \mathcal{H}_{-n+1}$. The aim consists in providing a rigourous meaning to the expression $A(.) + \alpha\varphi(.)\varphi, \alpha \in \mathbb{R}$, as some self-adjoint operator A_θ on a suitable Hilbert space. We define the minimal operator A_{min} as the restriction of A to $Dom\,(A_{min}) := \{u \in \mathcal{H}_n;\ \varphi(u) = 0\}$. A_{min} is essentially self-adjoint on $\mathcal{H}_0$, and is symmetric on $\mathcal{H}_{n-2}$ with deficiency indices $(1, 1)$. We look for some self-adjoint extensions of this minimal operator. The construction of Kurasov is the following. We choose $n-1$ arbitrary pairwise different real numbers $\mu_j < 0, j = 0, ..., n-2$, and $n-2$ arbitrary positive numbers $\gamma_j > 0, j = 1, ..., n-2$. We introduce the Hilbert space $\mathbb{H}_0$ spanned by $\mathcal{H}_{n-2}$ and $\varphi_j := \left(A - \mu_j\right)^{-1}\varphi \in \mathcal{H}_{-n+2}, j = 1, ..., n-2$. Any $u \in \mathbb{H}_0$ is uniquely expressed by $u = U_r + \sum_{j=1}^{n-2} u_j\varphi_j$ with $U_r \in \mathcal{H}_{n-2}, u_j \in \mathbb{C}$, and $\mathbb{H}_0$ is a Hilbert space for the norm $\|u\|^2_{\mathbb{H}_0} := \|U_r\|^2_{\mathcal{H}_{n-2}} + \sum_{j=1}^{n-2}\gamma_j \mid u_j \mid^2$. The desired extensions A_θ are constructed as follows. We put

$$\Phi_0 := \left[\prod_{j=0}^{n-2}\left(A - \mu_j\right)^{-1}\right]\varphi,\ \ b_j := \prod_{i,\, i\neq j}(\mu_j - \mu_i)^{-1},\ \ j = 1, ..., n-2.$$

For any $\theta \in [0, \pi)$, the operator A_θ is defined by the domain

$$Dom\,(A_\theta) := \left\{u = U_r + u_0\Phi_0 + \sum_{j=1}^{n-2} u_j\varphi_j,\ U_r \in \mathcal{H}_n,\ u_j \in \mathbb{C},\ \mathcal{B}_\theta(u) = 0\right\},$$

associated with the "boundary condition"

$$\mathcal{B}_\theta(u) := \varphi(U_r)\sin\theta + u_0\cos\theta - \sin\theta\sum_{j=1}^{n-2} b_j\gamma_j u_j,$$

and its action is given by

$$A_\theta u := A(U_r) + \mu_0 u_0\Phi_0 + \sum_{j=1}^{n-2}(\mu_j u_j + b_j u_0)\varphi_j.$$

Then A_θ is a self-adjoint operator on $\mathbb{H}_0$. We shall use these techniques in the case $H = L^2(\mathbb{R}^6)$, $A = -\Delta, \varphi = \delta_0, n = 4$, in order to give a sense to $-\Delta u + L(u)\delta_0$ for the linear form L on $\mathbb{H}_0$ defined by $L(u) = -u_1 - u_2$. We now return to our problem.

We want to investigate the wave equation on the Minkowski space-time $\mathbb{R}_t \times \mathbb{R}^6_Z$ with a supersingular perturbation localized at $Z = 0$. More precisely, given $m \geq 0$, we shall consider the abstract Klein-Gordon equation

$$\partial_t^2 u + \mathbb{A}u + m^2 u = 0, \tag{2.1}$$

where $\mathbb{A}$ is a densely defined selfadjoint operator on a Hilbert space $\mathbb{H}_0$ of distributions on $\mathbb{R}^6$, such that

$$C_0^\infty\left(\mathbb{R}^6\setminus\{0\}\right)\subset Dom(\mathbb{A}),\ \ \forall\varphi\in C_0^\infty\left(\mathbb{R}^6\setminus\{0\}\right),\ \ \mathbb{A}\varphi=-\Delta\varphi.$$

In fact, we choose a very simple point-like interaction at the origin, so for all $u\in Dom(\mathbb{A})$, $\mathbb{A}u$ has the form

$$\mathbb{A}u=-\Delta u+L(u)\delta_0, \tag{2.2}$$

where L is a continuous linear form on the space $\mathbb{H}_0$ defined below by (2.3), equal to zero on $C_0^\infty\left(\mathbb{R}^6\setminus\{0\}\right)$. This constraint yields a character very singular to the perturbation and the Cauchy problem cannot be solved as usual in a scale of Sobolev spaces: if $u\in\cap_{k=0}^2C^k\left(\mathbb{R}_t;\,H^{s-k}(\mathbb{R}^6)\right)$ is solution of (2.1) and (2.2) with $L(u)\neq 0$, then $s<-1$ since $\delta_0\in H^\sigma(\mathbb{R}^6)$ iff $\sigma<-3$. Hence a contradiction appears since $C_0^\infty(\mathbb{R}^6\setminus\{0\})$ is dense in $H^s(\mathbb{R}^6)$, $s\leq 3$, and as a consequence $L(u)=0$. Therefore we have to introduce some functional spaces, in which $C_0^\infty(\mathbb{R}^6\setminus\{0\})$ is not dense. We want also to recover the static solutions $u_{stat}(t,Z)=|Z|^{-4}$ for $m=0$, and $u_{stat}(t,Z)=\frac{m^2K_2(m|Z|)}{2|Z|^2}$ when $m>0$, where K_2 is the classical modified Bessel function (see below), that are solutions of (2.1) and (2.2) with $L(u_{stat})=-4\pi^3$. On the other hand we know (see Lemma 2.2) that

$$\frac{m^2K_2(m\,|Z|)}{2\,|Z|^2}=\frac{1}{|Z|^4}-\frac{m^2}{4\,|Z|^2}-\frac{m^4}{16}\log|Z|\,+O(1),\quad Z\to 0.$$

All these properties suggest to consider Hilbert spaces of distributions, spanned by $|Z|^{-4}$, $|Z|^{-2}$, $\log|Z|$ and some usual Sobolev spaces. In fact it is just the spaces of the abstract setting of [11]. More precisely we take $\chi\in C_0^\infty(\mathbb{R}^6_Z)$ satisfying for some $\rho>0$, $\chi(Z)=1$ when $|Z|\leq\rho$. We introduce the spaces

$$\mathbb{H}_k:=\left\{u=v_r+v_1\frac{\chi(Z)}{|Z|^2}+v_2\frac{\chi(Z)}{|Z|^4},\ v_r\in H^{k+2}(\mathbb{R}^6_Z),\ \ v_j\in\mathbb{C}\right\},\quad k=-1,0, \tag{2.3}$$

$$\mathbb{H}_k:=\left\{u=V_r+v_0\chi(Z)\log(|Z|)+v_1\frac{\chi(Z)}{|Z|^2}+v_2\frac{\chi(Z)}{|Z|^4},\ V_r\in H^{k+2}(\mathbb{R}^6_Z),\ v_j\in\mathbb{C}\right\},\quad k=1,2, \tag{2.4}$$

where $H^m(\mathbb{R}^6)$ are the usual Sobolev spaces of functions $v\in L^2$ such that $(-\Delta+1)^{\frac{m}{2}}v\in L^2$. The link with the space $\mathbb{H}_0$ introduced in [11], as previously described, will be explained by Lemma 2.2. It is clear that these spaces do not depend on the choice of function χ, and given u, the coordinates v_j, $0\leq j\leq 2$, and $V_r(0)$ when $k=2$, are also independent of χ. We easily check that in the sense of the distributions on $\mathbb{R}^6_Z$ we have

$$\Delta_Z\log(|Z|)=\frac{4}{|Z|^2},\quad \Delta_Z\left(\frac{1}{|Z|^2}\right)=-\frac{4}{|Z|^4},\quad \Delta_Z\left(\frac{1}{|Z|^4}\right)=-4\pi^3\delta_0(Z). \tag{2.5}$$

Since for any $\epsilon>0$, $\delta_0\in H^{-3-\epsilon}(\mathbb{R}^6)\setminus H^{-3}(\mathbb{R}^6)$, we have

$$\frac{\chi(Z)}{|Z|^4}\in H^{-1-\epsilon}(\mathbb{R}^6)\setminus H^{-1}(\mathbb{R}^6),\quad \frac{\chi(Z)}{|Z|^2}\in H^{1-\epsilon}(\mathbb{R}^6)\setminus H^1(\mathbb{R}^6),$$
$$\chi(Z)\log(|Z|)\in H^{3-\epsilon}(\mathbb{R}^6)\setminus H^3(\mathbb{R}^6).$$

We deduce that $\mathbb{H}_2 \subset \mathbb{H}_1 \subset \mathbb{H}_0 \subset \mathbb{H}_{-1} \subset L^1_{loc}(\mathbb{R}^6)$. Now we take two real μ_1, μ_2, such that

$$\mu_j < 0, \quad \mu_1 \neq \mu_2, \tag{2.6}$$

and we choose on $H^2(\mathbb{R}^6)$ the norm given by:

$$\|v_r\|_{H^2} := \|(-\Delta - \mu_1)^{\frac{1}{2}}(-\Delta - \mu_2)^{\frac{1}{2}} v_r\|_{L^2} .$$

The other spaces H^m are endowed with the norm $\| v_r \|_{H^m} := \| (-\Delta + 1)^{\frac{m}{2}} v_r \|_{L^2}$. If we put

$$\|u\|_{\mathbb{H}_k} := \left(\|v_r\|^2_{H^{k+2}} + \sum_{j=1}^{2} |v_j|^2 \right)^{\frac{1}{2}}, \quad k = -1, 0, \tag{2.7}$$

$$\|u\|_{\mathbb{H}_k} := \left(\|V_r\|^2_{H^{k+2}} + \sum_{j=0}^{2} |v_j|^2 \right)^{\frac{1}{2}}, \quad k = 1, 2, \tag{2.8}$$

we can see that $\|.\|_{\mathbb{H}_j}$ is a norm on $\mathbb{H}_j$ and $(\mathbb{H}_j, \ \| \ . \ \|_{\mathbb{H}_j})$ is a Hilbert space, and $\mathbb{H}_i$ is dense in $\mathbb{H}_j$ for $j \leq i$. Since $H^{3+\epsilon}(\mathbb{R}^6) \subset C^0(\mathbb{R}^6)$, $V_r(0)$ is well defined for any $u \in \mathbb{H}_2$. Then given a linear form q on $\mathbb{C}^4$, we introduce the closed subspace of $\mathbb{H}_2$,

$$\mathbb{D}(q) := \{u \in \mathbb{H}_2; \quad q(V_r(0), v_0, v_1, v_2) = 0\} .$$

$C_0^\infty(\mathbb{R}^6 \setminus \{0\})$ is a subspace of $\mathbb{D}(q)$. We denote $\mathcal{D}'(\mathbb{R}_t; \mathbb{D}(q_\lambda))$ the space of the $\mathbb{D}(q_\lambda)$-valued vector distributions on $\mathbb{R}_t$. Finally we have to choose the linear form L on $\mathbb{H}_k$. Since we want that $\mathbb{A}u$ given by (2.2) belongs to $L^1_{loc}(\mathbb{R}^6)$, we note that (2.5) imposes to take:

$$L(u) = -4\pi^3 v_2. \tag{2.9}$$

We emphasize that $u \mapsto L(u)\delta_0$ is a local perturbation since when $u = 0$ in a neighborhood of 0, then $v_2 = 0$, and so $L(u)\delta_0 = 0$. The Cauchy problem is solved by the following theorem.

Theorem 2.1. *For all μ_1, μ_2 satisfying (2.6), there exists a continuous family $(q_\lambda)_{\lambda \in \mathbb{R}^3}$ of linear forms on $\mathbb{C}^4$ such that $\mathbb{D}(q_\lambda)$ is dense in $\mathbb{H}_1$, and for any $m \geq 0$, $f \in \mathbb{H}_1$, $g \in \mathbb{H}_0$, there exists a unique u_λ satisfying*

$$u_\lambda \in C^2(\mathbb{R}_t; \mathbb{H}_{-1}) \cap C^1(\mathbb{R}_t; \mathbb{H}_0) \cap C^0(\mathbb{R}_t; \mathbb{H}_1) \cap \mathcal{D}'(\mathbb{R}_t; \mathbb{D}(q_\lambda)), \tag{2.10}$$

$$\partial_t^2 u_\lambda - \Delta_Z u_\lambda + m^2 u_\lambda + L(u_\lambda)\delta_0 = 0, \tag{2.11}$$

$$u_\lambda(0, Z) = f(Z), \quad \partial_t u_\lambda(0, Z) = g(Z). \tag{2.12}$$

The solution depends continuously on the initial data: there exists $C, K > 0$, depending on λ but independent of m, such that

$$\begin{aligned} &\| u_\lambda(t) \|_{\mathbb{H}_1} + m \| u_\lambda(t) \|_{\mathbb{H}_0} + \| \partial_t u_\lambda(t) \|_{\mathbb{H}_0} \\ &\quad \leq C \left(\| f \|_{\mathbb{H}_1} + m \| f \|_{\mathbb{H}_0} + \| g \|_{\mathbb{H}_0} \right) e^{(K - m^2)_+ |t|}, \end{aligned} \tag{2.13}$$

where $x_+ = x$ *when* $x > 0$ *and* $x_+ = 0$ *when* $x \leq 0$*, and for all* $\Theta \in C_0^\infty(\mathbb{R}_t)$ *we have:*

$$\| \int \Theta(t) u_\lambda(t) dt \|_{\mathbb{H}_2} \leq C \left(\|f\|_{\mathbb{H}_1} + m\|f\|_{\mathbb{H}_0} + \|g\|_{\mathbb{H}_0} \right) \int \left(|\Theta(t)| + |\Theta''(t)| \right) e^{(K-m^2)_+|t|} dt. \quad (2.14)$$

There exists a conserved energy, i.e. a non-trivial, continuous quadratic form $\mathcal{E}_\lambda$ *defined on* $\mathbb{H}_1 \oplus \mathbb{H}_0$*, that satisfies:*

$$\forall t \in \mathbb{R}, \quad \mathcal{E}_\lambda \left(u_\lambda(t), \partial_t u_\lambda(t) \right) = \mathcal{E}_\lambda(f, g). \quad (2.15)$$

This energy is not positive definite but $\mathcal{E}_\lambda$ *is given on* $C_0^\infty(\mathbb{R}^6 \setminus \{0\}) \oplus C_0^\infty(\mathbb{R}^6 \setminus \{0\})$ *by:*

$$\mathcal{E}_\lambda(f, g) = \|\nabla f\|^2_{H^2} + m^2 \, \|f\|^2_{H^2} + \|g\|^2_{H^2} \, . \quad (2.16)$$

When $f \in \mathbb{D}(q_\lambda)$, $g \in \mathbb{H}_1$*, then* u_λ *is a strong solution in the sense that:*

$$u_\lambda \in C^2 \left(\mathbb{R}_t; \mathbb{H}_0 \right) \cap C^1 \left(\mathbb{R}_t; \mathbb{H}_1 \right) \cap C^0 \left(\mathbb{R}_t; \mathbb{D}(q_\lambda) \right), \quad (2.17)$$

and there exists $C, K > 0$*, depending on* λ *but independent of* m*, such that*

$$\|u_\lambda(t)\|_{\mathbb{H}_2} + m \, \|u_\lambda(t)\|_{\mathbb{H}_1} + \|\partial_t u_\lambda(t)\|_{\mathbb{H}_1} \leq C \left(\|f\|_{\mathbb{H}_2} + m \, \|f\|_{\mathbb{H}_1} + \|g\|_{\mathbb{H}_1} \right) e^{(K-m^2)_+|t|}. \quad (2.18)$$

We prove in the next theorem that these linear forms q_λ are pairwise different. This large family is described below by (2.55) and Theorem 2.5. The terrific expression of the high order energy is given by (2.45) and (2.46). The strategy of the proof consists in introducing a suitable hermitian product $<,>_0$ on $\mathbb{H}_0$ such that $\mathbb{A}$ endowed with $\mathbb{D}(q)$ as domain, is a densely defined self-adjoint operator. Then the energy is simply

$$\mathcal{E}_\lambda(f, g) = \|g\|_0^2 + \langle \mathbb{A} f, f \rangle_0 + m^2 \|f\|_0^2.$$

Proof of Theorem 2.1. It will be convenient to use an alternative definition of the spaces $\mathbb{H}_k$. We take a third real number μ_0 and we assume that

$$\mu_0 < 0, \quad \mu_0 \neq \mu_1, \quad \mu_0 \neq \mu_2. \quad (2.19)$$

We introduce the distributions

$$\Phi_0 := (-\Delta - \mu_0)^{-1} (-\Delta - \mu_1)^{-1} (-\Delta - \mu_2)^{-1} \delta_0 \in H^{3-\epsilon}(\mathbb{R}^6) \setminus H^3(\mathbb{R}^6),$$
$$\varphi_j := (-\Delta - \mu_j)^{-1} \delta_0 \in H^{-1-\epsilon}(\mathbb{R}^6) \setminus H^{-1}(\mathbb{R}^6).$$

By the elliptic regularity, all these functions belong to $C^\infty \left(\mathbb{R}^6 \setminus \{0\} \right)$, and an explicit calculation give the structure near $Z = 0$. Hence we can recover the Hilbert spaces introduced in [11]:

Lemma 2.2. *Φ_0 and φ_j belong to $L^1(\mathbb{R}^6)$ and can be written as*

$$\varphi_j(Z) = \frac{\chi(Z)}{4\pi^3}\left(\frac{1}{|Z|^4} + \frac{\mu_j}{4\,|Z|^2} - \frac{\mu_j^2}{16}\log(|Z|)\right) + F_j(Z), \tag{2.20}$$

$$\Phi_0(Z) = \frac{1}{32\pi^3}\chi(Z)\log(|Z|) + G_0(Z), \tag{2.21}$$

where F_j and G_0 are functions of $H^4(\mathbb{R}^6)$, satisfying

$$F_j(0) = \frac{\mu_j^2}{256\pi^3}(4\log 2 + 3 - 4\gamma - 2\log|\mu_j|), \tag{2.22}$$

$$\begin{aligned} G_0(0) = &-\frac{4\log 2 + 3 - 4\gamma}{128\pi^3} \\ &-\frac{\mu_1^2(\mu_2-\mu_0)\log|\mu_1| + \mu_2^2(\mu_0-\mu_1)\log|\mu_2| + \mu_0^2(\mu_1-\mu_2)\log|\mu_0|}{64\pi^3(\mu_0-\mu_1)(\mu_1-\mu_2)(\mu_2-\mu_0)}, \end{aligned} \tag{2.23}$$

where γ is the Euler's constant.

As a consequence, we have the following characterization of spaces $\mathbb{H}_k$:

$$\mathbb{H}_k = \left\{u = u_r + u_1\varphi_1(Z) + u_2\varphi_2(Z),\ u_r \in H^{k+2}(\mathbb{R}^6_Z),\ \ u_j \in \mathbb{C}\right\},\quad k = -1, 0, \tag{2.24}$$

$$\mathbb{H}_k = \left\{u = U_r + u_0\Phi_0(Z) + u_1\varphi_1(Z) + u_2\varphi_2(Z),\ U_r \in H^{k+2}(\mathbb{R}^6_Z),\ \ u_j \in \mathbb{C}\right\},\quad k = 1, 2, \tag{2.25}$$

where the coordinates u_0, u_1, u_2 do not depend on the choice of μ_0, and the norms

$$|u|_k := \left(\|u_r\|^2_{H^{k+2}} + \sum_{j=1}^{2}|u_j|^2\right)^{\frac{1}{2}},\quad k = -1, 0, \tag{2.26}$$

$$|u|_k := \left(\|U_r\|^2_{H^{k+2}} + \sum_{j=0}^{2}|u_j|^2\right)^{\frac{1}{2}},\quad k = 1, 2, \tag{2.27}$$

are equivalent to the $\|.\|_{\mathbb{H}_k}$-norms (2.7), (2.8).

The proof of lemma is based on explicit calculations of integrals involving special functions, that are detailled in the Appendix.

Now we take two real numbers $\gamma_j > 0$ and for $u \in \mathbb{H}_0$ we put

$$\|u\|_0 := \left(\|u_r\|^2_{H^2} + \sum_{j=1}^{2}\gamma_j\,|u_j|^2\right)^{\frac{1}{2}}, \tag{2.28}$$

that is clearly equivalent to the $\|.\|_{\mathbb{H}_0}$-norm. We choose $\theta \in [0, \pi[$ and we put

$$\lambda := (\lambda_0, \lambda_1, \lambda_2) = (\cot\theta, \log\gamma_1, \log\gamma_2) \in]-\infty, \infty] \times \mathbb{R} \times \mathbb{R}.$$

We introduce the operator $\mathbb{A}$ defined by:

$$\mathbb{A}u := -\Delta U_r + \mu_0 u_0 \Phi_0 + \left(\mu_1 u_1 + \frac{u_0}{\mu_1 - \mu_2}\right)\varphi_1 + \left(\mu_2 u_2 + \frac{u_0}{\mu_2 - \mu_1}\right)\varphi_2.$$

This operator is a continuous linear map from $\mathbb{H}_k$ to $\mathbb{H}_{k-2}$ for $k = 1, 2$. Now we define $\mathbb{A}_\lambda$ as its restriction to the domain $Dom(\mathbb{A}_\lambda)$ defined by

$$Dom(\mathbb{A}_\lambda) := \left\{ u \in \mathbb{H}_2 \ ; \quad U_r(0)\sin\theta + u_0\cos\theta - (\gamma_1 u_1 - \gamma_2 u_2)\frac{\sin\theta}{\mu_1 - \mu_2} = 0 \right\}. \tag{2.29}$$

We consider the Cauchy problem associated to (2.12) and

$$\partial_t^2 u_\lambda + \mathbb{A}_\lambda u_\lambda + m^2 u_\lambda = 0. \tag{2.30}$$

We show that this equation is just (2.11). We have $(-\Delta - \mu_0)\Phi_0 = (-\Delta - \mu_1)^{-1}\varphi_2 = \frac{\varphi_1 - \varphi_2}{\mu_1 - \mu_2}$. Hence we get

$$\mathbb{A}u = -\Delta u - (u_1 + u_2)\delta_0. \tag{2.31}$$

Since (5.6) implies

$$u_1 + u_2 = 4\pi^3 v_2 = -L(u), \tag{2.32}$$

Eqs. (2.30) and (2.11) are equivalent to

$$\partial_t^2 u - \Delta u + m^2 u + L(u)\delta_0 = 0. \tag{2.33}$$

The Cauchy problem for this equation has to be completed by the "boundary condition at $Z = 0$" specified by the domain of $\mathbb{A}_\lambda$:

$$U_r(0)\sin\theta + u_0\cos\theta - (\gamma_1 u_1 - \gamma_2 u_2)\frac{\sin\theta}{\mu_1 - \mu_2} = 0. \tag{2.34}$$

Thanks to (5.6), this constraint can be associated with a linear form $q_\lambda\ (V_r(0), v_0, v_1, v_2)$ defined on $\mathbb{C}^4$ and $\mathbb{D}(q_\lambda) = Dom(\mathbb{A}_\lambda)$. Therefore to prove the theorem, it is sufficient to investigate the Cauchy problem (2.12), (2.30).

The case $\theta = 0$ that corresponds to $u_0 = 0$ or $16v_0 + 4(\mu_1 + \mu_2)v_1 - \mu_1\mu_2 v_2 = 0$, is rather peculiar since $Dom(\mathbb{A}_\lambda)$ is not dense in $\mathbb{H}_1$. It corresponds simply to the operator defined on $H^4(\mathbb{R}^6) \oplus \mathbb{C}\varphi_1 \oplus \mathbb{C}\varphi_2$ by

$$\forall U_r \in H^4(\mathbb{R}^6), \quad \mathbb{A}_0 U_r := -\Delta U_r, \quad \mathbb{A}_0 \varphi_j = \mu_j \varphi_j, \quad j = 1, 2. \tag{2.35}$$

In this case the dynamics is uncoupled between the regular and singular parts of the field: given $f = F_r + f_1\varphi_1 + f_2\varphi_2$, $g = G_r + g_1\varphi_1 + g_2\varphi_2$, $F_r, G_r \in H^4(\mathbb{R}^6)$, $f_j, g_j \in \mathbb{C}$, the Cauchy problem is easily solved by

$$u_\lambda(t, Z) = U_r(t, Z) + f_1(t)\varphi_1(Z) + f_2(t)\varphi_2(Z),$$

where U_r is the solution of the free Klein-Gordon equation $\partial_t^2 U_r - \Delta U_r + m^2 U_r = 0$ with $U_r(0) = F_r$, $\partial_t U_r(0) = G_r$, and $f_j(t)$ is a solution of the harmonic oscillator $\ddot{f}_j + (m^2 + \mu_j) f_j = 0$, with $f_j(0) = f_j$, $\dot{f}_j(0) = g_j$.

In the sequel, we consider the case $\theta \neq 0$, i.e. $\lambda \in \mathbb{R}^3$ and the family of linear forms is given by

$$q_\lambda(V_r(0), v_0, v_1, v_2) := U_r(0) + \lambda_0 u_0 - \frac{e^{\lambda_1}}{\mu_1 - \mu_2} u_1 - \frac{e^{\lambda_2}}{\mu_2 - \mu_1} u_2. \tag{2.36}$$

The crucial point is the selfadjointness of the operator $\mathbb{A}_\lambda$ and some energy estimates.

Lemma 2.3. $(\mathbb{A}_\lambda, Dom(\mathbb{A}_\lambda))$ *is a densely defined selfadjoint operator on* $(\mathbb{H}_0, \| \,.\, \|_0)$ *and* $Dom(\mathbb{A}_\lambda)$ *is dense in* $\mathbb{H}_1$. *Moreover* $\mathbb{A}_\lambda$ *is bounded from below and there exists* $M_\lambda, \alpha > 0$, $c(\lambda) > 0$ *such that for all* $u \in Dom(\mathbb{A}_\lambda)$ *we have*

$$\langle \mathbb{A}_\lambda u, u\rangle_0 + M_\lambda \ \|u\|_0^2 \geq \alpha \ \|u\|^2_{\mathbb{H}_1}, \tag{2.37}$$

$$\|u\|_{\mathbb{H}_2} \leq c(\lambda) \|(\mathbb{A}_\lambda + M_\lambda)\, u\|_{\mathbb{H}_0} \leq \frac{1}{c(\lambda)} \|u\|_{\mathbb{H}_2}. \tag{2.38}$$

Proof of Lemma 2.3. Thanks to Lemma 2.2 our Hilbert spaces $\mathbb{H}_k$ coincide with the the spaces introduced by Kurasov and it was proved in [11] that $(\mathbb{A}_\lambda, Dom(\mathbb{A}_\lambda))$ is a selfadjoint operator on $(\mathbb{H}_0, \| \,.\, \|_0)$. First we prove that $Dom(\mathbb{A}_\lambda)$ is dense in $\mathbb{H}_1$. Given $u = U_r + u_0\Phi_0 + u_1\varphi_1 + u_2\varphi_2 \in \mathbb{H}_1$, we pick a sequence $\psi^n \in C_0^\infty(\mathbb{R}^6 \setminus \{0\})$ converging to U_r in $H^3(\mathbb{R}^6)$, and a sequence $\chi_n \in C_0^\infty(\mathbb{R}^6 \setminus \{0\})$ converging to χ in $H^3(\mathbb{R}^6)$. We put $U_r^n := \psi_n + \left(\frac{\gamma_1\mu_1 - \gamma_2\mu_2}{\mu_1-\mu_2} - u_0 \cot\theta\right)(\chi - \chi_n)$. Then $u^n := U_r^n + u_0\Phi_0 + u_1\varphi_1 + u_2\varphi_2$ belongs to $Dom(\mathbb{A}_\lambda)$ and tends to u in $\mathbb{H}_1$ as n tends to infinity.

Now we investigate the quadratic form associated with the operator $\mathbb{A}_\lambda$. We use the fact that $\langle(-\Delta - \mu_0)U_r, \Phi_0\rangle_{H^2} = U_r(0) = \frac{\gamma_1 u_1 - \gamma_2 u_2}{\mu_1 - \mu_2} - u_0 \cot\theta$ to evaluate:

$$\begin{aligned}
\langle \mathbb{A}_\lambda u, u\rangle_0 =& \mu_0 \ \|\Phi_0\|^2_{H^2} |u_0|^2 + \mu_1\gamma_1 \ |u_1|^2 + \mu_2\gamma_2 \ |u_2|^2 + \frac{\gamma_1}{\mu_1 - \mu_2} u_0\overline{u_1} - \frac{\gamma_2}{\mu_1-\mu_2} u_0 \overline{u_2} \\
&+ \|\nabla U_r\|^2_{H^2} + \langle(-\Delta - \mu_0)U_r, u_0\Phi_0\rangle_{H^2} + 2\mu_0 \Re \langle U_r, u_0\Phi_0\rangle_{H^2} \\
=& (-\mu_0 \ \|\Phi_0\|^2_{H^2} - \cot\theta) \ |u_0|^2 + \mu_1\gamma_1 \ |u_1|^2 + \mu_2\gamma_2 \ |u_2|^2 \\
&+ 2\Re\left(\overline{u_0} \frac{\gamma_1 u_1 - \gamma_2 u_2}{\mu_1 - \mu_2}\right) + \|\nabla U_r\|^2_{H^2} - 2\mu_0 \ \|U_r\|^2_{H^2} + 2\mu_0 \ \|U_r + u_0\Phi_0\|^2_{H^2}.
\end{aligned} \tag{2.39}$$

We see that $u \mapsto \langle \mathbb{A}_\lambda u, u\rangle_0$ is a continuous sesquilinear form on $Dom(\mathbb{A}_\lambda)$ endowed with the $\mathbb{H}_1$-norm. Moreover for any $M \geq 0$ we have

$$\begin{aligned}
&\langle \mathbb{A}_\lambda u, u\rangle_0 + M \ \|u\|_0^2 \\
&\quad \geq (-\mu_0 \ \|\Phi_0\|^2_{H^2} - \cot\theta - 1 + M) \ |u_0|^2 \\
&\quad + \gamma_1 \left(\mu_1 - \frac{\gamma_1}{(\mu_1 - \mu_2)^2} + M\right) |u_1|^2 + \gamma_2 \left(\mu_2 - \frac{\gamma_2}{(\mu_1 - \mu_2)^2} + M\right) |u_2|^2 \\
&\quad + \|\nabla U_r\|^2_{H^2} - 2\mu_0 \ \|U_r\|^2_{H^2} + (2\mu_0 + M) \ \|U_r + u_0\Phi_0\|^2_{H^2}\,.
\end{aligned} \tag{2.40}$$

We deduce that for $M = M_\lambda$ large enough, there exists $\alpha > 0$ such that for all $u \in Dom(\mathbb{A}_\lambda)$, Equation (2.37) holds. We conclude that $\mathbb{A}_\lambda$ is bounded from below, $\|(\mathbb{A}_\lambda + M_\lambda)^{\frac{1}{2}}\, u\|_0$ is a norm equivalent to the $\mathbb{H}_1$ norm, the domain of the sesquilinear form is just $\mathbb{H}_1$, and $(\mathbb{A}_\lambda + M_\lambda)^{-\frac{1}{2}}$ is a continuous linear map from $\mathbb{H}_0$ to $\mathbb{H}_1$. Equation (2.37) implies also that

$$\alpha \ \|u\|_{\mathbb{H}_1} \leq \ \|(\mathbb{A}_\lambda + M_\lambda)\, u\|_{\mathbb{H}_0},$$

hence

$$\sum_{j=0}^{2} |u_j| \leq \ \kappa \|(\mathbb{A}_\lambda + M_\lambda)\, u\|_{\mathbb{H}_0}.$$

We have also

$$\|U_r\|_{H^4} \leq C\|(-\Delta + M_\lambda)U_r\|_{H^2} \leq C\left(\|(\mathbb{A}_\lambda + M_\lambda)\, u\|_{\mathbb{H}_0} + |u_0| \ \|\Phi_0\|_{H^2}\right).$$

Therefore we conclude that there exists $c(\lambda) > 0$ such that for all $u \in Dom(\mathbb{A}_\lambda)$ we have (2.38). □

Then it is well-known that for $f \in Dom(\mathbb{A}_\lambda)$, $g \in \mathbb{H}_1$, the Cauchy problem (2.12), (2.30) has a unique solution $u_\lambda \in C^2(\mathbb{R}_t; \mathbb{H}_0) \cap C^1(\mathbb{R}_t; \mathbb{H}_1) \cap C^0(\mathbb{R}_t; Dom(\mathbb{A}_\lambda))$ and this solution depends continuously on the initial data (see e.g. Theorem 7.8, p. 114 in [9]). Nevertheless, since we need to carefully control the constants with respect to the mass m, we present some details. If $m^2 \geq M_\lambda$, we have simply $u_\lambda(t) = \cos\left(t\sqrt{\mathbb{A}_\lambda + m^2}\right) f + \frac{\sin\left(t\sqrt{\mathbb{A}_\lambda + m^2}\right)}{\sqrt{\mathbb{A}_\lambda + m^2}} g$, hence (2.37) and (2.38) imply:

$$\|\partial_t u_\lambda(t)\|_{\mathbb{H}_k} + \|u_\lambda(t)\|_{\mathbb{H}_{k+1}} \leq C\left(\|f\|_{\mathbb{H}_{k+1}} + \|g\|_{\mathbb{H}_k}\right), \quad k = 0, 1. \tag{2.41}$$

When $m^2 < M_\lambda$, we can construct u_λ by solving the following integral equation thanks to Picard's iterates:

$$u_\lambda(t) = \cos\left(t\sqrt{\mathbb{A}_\lambda + M_\lambda}\right) f + \frac{\sin\left(t\sqrt{\mathbb{A}_\lambda + M_\lambda}\right)}{\sqrt{\mathbb{A}_\lambda + M_\lambda}} g$$
$$+(M_\lambda - m^2)\int_0^t \frac{\sin\left((t-s)\sqrt{\mathbb{A}_\lambda + M_\lambda}\right)}{\sqrt{\mathbb{A}_\lambda + M_\lambda}} u_\lambda(s)ds.$$

The Gronwall lemma gives

$$\|u_\lambda(t)\|_{\mathbb{H}_1} + \|\partial_t u_\lambda(t)\|_{\mathbb{H}_0} \leq C(\lambda)\left(\|f\|_{\mathbb{H}_1} + \|g\|_{\mathbb{H}_0}\right) e^{|t|(M_\lambda - m^2)}, \tag{2.42}$$

and by applying $\mathbb{A}_\lambda + M_\lambda$ to the integral equation, using (2.38) and the Gronwall lemma again, we get

$$\|u_\lambda(t)\|_{\mathbb{H}_2} + \|\partial_t u_\lambda(t)\|_{\mathbb{H}_1} \leq C(\lambda)\left(\|f\|_{\mathbb{H}_1} + \|g\|_{\mathbb{H}_0}\right) e^{|t|(M_\lambda - m^2)}. \tag{2.43}$$

Now we have to control $m\|u_\lambda(t)\|_{\mathbb{H}_k}$, $k = 0, 1$. We start by noting that the following energy is conserved:

$$\|\partial_t u_\lambda(t)\|_0^2 + \langle \mathbb{A}_\lambda u_\lambda(t), u_\lambda(t)\rangle_0 + m^2 \ \|u_\lambda(t)\|_0^2$$
$$= \|g\|_0^2 + \langle \mathbb{A}_\lambda f, f\rangle_0 + m^2 \ \|f\|_0^2 := \mathcal{E}_\lambda(f, g), \tag{2.44}$$

hence (2.37) and (2.42) imply (2.13) with $K := M_\lambda + 1$ when $m^2 \geq K$. Furthermore, we get its expression with (2.39): given $f = F_r + f_0\Phi_0 + f_1\varphi_1 + f_2\varphi_2 \in \mathbb{H}_1$, $g = g_r + g_1\varphi_1 + g_2\varphi_2 \in \mathbb{H}_0$, $F_r \in H^3(\mathbb{R}^6)$, $g_r \in H^2(\mathbb{R}^6)$, $f_j, g_j \in \mathbb{C}$, we have:

$$\begin{aligned}
\mathcal{E}_\lambda(f,g) &= \sum_{j=1}^{2} e^{\lambda_j}\left[(\mu_j + m^2)\mid f_j\mid^2 + \mid g_j\mid^2\right] + \left(-\mu_0 \parallel \Phi_0\parallel_{H^2}^2 - \lambda_0\right)\mid f_0\mid^2 \\
&\quad + 2\Re\left(\overline{f_0}\frac{e^{\lambda_1} f_1 - e^{\lambda_2} f_2}{\mu_1 - \mu_2}\right) + \|(-\Delta - \mu_1)^{\frac{1}{2}}(-\Delta - \mu_2)^{\frac{1}{2}} g_r\|_{L^2}^2 \\
&\quad + (m^2 + 2\mu_0)\ \|(-\Delta - \mu_1)^{\frac{1}{2}}(-\Delta - \mu_2)^{\frac{1}{2}}(F_r + f_0\Phi_0)\|_{L^2}^2 \\
&\quad + \|\nabla(-\Delta - \mu_1)^{\frac{1}{2}}(-\Delta - \mu_2)^{\frac{1}{2}} F_r\|_{L^2}^2 - 2\mu_0\ \|(-\Delta - \mu_1)^{\frac{1}{2}}(-\Delta - \mu_2)^{\frac{1}{2}} F_r\|_{L^2}^2 \\
&= \sum_{j=1}^{2} e^{\lambda_j}\left[(\mu_j + m^2)\ |f_j|^2 + \mid g_j\mid^2\right] + \left((m^2 + \mu_0)\parallel \Phi_0\parallel_{H^2}^2 - \lambda_0\right)\mid f_0\mid^2 \\
&\quad + 2\Re\left(\overline{f_0}\frac{e^{\lambda_1} f_1 - e^{\lambda_2} f_2}{\mu_1 - \mu_2}\right) + \parallel(-\Delta - \mu_1)^{\frac{1}{2}}(-\Delta - \mu_2)^{\frac{1}{2}} g_r\|_{L^2}^2 \\
&\quad + 2(m^2 + 2\mu_0)\Re\left(\overline{f_0}(-\Delta - \mu_0)^{-1} F_r(0)\right) \\
&\quad + \parallel\nabla(-\Delta - \mu_1)^{\frac{1}{2}}(-\Delta - \mu_2)^{\frac{1}{2}} F_r\|_{L^2}^2 + m^2\ \|(-\Delta - \mu_1)^{\frac{1}{2}}(-\Delta - \mu_2)^{\frac{1}{2}} F_r\|_{L^2}^2,
\end{aligned} \tag{2.45}$$

where we can compute

$$\begin{aligned}
\|\Phi_0\|_{H^2}^2 &= \frac{1}{8}\int_0^\infty \frac{\rho^5}{(\rho^2 - \mu_1)(\rho^2 - \mu_2)(\rho^2 - \mu_0)^2} d\rho \\
&= \frac{1}{16}\left(\frac{\mu_1^2\log(-\mu_1)}{(\mu_2 - \mu_1)(\mu_1 - \mu_0)^2} + \frac{\mu_2^2\log(-\mu_2)}{(\mu_1 - \mu_2)(\mu_2 - \mu_0)^2}\right. \\
&\qquad \left. + \frac{(\mu_1\mu_0^2 + \mu_2\mu_0^2 - 2\mu_0\mu_1\mu_2)\log(-\mu_0)}{(\mu_1 - \mu_0)^2(\mu_2 - \mu_0)^2} - \frac{\mu_0}{(\mu_1 - \mu_0)(\mu_2 - \mu_0)}\right).
\end{aligned} \tag{2.46}$$

When $f_0 = f_1 = f_2 = g_1 = g_2 = 0$, in particular when $f, g \in C_0^\infty(\mathbb{R}^6 \setminus \{0\})$, then $\mathcal{E}_\lambda(f,g)$ is given by (2.16). To prove (2.18) when $m^2 \geq M_\lambda + 1$, we consider for $h \neq 0$, $v_h(t) := h^{-1}[u_\lambda(t+h) - u_\lambda(t)]$ that tends to $\partial_t u_\lambda(t)$ in $C^0(\mathbb{R}_t; \mathbb{H}_1) \cap C^1(\mathbb{R}_t; \mathbb{H}_0)$ as $h \to 0$. We apply estimate (2.44) to v_h and we get

$$\begin{aligned}
&\|\partial_t v_h(t)\|_0^2 + \left\|(\mathbb{A}_\lambda + M_\lambda)^{\frac{1}{2}}\ v_h\right\|_0^2 + (m^2 - M_\lambda)\|v_h(t)\|_0^2 \\
&\quad = \left\|\frac{\partial_t u_\lambda(h) - g}{h}\right\|_0^2 + \left\|(\mathbb{A}_\lambda + M_\lambda)^{\frac{1}{2}}\left(\frac{u_\lambda(h) - f}{h}\right)\right\|_0^2 + (m^2 - M)\left\|\frac{u_\lambda(h) - f}{h}\right\|_0^2,
\end{aligned}$$

and taking the limit as h tends to zero we obtain

$$\begin{aligned}
&\left\|(\mathbb{A}_\lambda + M_\lambda)\ u_\lambda(t) + (m^2 - M_\lambda) u_\lambda(t)\right\|_0^2 + \left\|(\mathbb{A}_\lambda + M_\lambda)^{\frac{1}{2}}\ \partial_t u_\lambda(t)\right\|_0^2 + (m^2 - M_\lambda)\|\partial_t u_\lambda(t)(t)\|_0^2 \\
&\quad = \left\|\mathbb{A}_\lambda f + m^2 f\right\|_0^2 + \left\|(\mathbb{A}_\lambda + M_\lambda)^{\frac{1}{2}}\ g\right\|_0^2 + (m^2 - M)\ \|g\|_0^2.
\end{aligned}$$

We deduce from this equality and with (2.13) and (2.38), that (2.18) is satisfied with $K = M_\lambda + 1$ when $m^2 \geq M_\lambda + 1$. It remains to study the case $0 \leq m^2 \leq M_\lambda + 1$. We simply use (2.42) and (2.43) to write

$$\sup_{m^2 \leq M_\lambda + 1} m\|u_\lambda(t)\|_{\mathbb{H}_k} \leq K\left(\|f\|_{\mathbb{H}_k} + \left|\int_0^t \|\partial_t u_\lambda(s)\|_{\mathbb{H}_k} ds\right|\right)$$
$$\leq C'(\lambda)\left(\|f\|_{\mathbb{H}_{k+1}} + \|g\|_{\mathbb{H}_k}\right) e^{|t|(K-m^2)}.$$

Now (2.13) and (2.18) are straight consequences of this estimate and (2.42) and (2.43).

To solve the Cauchy problem when $(f, g) \in \mathbb{H}_1 \oplus \mathbb{H}_0$, we pick a sequence $(f^n, g^n) \in Dom(\mathbb{A}_\lambda) \oplus \mathbb{H}_1$ that tends to (f, g) in $\mathbb{H}_1 \oplus \mathbb{H}_0$ as $n \to \infty$. Estimation (2.13) assures that the solution $u^n \in C^2(\mathbb{R}_t; \mathbb{H}_0) \cap C^1(\mathbb{R}_t; \mathbb{H}_1) \cap C^0(\mathbb{R}_t; Dom(\mathbb{A}_\lambda))$ of the Cauchy problem with initial data (f^n, g^n) tends to a function $u \in C^1(\mathbb{R}_t; \mathbb{H}_0) \cap C^0(\mathbb{R}_t; \mathbb{H}_1)$ that is a solution of (2.12), (2.30) and satisfies (2.44). Since $\mathbb{A}$ is continuous from $\mathbb{H}_1$ to $\mathbb{H}_{-1}$, the equation gives $u \in C^2(\mathbb{R}_t; \mathbb{H}_{-1})$. To prove that u is a distribution of $\mathcal{D}'(\mathbb{R}_t; Dom(\mathbb{A}_\lambda))$, we take $\Theta \in C_0^\infty(\mathbb{R}_t)$ and we consider $F := \int u(t)\Theta(t)dt \in \mathbb{H}_1$ and $F^n := \int u^n(t)\Theta(t)dt \in Dom(\mathbb{A}_\lambda)$. By the previous argument F^n tends to F in $\mathbb{H}_1$ as $n \to \infty$. Moreover $\mathbb{A}_\lambda F^n = -\int u^n(t)(\Theta''(t) + m^2\Theta(t))dt$ that converges to $-\int u(t)(\Theta''(t) + m^2\Theta(t))dt$ in $\mathbb{H}_1$. We conclude with (2.38) and (2.13) that $F \in Dom(\mathbb{A}_\lambda)$, i.e. u is a $Dom(\mathbb{A}_\lambda)$-valued distribution on $\mathbb{R}_t$ and (2.14) is established.

To prove the uniqueness, we consider a solution u satisfying (2.10), (2.30) and (2.12) with $f = g = 0$. We take a test function $\Theta \in C_0^\infty(\mathbb{R}_t)$, $0 \leq \Theta$, $\int \Theta(t)dt = 1$, and we define

$$u^n(t) := n\int \Theta(ns)u(t+s)ds. \tag{2.47}$$

u^n tends to u in $C^1(\mathbb{R}_t; \mathbb{H}_0) \cap C^0(\mathbb{R}_t; \mathbb{H}_1)$ as $n \to \infty$, hence we have $\|u^n(0)\|_{\mathbb{H}_1} \to 0$, $\|\partial_t u^n(0)\|_{\mathbb{H}_0} \to 0$. Moreover u^n is a strong solution satisfying (2.17) and (2.13). Therefore u^n tend to 0 in $C^1(\mathbb{R}_t; \mathbb{H}_0) \cap C^0(\mathbb{R}_t; \mathbb{H}_1)$, and finally $u = 0$. □

We now describe some important properties of these new dynamics.

Theorem 2.4. *The propagation is causal, i.e.*

$$supp(u_\lambda(t,.)) \subset \{Z;\ |Z| \leq |t|\} + [supp(f) \cup supp(g)]. \tag{2.48}$$

The dynamics is non-trivial: for all $f \in \mathbb{H}_1$, $g \in \mathbb{H}_0$, if f and g are spherically symmetric, then $L(u_\lambda(t)) \neq 0$ for some time t, except if $f = g = 0$.

If $\lambda \neq \lambda'$ the dynamics are different: given two spherically symmetric functions f, g in $C_0^\infty(\mathbb{R}^6 \setminus \{0\})$, $(f, g) \neq (0, 0)$, the solutions u_λ and $u_{\lambda'}$ of (2.10), (2.11), (2.12) are different.

For all $m \geq 0$, $m \neq \sqrt{-\mu_1}$, $m \neq \sqrt{-\mu_2}$, there exists a smooth surface $\Sigma(m) \subset \mathbb{R}^3$ such that for any $\lambda \in \Sigma(m)$, the static solutions $u(t, Z) = |Z|^{-4}$ for $m = 0$, and $u(t, Z) = |Z|^{-2} K_2(m|Z|)$ when $m > 0$, belong to $\mathbb{D}(q_\lambda)$. For all $m \geq 0$ and all $\lambda \in \Sigma(0)$, $u_\pm(t, Z) := \frac{e^{\pm imt}}{|Z|^4}$ is a time periodic strong solution of (2.11), (2.17).

We remark that the map

$$(f, g) \in \mathbb{H}_1 \times \mathbb{H}_0 \longmapsto v_2 = -\frac{1}{4\pi^3} L(u_\lambda) \in C^2(\mathbb{R}_t) \tag{2.49}$$

is linear and continuous. Its kernel contains the odd Cauchy data. The interesting point is that, if $R\mathbb{H}_k$ denotes the spherically symmetric elements of $\mathbb{H}_k$, this map is one-to-one on $R\mathbb{H}_1 \times R\mathbb{H}_0$. Then the leading term $v_2(t)\,|Z|^{-4}$ characterizes the spherically symmetric solutions u_λ. We shall see in the last section, that this property yields a holographic principle for the new dynamics in AdS^5.

Proof of Theorem 2.4. To prove that the propagation is causal, we write $u_\lambda = W+w$, where W is a solution of the free Klein-Gordon equation $(\partial_t^2 - \Delta + m^2)W = 0$ with $W(0) = u_\lambda(0)$, $\partial_t W(0) = \partial_t u_\lambda(0)$. Then $(\partial_t^2 - \Delta + m^2)w = -L(u_\lambda)\delta_0$ with $w(0) = \partial_t w(0) = 0$. We have $supp(W(t,.)) \subset \{Z;\ |Z| \le |t|\} + [supp(f) \cup supp(g)]$, $supp(w(t,.)) \subset \{Z;\ |Z| \le |t|\}$. When $0 \in supp(f) \cup supp(g)$, $supp(w(t,.)) \subset supp(W(t,.))$ and (2.48) is established. When $0 \notin supp(f) \cup supp(g)$, we consider firstly the case $(f,g) \in Dom(\mathbb{A}_\lambda) \oplus \mathbb{H}_1$. Then necessarily $u_0(0) = \dot{u}_0(0) = u_1(0) = \dot{u}_1(0) = u_2(0) = \dot{u}_2 = 0$, hence $(f,g) \in H^4(\mathbb{R}^6) \times H^3(\mathbb{R}^6)$. We denote $\tau > 0$ the distance between 0 and $supp(f) \cup supp(g)$. For $|t| \le \tau$, $W(t)$ satisfies trivially the boundary constraint $q_\lambda(W(t)) = 0$, hence $W(t) = u_\lambda(t)$. As a consequence $L(u(t)) = 0$ for $|t| \le \tau$, and for all t, $supp(w(t)) \subset \{Z;\ \ |Z| \le |t| - \tau\}$. Since $0 \in \{Z;\ \ |Z| \le |\tau|\} + [supp(f) \cup supp(g)]$, we conclude that (2.48) is satisfied again. When $(f,g) \in H^3(\mathbb{R}^6) \oplus H^2(\mathbb{R}^6)$ and $0 \notin supp(f) \cup supp(g)$, we choose a sequence $(f^n, g^n) \in H^4(\mathbb{R}^6) \times H^3(\mathbb{R}^6)$ that tends to (f,g) in $H^3(\mathbb{R}^6) \oplus H^2(\mathbb{R}^6)$, and $supp(f^n) \cup supp(g^n) \subset \{Z;\ \ |Z| \le \frac{1}{n}\} + [supp(f) \cup supp(g)]$. The previous result assures that $supp(u_\lambda^n(t,.)) \subset \{Z;\ \ |Z| \le |t| + \frac{1}{n}\} + [supp(f) \cup supp(g)]$, where u_λ^n is the strong solution with initial data (f^n, g^n). Now (2.48) follows from the convergence of u_λ^n to u_λ in $C^0(\mathbb{R}_t; \mathbb{H}_1)$ as $n \to \infty$.

To show that the dynamics is not trivial and involves a singular part in $|Z|^{-4}$ even for smooth initial data, we first consider a strong solution $u = U_r + u_0\Phi_0 + u_1\varphi_1 + u_2\varphi_2$ with Cauchy data $f \in \mathbb{H}_2$, $g \in \mathbb{H}_1$. Since $u \in C^2(\mathbb{R}_t; \mathbb{H}_0)$, we have $u_r := U_r + u_0\Phi_0 \in C^2(\mathbb{R}_t; H^2(\mathbb{R}^6))$, $u_1\ u_2 \in C^2(\mathbb{R})$. From $u \in C^1(\mathbb{R}_t; \mathbb{H}_1)$ we deduce that $u_0 \in C^1(\mathbb{R})$ and $U_r \in C^1(\mathbb{R}_t; H^3(\mathbb{R}^6))$. Finally $u \in C^0(\mathbb{R}_t; \mathbb{D}(q_\lambda))$ implies $U_r \in C^0(\mathbb{R}_t; H^4(\mathbb{R}^6))$. Furthermore (2.11) implies that

$$\begin{aligned}
&\partial_t^2 u_r + \ddot{u}_1\varphi_1 + \ddot{u}_2\varphi_2 - \Delta U_r + \mu_0 u_0 \Phi_0 + m^2 u_r \\
&\quad + \left(\mu_1 u_1 + m^2 u_1 + \frac{u_0}{\mu_1 - \mu_2}\right)\varphi_1 + \left(\mu_2 u_2 + m^2 u_2 + \frac{u_0}{\mu_2 - \mu_1}\right)\varphi_2 = 0,
\end{aligned}$$

where $\ddot{u}_j$ denotes the second derivative in time. By examining the regularity of each term, we obtain:

$$\begin{aligned}
\partial_t^2 u_r - \Delta U_r + \mu_0 u_0 \Phi_0 + m^2 u_r &= 0, \\
\ddot{u}_1 + (\mu_1 + m^2)u_1 + \frac{u_0}{\mu_1 - \mu_2} &= 0, \\
\ddot{u}_2 + (\mu_2 + m^2)u_2 + \frac{u_0}{\mu_2 - \mu_1} &= 0,
\end{aligned} \tag{2.50}$$

and the boundary condition at $Z = 0$:

$$U_r(t,0) + \lambda_0 u_0(t) - \frac{e^{\lambda_1}}{\mu_1 - \mu_2} u_1(t) - \frac{e^{\lambda_2}}{\mu_2 - \mu_1} u_2(t) = 0. \tag{2.51}$$

If we assume that $L(u(t)) = 0$ for any time t, then $u_1 + u_2 = 0$ and with the two last equations of (2.50) we successively get $\mu_1 u_1 + \mu_2 u_2 = 0$, $u_1 = u_2 = 0$, and finally $u_0 = 0$. We conclude that $u = U_r$ is a solution of the free Klein-Gordon equation $\partial_t^2 u - \Delta u + m^2 u = 0$ with $f \in H^4(\mathbb{R}^6)$, $g \in H^3(\mathbb{R}^6)$, and (2.51) assures that $u(t,0) = 0$. Moreover when the Cauchy data are spherically symmetric, the Fourier transform of u is given by the classical formula

$$\mathcal{F}(u)(t,\zeta) = \sum_{\pm} e^{\pm it\sqrt{|\zeta|^2+m^2}} A_\pm(|\zeta|), \quad (1+r^2)^2 A_\pm(r) \in L^2(\mathbb{R}_r^+, r^5 dr).$$

Then $u(t,0) = 0$ implies that for any $t \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} e^{itr} \left(A_+(\sqrt{r^2-m^2})\mathbf{1}_{[m,\infty[}(r) + A_-(\sqrt{r^2-m^2})\mathbf{1}_{]-\infty,-m]}(r) \right) (r^2-m^2)^2 r\,dr = 0.$$

We conclude that $A_\pm = 0$ and finally $f = g = 0$. In the general case where $f \in \mathbb{H}_1$ and $g \in \mathbb{H}_0$, we use the regularized solution (2.47). $L(u) = 0$ implies $L(u^n) = 0$ and we have $u^n = 0$ by the previous result. Since u^n strongly tends to u as $n \to \infty$, we conclude that $u = 0$.

Now we show that different λ yield different dynamics. We assume that $u = U_r + u_0\Phi_0 + u_1\varphi_1 + u_2\varphi_2$ is a solution of (2.11), (2.12), (2.17) for some λ and λ' in $\mathbb{R}^3$ with spherically symmetric initial data $f, g \in C_0^\infty(\mathbb{R}^6 \setminus \{0\})$, $(f,g) \neq (0,0)$. u satisfies system (2.50) that has to be completed by the initial data

$$U_r(0) = f, \quad u_0(0) = u_1(0) = u_2(0) = 0, \quad \partial_t U_r(0) = g, \quad \dot{u}_0(0) = \dot{u}_1(0) = \dot{u}_2(0) = 0,$$

and the two boundary conditions at $Z = 0$:

$$\begin{aligned} &U_r(t,0) + \lambda_0 u_0(t) - \frac{e^{\lambda_1}}{\mu_1-\mu_2} u_1(t) - \frac{e^{\lambda_2}}{\mu_2-\mu_1} u_2(t) \\ &= U_r(t,0) + \lambda_0' u_0(t) - \frac{e^{\lambda_1'}}{\mu_1-\mu_2} u_1(t) - \frac{e^{\lambda_2'}}{\mu_2-\mu_1} u_2(t) = 0. \end{aligned}$$

We get from both these constraints that

$$(\lambda_0 - \lambda_0')u_0(t) = \frac{e^{\lambda_1} - e^{\lambda_1'}}{\mu_1-\mu_2} u_1(t) + \frac{e^{\lambda_2} - e^{\lambda_2'}}{\mu_2-\mu_1} u_2(t). \tag{2.52}$$

First we assume that $\lambda_0 \neq \lambda_0'$. Then $u_0 = \frac{\lambda_0-\lambda_0'}{\mu_1-\mu_2}\left([e^{\lambda_1} - e^{\lambda_1'}]u_1 - [e^{\lambda_2} - e^{\lambda_2'}]u_2 \right)$, hence

$$\begin{aligned} &\ddot{u}_1 + (\mu_1 + m^2)u_1 + \frac{\lambda_0-\lambda_0'}{(\mu_1-\mu_2)^2}\left([e^{\lambda_1} - e^{\lambda_1'}]u_1 - [e^{\lambda_2} - e^{\lambda_2'}]u_2 \right) = 0, \\ &\ddot{u}_2 + (\mu_2 + m^2)u_2 - \frac{\lambda_0-\lambda_0'}{(\mu_1-\mu_2)^2}\left([e^{\lambda_1} - e^{\lambda_1'}]u_1 - [e^{\lambda_2} - e^{\lambda_2'}]u_2 \right) = 0. \end{aligned} \tag{2.53}$$

Since the initial data for u_j are zero, we deduce that $u_1(t) = u_2(t) = 0$ for all t, that is a contradiction with the fact that $u_1 + u_2$ is not identically zero. We conclude that $\lambda_0 = \lambda_0'$. As a consequence of (2.52), we get

$$\left(e^{\lambda_1} - e^{\lambda_1'} \right) u_1(t) = \left(e^{\lambda_2} - e^{\lambda_2'} \right) u_2(t).$$

We assume that $\lambda_1 \neq \lambda_1'$, hence we can express u_1 in terms of u_2. Since (2.50) shows that $\ddot{u}_1 + \ddot{u}_2 + (\mu_1 + m^2)u_1 + (\mu_2 + m^2)u_2 = 0$, we deduce that u_2 is solution of

$$\left(\frac{e^{\lambda_2} - e^{\lambda_2'}}{e^{\lambda_1} - e^{\lambda_1'}} + 1\right)\ddot{u}_2 + \left((\mu_1 + m^2)\frac{e^{\lambda_2} - e^{\lambda_2'}}{e^{\lambda_1} - e^{\lambda_1'}} + (\mu_2 + m^2)\right)u_2 = 0, \quad u_2(0) = \dot{u}_2(0) = 0.$$

If $\frac{e^{\lambda_2} - e^{\lambda_2'}}{e^{\lambda_1} - e^{\lambda_1'}} \neq -1$, then $u_2(t) = 0$ for all t, hence u_1 is also zero, that is a contradiction as previous. If $\frac{e^{\lambda_2} - e^{\lambda_2'}}{e^{\lambda_1} - e^{\lambda_1'}} = -1$, then $(\mu_1 - \mu_2)u_2(t) = 0$ for all t, hence u_j is also zero, that is a contradiction again. We conclude that $\lambda_1 = \lambda_1'$. We can prove by the same way that $\lambda_2 = \lambda_2'$, and finally $\lambda = \lambda'$.

We now want to determine for which λ, the static solutions belong to $Dom(\mathbb{A}_\lambda)$. Such a solution is given by $u_{stat} := (-\Delta + m^2)^{-1}\delta_0$ which is equal to $\frac{m^2}{8\pi^3|Z|^2}K_2(m\,|Z|)$ when $m \neq 0$ and $-\frac{1}{4\pi^3|Z|^4}$ for $m = 0$, since $L(u_{stat}) = -1$. If we write $u_{stat} = U_r + u_0\Phi_0 + u_1\varphi_1 + u_2\varphi_2$, we deduce from (2.20) and (2.21) that $u_{stat} \in \mathbb{H}_2$, and its coordinates are given by:

$$u_1 = \frac{m^2 + \mu_2}{\mu_2 - \mu_1}, \quad u_2 = \frac{m^2 + \mu_1}{\mu_1 - \mu_2}, \quad u_0 = -\frac{m^4 + m^2(\mu_1 + \mu_2) + \mu_1\mu_2}{2},$$
$$U_r(0) = -\frac{m^4}{8\pi^3}\log m + \frac{m^4}{8\pi^3}F(0) - u_0G_0(0) - u_1F_1(0) - u_2F_2(0),$$

where $F(0)$, $G_0(0)$ and $F_j(0)$ are given by (2.22), (2.23) and (5.3). Since $\mu_j \neq -m^2$, then $u_0 \neq 0$. Therefore $u_{stat} \in Dom(\mathbb{A}_\lambda)$ iff

$$\lambda \in \Sigma(m) := \left\{\lambda \in \mathbb{R}^3;\ \ \lambda_0 = \frac{1}{u_0}\left(\frac{e^{\lambda_1}}{\mu_1 - \mu_2}u_1 + \frac{e^{\lambda_2}}{\mu_2 - \mu_1}u_2 - U_r(0)\right)\right\}.$$

At last it is clear that the time periodic solution $e^{\pm imt}\,|Z|^{-4}$ is in $Dom(\mathbb{A}_\lambda)$ iff $\lambda \in \Sigma(0)$. $\square$

The previous construction heavily depends on the choice of the different parameters $\mu_0, \mu_1, \mu_2, \theta, \gamma_1, \gamma_2$. We now want to make more clear the role of these parameters. First we note that the changing of μ_0 into μ_0', does not affect u_0, u_1, u_2 and it reduces to replace λ_0 by $\lambda_0 + G_0'(0) - G_0(0)$, where $G_0'(0)$ is defined by (2.23) with μ_0', μ_1, μ_2. Therefore, the set of all the linear forms

$$q_\lambda(V_r(0), v_0, v_1, v_2) = AV_r(0) + \alpha_0 v_0 + \alpha_1 v_1 + \alpha_2 v_2, \quad A, \alpha_j \in \mathbb{R}$$

is obtained by varying $\mu_1, \mu_2, \mu_1 \neq \mu_2, \mu_j < 0, \theta \in \mathbb{R}, \gamma_1, \gamma_2 > 0$.

As we have noticed, the case $\theta = 0$ in (2.34) is not very interesting since in this case, the dynamics is trivial for the initial data f, g in $C_0^\infty(\mathbb{R}^6 \setminus \{0\})$: the solution u satisfies $L(u) = 0$ and $\partial_t^2 u - \Delta u = 0$. It corresponds to the condition $u_0 = 0$ that becomes by (5.6) with $\mu_j' = \mu_j/4$,

$$A = 0, \quad v_0 + (\mu_1' + \mu_2')v_1 - \mu_1'\mu_2'v_2 = 0,$$

where $\mu'_j := \mu_j/4$ are any real numbers such that $\mu'_j < 0, \mu'_1 \neq \mu'_2$. If we put $\alpha_1 = \mu'_1+\mu'_2, \alpha_2 = -\mu'_1\mu'_2$, then μ'_j are solutions of the polynomial $\mu^2-\alpha_1\mu-\alpha_2 = 0$. This equation has two negative distinct solutions if and only if the coefficients α_j satisfy

$$\alpha_0 = 1, \quad \alpha_1 < 0, \quad -\alpha_1^2 < 4\alpha_2 < 0. \tag{2.54}$$

For $\theta \neq 0$, we describe in terms of the coordinates $(V_r(0), v_0, v_1, v_2)$, all the families of the linear forms that we have constructed. If we normalize by taking $A = 1$, (5.6) and (2.36) show that:

$$\begin{aligned}
\alpha_0 &= 32\pi^3\left(\lambda_0 - G_0(0)\right),\\
\alpha_1 &= (\mu_1+\mu_2)\left[8\pi^3\left(\lambda_0 - G_0(0)\right) - \frac{\log 2}{4} - \frac{3}{16} + \frac{\gamma}{4}\right]\\
&\quad + \frac{\mu_1^2\log(|\mu_1|) - \mu_2^2\log(|\mu_2|)}{8(\mu_1-\mu_2)} - 16\pi^3\frac{\gamma_1+\gamma_2}{(\mu_1-\mu_2)^2},\\
\alpha_2 &= \mu_1\mu_2\left[-2\pi^3\left(\lambda_0 - G_0(0)\right) + \frac{\log 2}{16} + \frac{3}{64} - \frac{\gamma}{16}\right.\\
&\quad \left. - \frac{\mu_1\log(\mid\mu_1\mid) - \mu_2\log(\mid\mu_2\mid)}{32(\mu_1-\mu_2)}\right] + 4\pi^3\frac{\mu_1\gamma_1+\mu_2\gamma_2}{(\mu_1-\mu_2)^2},
\end{aligned} \tag{2.55}$$

where $G_0(0)$ can be explicitly expressed by the formula (2.23) involving the Euler's constant γ, and μ_0, μ_1, μ_2.

Conversely, we want to determine for which $\alpha := (\alpha_0, \alpha_1, \alpha_2)$, $V_r(0)+\alpha_0 v_0+\alpha_1 v_1+\alpha_2 v_2$ is a linear form q_λ associated with some $\mu_j < 0$, $\mu_1 \neq \mu_2$, and $\gamma_j > 0$.

Theorem 2.5. *The whole family of the linear forms*

$$q_\lambda(V_r(0), v_0, v_1, v_2) = V_r(0) + \alpha_0 v_0 + \alpha_1 v_1 + \alpha_2 v_2, \quad \alpha_j \in \mathbb{R}$$

of Theorem 2.1 obtained with all the values of $\mu_1, \mu_2 < 0$, $\mu_1 \neq \mu_2$, $\lambda \in \mathbb{R}^3$, is given by the set $\mathcal{A}$ of $\alpha \in \mathbb{R}^3$ satisfying firstly

$$\alpha_0 + \frac{\alpha_1}{\alpha_1+\sqrt{\alpha_1^2-4\alpha_2}} - \frac{\alpha_2}{\left(\alpha_1+\sqrt{\alpha_1^2-4\alpha_2}\right)^2} + \frac{1}{2}\log\left|\alpha_1+\sqrt{\alpha_1^2-4\alpha_2}\right| < \frac{3}{4} - \gamma, \tag{2.56}$$

and secondly

$$\left\{\begin{array}{c}
\alpha_2 < 0,\\
\textit{or}\\
\alpha_2 = 0, \quad \alpha_1 > 0,\\
\textit{or}\\
\left\{\begin{array}{l}
0 < \alpha_1, \quad 0 < 4\alpha_2 < \alpha_1^2,\\
\alpha_0 + \frac{\alpha_1}{\alpha_1-\sqrt{\alpha_1^2-4\alpha_2}} - \frac{\alpha_2}{\left(\alpha_1-\sqrt{\alpha_1^2-4\alpha_2}\right)^2} + \frac{1}{2}\log\left|\alpha_1-\sqrt{\alpha_1^2-4\alpha_2}\right| > \frac{3}{4} - \gamma.
\end{array}\right.
\end{array}\right. \tag{2.57}$$

If $(\alpha_0,\alpha_1,\alpha_2)\neq(\alpha_0',\alpha_1',\alpha_2')$*, the dynamics are different: given two spherically symmetric functions* f, g *in* $C_0^\infty(\mathbb{R}^6\setminus\{0\})$, $(f,g)\neq(0,0)$*, the solutions* u *and* u' *of (2.10), (2.11), (2.12) are different.*

Given $m>0$*, the static solution* $u(Z)=\frac{K_2(m|Z|)}{|Z|^2}$ *belongs to* $\mathbb{D}(q)$ *iff*

$$\alpha\in\Sigma(m)=\left\{\alpha\in\mathcal{A},\quad m^2\alpha_0+\frac{\alpha_1}{2}-\frac{2\alpha_2}{m^2}=m^2\left(\frac{4\log 2+3-4\gamma}{32}-\log m\right)\right\}, \tag{2.58}$$

and for $m\geq 0$*, the time-periodic solutions* $\frac{e^{\pm imt}}{|Z|^4}$ *belong to* $\mathbb{D}(q)$ *iff*

$$\alpha\in\Sigma(0)=\left\{(\alpha_0,\alpha_1,\alpha_2);\ \alpha_2=0,\ \alpha_1>0,\ \alpha_0+\frac{1}{2}\log\alpha_1<\frac{1}{4}-\frac{\log 2}{2}-\gamma\right\}. \tag{2.59}$$

Proof of Theorem 2.5. With (5.4) we can check that we have

$$q_\lambda(V_r(0,v_0,v_1,v_2)=U_r(0)+\lambda_0u_0-\frac{\gamma_1}{\mu_1-\mu_2}u_1-\frac{\gamma_2}{\mu_2-\mu_1}u_2$$

iff

$$\lambda_0=G_0(0)+\frac{\alpha_0}{32\pi^3}, \tag{2.60}$$

$$\gamma_1=(\mu_1-\mu_2)\left[\frac{\alpha_0}{64\pi^3}\mu_1^2-\frac{\alpha_1}{16\pi^3}\mu_1-\frac{\alpha_2}{4\pi^3}-F_1(0)\right], \tag{2.61}$$

$$\gamma_2=(\mu_2-\mu_1)\left[\frac{\alpha_0}{64\pi^3}\mu_2^2-\frac{\alpha_1}{16\pi^3}\mu_2-\frac{\alpha_2}{4\pi^3}-F_2(0)\right]. \tag{2.62}$$

Equation (2.60) yields no constraint on α_j since λ_0 is an arbitrary real number. In opposite, (2.61) and (2.62) show that $\alpha=(\alpha_0,\alpha_1,\alpha_2)$ defines a linear form of the families q_λ, if and only if we can find $\mu_1,\mu_2<0$, $\mu_1\neq\mu_2$, $\gamma_1,\gamma_2>0$ solutions of these equations that we can write as:

$$\gamma_1=\frac{\mu_1-\mu_2}{16\pi^3}\mu_1^2G_\alpha(\mu_1),\quad \gamma_2=\frac{\mu_2-\mu_1}{16\pi^3}\mu_2^2G_\alpha(\mu_2), \tag{2.63}$$

where

$$G_\alpha(\mu)=\frac{\alpha_0}{4}-\frac{\alpha_1}{\mu}-\frac{4\alpha_2}{\mu^2}+\frac{1}{8}\log(|\mu|)-\frac{\log 2}{4}-\frac{3}{16}+\frac{\gamma}{4}.$$

We note that the conditions $\gamma_j>0$ in (2.63) are equivalent to the constraint

$$\exists\mu_i,\mu_j;\quad \mu_i<\mu_j<0,\quad G_\alpha(\mu_i)<0<G_\alpha(\mu_j).$$

An elementary study of the function G_α shows that this case occurs iff

$$\exists\mu_*<0,\quad G_\alpha(\mu_*)=0,\quad G_\alpha'(\mu_*)>0. \tag{2.64}$$

Since $G_\alpha'(\mu)=\mu^{-3}(\frac{1}{8}\mu^2+\alpha_1\mu+8\alpha_2)$, we look for α such that

$$\exists\mu_*<0,\quad G_\alpha(\mu_*)=0,\quad \frac{1}{8}\mu_*^2+\alpha_1\mu_*+8\alpha_2<0. \tag{2.65}$$

If $\mu_\pm := 4\left(-\alpha_1 \pm \sqrt{\alpha_1^2 - 4\alpha_2}\right)$, an obvious equivalent condition is

$$\alpha_1^2 > 4\alpha_2, \quad \exists \mu_* \in \left]\mu_-, \mu_+\right[\cap \left]-\infty, 0\right[, \quad G_\alpha(\mu_*) = 0.$$

Therefore, taking account of the asymptotic behaviour of $G_\alpha(\mu)$ as $\mu \to 0^-$, we have to determine the set of α such that $G_\alpha(\mu_-) < 0$, and $G_\alpha(\mu_+) > 0$ when $\mu_+ < 0$. The constraints (2.56), (2.57) easily follow.

Now we prove that different α yield to different dynamics. We can see that

$$u(t, Z) = V_r(t, Z) + v_0(t)\chi(Z)\log(|Z|) + v_1(t)\frac{\chi(Z)}{|Z|^2} + v_2(t)\frac{\chi(Z)}{|Z|^4}$$

is solution of (2.11) iff

$$\begin{aligned} 0 = {} & \partial_t^2 V_r - \Delta V_r + m^2 V_r - \left(v_0 \log(|Z|) + \frac{v_1}{|Z|^2} + \frac{v_2}{|Z|^4}\right)\Delta\chi \\ & - \left(2\frac{v_0}{|Z|^2} - 4\frac{v_1}{|Z|^4} - 4\frac{v_2}{|Z|^6}\right) Z.\nabla\chi + (\ddot{v}_0 + m^2 v_0)\chi \log(|Z|) \\ & + (\ddot{v}_1 + m^2 v_1 - 4v_0)\frac{\chi}{|Z|^2} + (\ddot{v}_2 + m^2 v_2 + 4v_1)\frac{\chi}{|Z|^4}. \end{aligned}$$

When $u \in C^2(\mathbb{R}_t; \mathbb{H}_0)$, we have $V_r + v_0\chi\log(|Z|) \in C^2(\mathbb{R}_t; H^2(\mathbb{R}^6))$, $v_1, v_2 \in C^2(\mathbb{R})$. $u \in C^1(\mathbb{R}_t; \mathbb{H}_1)$ implies that $v_0 \in C^1(\mathbb{R})$ and $V_r \in C^1(\mathbb{R}_t; H^3(\mathbb{R}^6))$. Finally $u \in C^0(\mathbb{R}_t; \mathbb{D}(q_\lambda))$ yields $V_r \in C^0(\mathbb{R}_t; H^4(\mathbb{R}^6))$. Now we consider a strong solution u of which the initial data are two spherically symmetric functions f, g in $C_0^\infty(\mathbb{R}^6 \setminus \{0\})(f, g) \neq (0, 0)$. We know that there exists T such that $v_2(T) \neq 0$. Taking account of the regularity of each term in the previous equation, we get that

$$\begin{aligned} 0 = {} & \partial_t^2 V_r - \Delta V_r + m^2 V_r - \left(v_0 \log(|Z|) + \frac{v_1}{|Z|^2} + \frac{v_2}{|Z|^4}\right)\Delta\chi \\ & - \left(2\frac{v_0}{|Z|^2} - 4\frac{v_1}{|Z|^4} - 4\frac{v_2}{|Z|^6}\right) Z.\nabla\chi + (\ddot{v}_0 + m^2 v_0)\chi \log(|Z|), \end{aligned} \tag{2.66}$$

$$0 = \ddot{v}_1 + m^2 v_1 - 4v_0, \tag{2.67}$$

$$0 = \ddot{v}_2 + m^2 v_2 + 4v_1. \tag{2.68}$$

We assume that u is solution associated with two linear forms with $(\alpha_0, \alpha_1, \alpha_2)$ and $(\alpha_0', \alpha_1', \alpha_2')$. Then we have

$$(\alpha_0 - \alpha_0')v_0(t) + (\alpha_1 - \alpha_1')v_1(t) + (\alpha_2 - \alpha_2')v_2(t) = 0.$$

If $\alpha_0 \neq \alpha_0'$ we can express v_0 in terms of v_1 and v_2 in (2.67) and with (2.68) and the initial data $v_j(0) = \dot{v}_j(0) = 0$ we obtain $v_1(t) = v_2(t) = 0$ for all t, that is a contradiction with $v_2(T) \neq 0$. We deduce that $\alpha_0 = \alpha_0'$. Now if $\alpha_1 \neq \alpha_1'$, we express v_1 by $-\frac{\alpha_2 - \alpha_2'}{\alpha_1 - \alpha_1'}v_2$ in (2.68) and we obtain $v_2 = 0$ again, hence $\alpha_1 = \alpha_1'$ and $(\alpha_2 - \alpha_2')v_2 = 0$. Finally since $v_2(T) \neq 0$ we conclude that $\alpha_2 = \alpha_2'$.

Finally to determine $\Sigma(m)$ we use (5.2) to get the components of the static solution $|Z|^{-2}\, K_2(m\,|Z|) : v_2 = \frac{2}{m^2}$, $v_1 = -\frac{1}{2}$, $v_0 = -m^2$, $V_r(0) = -m^2 \log m + m^2 F(0)$, and the result follows from (5.3). To characterize $\Sigma(0)$, we note that $V_r(0) = v_0 = v_1 = 0$ and $v_2 = e^{imt}$ for the time periodic solution $u(t, Z) = |Z|^{-4}\, e^{\pm imt}$. Hence $u(t, .) \in \mathbb{D}(q)$ iff $\alpha_2 = 0$, and we conclude with (2.56) and (2.57).

3. Super-Singular Perturbations of The 1 + 1D-Klein-Gordon equation

In this section we investigate the Cauchy problem for some super-singular perturbations of the Klein-Gordon equation on the half line with a Bessel potential and a mass $m \geq 0$:

$$\begin{cases} \partial_t^2 \psi - \partial_z^2 \psi + \frac{15}{4z^2}\psi + m^2\psi = 0, & t \in \mathbb{R}, \quad z > 0, \\ \psi(0, z) = f(z), \quad \partial_t \psi(0, z) = g(z) & z > 0. \end{cases} \tag{3.1}$$

We recall some basic facts (see e.g. [4] p. 532). The Bessel operator

$$P_2 := -\frac{d^2}{dz^2} + \frac{15}{4z^2} \tag{3.2}$$

with domain $C_0^\infty(]0, \infty[)$ is essentially self-adjoint in $L^2(0, \infty)$ since $15/4 \geq 3/2$ and its unique self-adjoint extension is the Friedrichs extension $\mathbf{A}_F$ of which the domain is

$$\mathfrak{D}_F := \left\{ \psi \in L^2(0, \infty);\ P_2\psi \in L^2 \right\} = \left\{ \psi \in L^2(0, \infty);\ P_2\psi,\ \psi',\ z^{-1}\psi \in L^2 \right\}.$$

As a consequence, the Cauchy problem is well-posed for $f \in H_0^1(]0, \infty[)$, $g \in L^2(0, \infty)$ and the solution $\psi \in C^0(\mathbb{R}_t; H_0^1(]0, \infty[) \cap C^1(\mathbb{R}_t; L^2(0, \infty))$ is given by the standard formula

$$\psi(t) = \cos\left(t\sqrt{\mathbf{A}_F + m^2}\right) f + \frac{\sin\left(t\sqrt{\mathbf{A}_F + m^2}\right)}{\sqrt{\mathbf{A}_F + m^2}} g.$$

These solutions are called *"Friedrichs solutions"* of (3.1) and they satisfy the conservation of the natural energy

$$\mathcal{E}(\psi) := \int_0^\infty |\partial_t \psi(t, z)|^2 + |\partial_z \psi(t, z)|^2 + \left(m^2 + \frac{15}{4z^2}\right) |\psi(t, z)|^2 dz,$$

and the Dirichlet condition at the origin:

$$\psi(t, 0) = 0.$$

We want to construct other solutions of (3.1) associated with other energies and other constraints at $z = 0$. We could use the recent spectral results on the singular perturbations of the Bessel operators in [12] but an easier way consists in using the link of P_2 and the Laplace operator in $\mathbb{R}^6$,

$$-\Delta_Z = z^{-\frac{5}{2}} \left(P_2 - \frac{1}{z^2}\Delta_{S^5}\right) z^{\frac{5}{2}}.$$

In this way, we can apply the results of the previous section. Then the super-singular perturbations of Δ_Z restricted to the spherically symmetric functions, yield to hypersingular perturbations of P_2 in the spaces of the trace of the radial distributions (see [17] for an extensive study of these spaces).

Now we perform the suitable functional framework. We introduce the differential operators

$$P_1 := \frac{d}{dz} - \frac{5}{2z}, \quad P_1^* := -\frac{d}{dz} - \frac{5}{2z}, \tag{3.3}$$

and, for $1 \leq k \leq 4$, we define the Hilbert spaces $\mathbf{h}^k$ as the closure of $C_0^\infty(]0,\infty[)$ for the following norms:

$$k=1,2,\quad \|\psi\|^2_{\mathbf{h}^k} := \|\psi\|^2_{L^2} + \|P_k\psi\|^2_{L^2},\quad \|\psi\|^2_{\mathbf{h}^{k+2}} := \|\psi\|^2_{L^2} + \|P_kP_2\psi\|^2_{L^2}. \tag{3.4}$$

Given $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi(z) = 1$ in a neighborhood of $z = 0$, we introduce the spaces

$$k=-1,0,\quad \mathfrak{h}_k := \left\{\psi(z) = \psi_r(z) + v_1\chi(z)z^{\frac{1}{2}} + v_2\chi(z)z^{-\frac{3}{2}},\quad \psi_r \in \mathbf{h}^{k+2},\ v_j \in \mathbb{C}\right\}, \tag{3.5}$$

$$\mathfrak{h}_1 := \Big\{\psi(z) = \psi_r(z) + v_0\chi(z)z^{\frac{5}{2}}\log z + v_1\chi(z)z^{\frac{1}{2}} +v_2\chi(z)z^{-\frac{3}{2}},\quad \psi_r \in \mathbf{h}^3,\ v_j \in \mathbb{C}\Big\}, \tag{3.6}$$

$$\mathfrak{h}_2 := \Big\{\psi(z) = \psi_r(z) + v_{-1}\chi(z)z^{\frac{5}{2}} + v_0\chi(z)z^{\frac{5}{2}}\log z +v_1\chi(z)z^{\frac{1}{2}} + v_2\chi(z)z^{-\frac{3}{2}},\ \psi_r \in \mathbf{h}^4,\ v_j \in \mathbb{C}\Big\}, \tag{3.7}$$

and if X a space of distributions on $\mathbb{R}^6_Z$, we introduce the subspace RX of the spherically symmetric distributions of X:

$$RX := \left\{u \in X;\quad Z_i\partial_{Z_j}u - Z_j\partial_{Z_i}u = 0,\quad 1 \leq i < j \leq 6\right\}.$$

Given $u \in RL^2(\mathbb{R}^6_Z)$ we associate ψ_u defined on $]0,\infty[_z$ by

$$\psi_u(|Z|) := |Z|^{\frac{5}{2}}\ u(Z). \tag{3.8}$$

Lemma 3.1. *Given $\psi \in L^2(0,\infty)$, ψ belongs to $\mathbf{h}^k$ if and only if $u_\psi(Z) := |Z|^{-\frac{5}{2}}\ \psi(|Z|)$ belongs to $H^k(\mathbb{R}^6_Z)$ and $u_\psi(0) = 0$ for $k = 4$. As a consequence, we have*

$$\mathbf{h}^4 \subset \mathbf{h}^3 \subset \mathbf{h}^2 \subset \mathbf{h}^1,$$

$$\psi \in \mathbf{h}^1,\ |\psi(z)| \leq Cz^{\frac{1}{2}},\quad \psi \in \mathbf{h}^2,\ |\psi(z)| \leq Cz^{\frac{3}{2}}, \tag{3.9}$$

$$\psi \in \mathbf{h}^3,\ |\psi(z)| \leq Cz^{\frac{5}{2}}\sqrt{|\log z|},\quad \psi \in \mathbf{h}^4,\ \lim_{z\to 0^+} z^{-\frac{5}{2}}\psi(z) = 0, \tag{3.10}$$

$$-1 \leq k \leq 2,\quad \mathfrak{h}_k = \{\psi_u;\quad u \in R\mathbb{H}_k\}. \tag{3.11}$$

The coefficients v_j do not depend on the choice of the function χ and $v_{-1} = V_r(0)$ when $\psi \in \mathfrak{h}_2$ and $u_\psi = V_r + v_0\chi\log|Z| + v_1\chi\ |Z|^{-2} + v_2\chi\ |Z|^{-4}$. The spaces $\mathfrak{h}_k$ are Hilbert spaces for the norms

$$\|\psi\|^2_{\mathfrak{h}_k} := \|\psi_r\|^2_{\mathbf{h}^{k+2}} + \sum_j |v_j|^2\ . \tag{3.12}$$

Proof of Lemma 3.1. We remark that for $u \in RC_0^\infty(\mathbb{R}^6 \setminus \{0\})$ we have

$$\int_{\mathbb{R}^6} |u(Z)|^2\, dZ = \pi^3 \int_0^\infty |\psi_u(z)|^2\, dz, \quad \int_{\mathbb{R}^6} |\nabla_Z u(Z)|^2\, dZ$$
$$= \pi^3 \int_0^\infty |\psi_u'(z) - \frac{5}{2z}\psi_u(z)|^2\, dz, \tag{3.13}$$

$$\int_{\mathbb{R}^6} |\Delta_Z u(Z)|^2\, dZ = \pi^3 \int_0^\infty |\psi_u''(z) - \frac{15}{4z^2}\psi_u(z)|^2\, dz.$$

We deduce that $u \mapsto \pi^{-\frac{3}{2}}\psi_u$ is an isometry from $RC_0^\infty(\mathbb{R}^6\setminus\{0\})$ endowed with a suitable $H^k(\mathbb{R}^6)$-norm, into $RC_0^\infty(]0,\infty[)$ endowed with the $\mathbf{h}_k$-norm. Since $RC_0^\infty(\mathbb{R}^6 \setminus \{0\})$ is dense in $H^m(\mathbb{R}^6)$ for $m \le 3$, we conclude that

$$k = -1, 0, 1, \quad \mathbf{h}_{k+2} = \left\{\psi_u; \quad u \in RH^{k+2}(\mathbb{R}^6)\right\}, \quad \mathfrak{h}_k = \{\psi_u; \quad u \in R\mathbb{H}_k\} \tag{3.14}$$

and (3.12) defines a norm $\|\psi_u\|_{\mathfrak{h}_k} \sim \|u\|_{\mathbb{H}_k}$, for which $\mathfrak{h}_k$ is a Hilbert space.

On the other hand, the Sobolev embedding $H^4(\mathbb{R}^6) \subset C^0(\mathbb{R}^6)$ implies that the closure of $RC_0^\infty(\mathbb{R}^6 \setminus \{0\})$ in $RH^4(\mathbb{R}^6)$ is the set of functions $u \in RH^4$ that are zero at $Z = 0$, and $RH^4(\mathbb{R}^6) = \overline{RC_0^\infty(\mathbb{R}^6 \setminus \{0\})} \oplus \mathbb{C}\chi(|Z|)$. We conclude that $\lim_{z\to 0^+} z^{-\frac{5}{2}}\psi(z) = 0$ when $\psi \in \mathbf{h}_4$ and

$$\mathbf{h}_4 = \left\{\psi_u; \quad u \in RH^4(\mathbb{R}^6),\ u(0) = 0\right\}, \ \mathbf{h}_4 \oplus \mathbb{C}\chi(z)z^{\frac{5}{2}} = \left\{\psi_u;\ u \in RH^4(\mathbb{R}^6)\right\},$$
$$\mathfrak{h}_2 = \{\psi_u; \quad u \in R\mathbb{H}_2\}.$$

The decay near the origin (3.9), (3.10) for $k = 1, 2, 3$ are consequences of Theorems 13 and 14 of [17]. To achieve the proof of the lemma, we remark that $\chi(z)z^{\frac{1}{2}} \notin \mathbf{h}_1$, $\chi(z)z^{\frac{5}{2}}\log z \in \mathbf{h}_2 \setminus \mathbf{h}_3$, $\chi(z)z^{\frac{5}{2}} \in \mathbf{h}_3 \setminus \mathbf{h}_4$. Then the coefficients v_j only depend on ψ and since $V_r(Z) = |Z|^{-\frac{5}{2}}\ \psi_r(|Z|) = +v_{-1}\chi(|Z|)$, we have $v_{-1} = V_r(0)$. Finally since $\|u\|_{H^4} \sim \|\psi_r\|_{\mathbf{h}_4} + |\ v_{-1}\ |$ for $u \in RH^4(\mathbb{R}^6)$, we have $\|u\|_{\mathbb{H}_2} \sim \|\psi_u\|_{\mathfrak{h}_2}$ and it is clear that (3.12) defines a norm for which $\mathfrak{h}_2$ is a Hilbert space.

We now introduce the "boundary conditions". Given $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{R}^3$, we consider the Hilbert subspace

$$\mathfrak{d}_\alpha := \{\psi \in \mathfrak{h}_2; \quad v_{-1} + \alpha_0 v_0 + \alpha_1 v_1 + \alpha_2 v_2 = 0\}, \tag{3.15}$$

and we denote $\mathbf{A}_\alpha$ the differential operator P_2 endowed with $\mathfrak{d}_\alpha$ as domain. The existence of super-singular perturbations of the Bessel operator P_2 is stated by the following:

Proposition 3.2. *For all $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{R}^3$ satisfying the constraints (2.56) and (2.57), there exists a hermitian product on $\mathfrak{h}_0$, equivalent to the initial $\|.\|_{\mathfrak{h}_0}$-scalar product, for which $\mathbf{A}_\alpha$ is a semi-bounded from below, self-adjoint operator on $\mathfrak{h}_0$. Its essential spectrum is $[0, \infty[$. Its point spectrum is a set of 0, 1, 2 or 3 non positive eigenvalues $-\lambda_j^2$, associated with eigenfunctions $\psi_j(z) = \sqrt{z}K_2(\lambda_j z)$ if $\lambda_j > 0$, $\psi_j(z) = z^{-\frac{3}{2}}$ if $\lambda_j = 0$. Moreover $\lambda_j^2 > 0$ are the roots of the equation:*

$$\log x + 2\alpha_0 + \frac{8\alpha_1}{x} - \frac{32\alpha_2}{x^2} = 0, \tag{3.16}$$

and 0 is eigenvalue iff α belongs to $\Sigma(0)$ defined by (2.59). In particular, the point spectrum is empty for all α such that

$$\alpha_2 < 0, \quad -\log 2 < \alpha_0 + \frac{\alpha_1}{\alpha_1 + \sqrt{\alpha_1^2 - 4\alpha_2}} - \frac{\alpha_2}{\left(\alpha_1 + \sqrt{\alpha_1^2 - 4\alpha_2}\right)^2}$$
$$+\frac{1}{2}\log\left(\alpha_1 + \sqrt{\alpha_1^2 - 4\alpha_2}\right) < \frac{3}{4} - \gamma, \tag{3.17}$$

and 0 is the unique eigenvalue when

$$\alpha_2 = 0 < \alpha_1, \ -\frac{1}{2} - \frac{3}{2}\log 2 < \alpha_0 + \frac{1}{2}\log\alpha_1 < \frac{1}{4} - \frac{1}{2}\log 2 - \gamma, \tag{3.18}$$

Proof of Proposition 3.2. The previous lemma assures that the map $\psi \mapsto u_\psi$ defined by

$$\psi(z) = \psi_r(z) + v_1\chi(z)z^{\frac{1}{2}} + v_2\chi(z)z^{-\frac{3}{2}} \longmapsto u_\psi(Z) = \frac{\psi_r(|Z|)}{|Z|^{\frac{5}{2}}} + v_1\frac{\chi(|Z|)}{|Z|^2} + v_2\frac{\chi(|Z|)}{|Z|^4} \tag{3.19}$$

is an isometry from $\mathfrak{h}_0$ onto $R\mathbb{H}_0$, where $\mathbb{H}_0$ is the space (2.3) endowed with the equivalent norm

$$\pi^{-\frac{3}{2}}\left(\|v_r\|^2_{L^2(\mathbb{R}^6)} + \|\Delta v_r\|^2_{L^2(\mathbb{R}^6)} + \sum_{j=1}^{2}|v_j|^2\right)^{\frac{1}{2}}.$$

Moreover we have for any $\psi \in \mathbf{h}^1$

$$u_{P_2\psi} = -\Delta u_\psi - 4\pi^3 v_2\delta_0(Z).$$

Now we consider $\mu_1, \mu_2 < 0$, $\mu_1 \neq \mu_2$, $\lambda_0 \in \mathbb{R}$, $\gamma_1, \gamma_2 > 0$ associated with α by Theorem 2.5, and we endow $\mathbb{H}_0$ with the norm $\|.\|_0$ given by (2.28) for which $\mathbb{A}_\lambda$ defined by (2.29) and (2.31) is semi-bounded from below, self-adjoint. We remark that

$$\left(Z_i\partial_j - Z_j\partial_i\right)(-\Delta_Z + L(u)\delta_0) = -\Delta_Z = (-\Delta_Z + L(u)\delta_0)\left(Z_i\partial_j - Z_j\partial_i\right),$$

hence the restriction of $\mathbb{A}_\lambda$ to $R\mathbb{H}_0$ with the domain $RDom(\mathbb{A}_\lambda)$ is a densely defined self-adjoint operator that we denote $R\mathbb{A}_\lambda$. Since

$$\mathfrak{d}_\alpha = \{\psi_u;\ \ u \in RDom(\mathbb{A}_\lambda)\}, \quad \mathbb{A}_\lambda u_\psi = u_{P_2\psi},$$

we conclude that if $\mathfrak{h}_0$ is endowed with the equivalent norm

$$\|\psi\|_0 := \|u_\psi\|_0, \tag{3.20}$$

where $\|u_\psi\|_0$ is defined by (2.28), then $\mathbf{A}_\alpha$ is unitarily equivalent to $R\mathbb{A}_\lambda$. Therefore it is semi-bounded from below, and self-adjoint on $\mathfrak{h}_0$. We introduce the operator $\mathbf{A}_0$ defined as the differential operator P_2 provided with

$$\begin{aligned}\mathfrak{d}_0 &:= \{\psi_u;\ u \in R\mathbb{H}_2,\ u_0 = 0\}\\ &= \left\{\psi_u;\ u = U_r + u_1\varphi_1(Z) + u_2\varphi_2(Z),\ U_r \in RH^4(\mathbb{R}^6_Z),\ u_j \in \mathbb{C}\right\}.\end{aligned} \tag{3.21}$$

Then $\mathbf{A}_0$ is unitarily equivalent to $R\mathbb{A}_0$ where $\mathbb{A}_0$ is given by (2.35). Since the essential spectrum of the Laplacien considered as an operator on $RH^2(\mathbb{R}^6)$ endowed with its natural domain $RH^4(\mathbb{R}^6)$ is $[0,\infty[$, and $(\mathbf{A}_0+i)^{-1}-(\mathbf{A}_\alpha+i)^{-1}$ is finite rank, we conclude by the Weyl theorem that $\sigma_{ess}(\mathbf{A}_\alpha)=[0,\infty[$.

Now given $\lambda>0$, the solutions of $P_2\psi=\lambda^2\psi$ are given by $\psi(z)=A\sqrt{z}J_2(\lambda z)+B\sqrt{z}Y_2(\lambda z)$. Since $\psi(z)\sim-\sqrt{\frac{2}{\pi}}\left[A\cos(z-\frac{\pi}{4})+B\sin(z-\frac{\pi}{4})\right]$, ψ does not belong to $\mathfrak{h}_0$ when $(A,B)\neq(0,0)$. We conclude that the eigenvalues of P_2 are non positive. On the other hand, the solutions of $P_2\psi=-\lambda^2\psi$ are given by $\psi(z)=A\sqrt{z}I_2(\lambda z)+B\sqrt{z}K_2(\lambda z)$. Since $I_2(z)\sim\frac{1}{\sqrt{2\pi z}}e^z$ as $z\to\infty$, and taking account of (5.2), the eigenfunction in $\mathfrak{h}_0$ is

$$\psi(z)=\sqrt{z}K_2(\lambda z)=\lambda^4z^{\frac{9}{2}}G(\lambda^2z^2)\log(\lambda z)+\lambda^2z^{\frac{5}{2}}F(\lambda^2z^2)-\left(\frac{\lambda^2}{8}\log\lambda\right)z^{\frac{5}{2}}$$
$$-\frac{\lambda^2}{8}z^{\frac{5}{2}}\log z-\frac{1}{2}z^{\frac{1}{2}}+\frac{2}{\lambda^2}z^{-\frac{3}{2}}.$$

Then $v_{-1}=-\frac{\lambda^2}{8}\log\lambda$, $v_0=-\frac{\lambda^2}{8}$, $v_1=\frac{1}{2}$, $v_2=\frac{2}{\lambda^2}$ satisfy $v_{-1}+\alpha_0v_0+\alpha_1v_1+\alpha_2v_2=0$ iff λ^2 is a strictly positive solution of (3.16). To determine the number of these roots, we study the function $h(x):=\log x+2\alpha_0+\frac{8\alpha_1}{x}-\frac{32\alpha_2}{x^2}$. When $\alpha_2<0$, or when $\alpha_2=0$ and $\alpha_1>0$, h is decreasing from $+\infty$ to $\inf h=2\left(\alpha_0+\frac{\alpha_1}{\alpha_1+\sqrt{\alpha_1^2-4\alpha_2}}-\frac{\alpha_2}{(\alpha_1+\sqrt{\alpha_1^2-4\alpha_2})^2}+\frac{1}{2}\log(\alpha_1+\sqrt{\alpha_1^2-4\alpha_2})+log2\right)$ when $x\in]0,4(\alpha_1+\sqrt{\alpha_1^2-4\alpha_2})]$, and from $\inf h$ to $+\infty$ for $x\in[4(\alpha_1+\sqrt{\alpha_1^2-4\alpha_2}),\infty[$. We deduce that there exists 0, 1 or 2 roots according to $\inf h>0$, $\inf h=0$, $\inf h<0$. Then (3.17) and (3.18) follow from (2.56) and (2.59). Finally when $0<4\alpha_2<\alpha_1^2$ and $0<\alpha_1$, h is increasing from $-\infty$ to $2\left(\alpha_0+\frac{\alpha_1}{\alpha_1-\sqrt{\alpha_1^2-4\alpha_2}}-\frac{\alpha_2}{(\alpha_1-\sqrt{\alpha_1^2-4\alpha_2})^2}+\frac{1}{2}\log(\alpha_1-\sqrt{\alpha_1^2-4\alpha_2})+log2\right)$ when $x\in]0,4(\alpha_1-\sqrt{\alpha_1^2-4\alpha_2})]$, decreasing for $x\in]4(\alpha_1-\sqrt{\alpha_1^2-4\alpha_2}),4(\alpha_1+\sqrt{\alpha_1^2+4\alpha_2})]$, and increasing to $+\infty$ for $x>4(\alpha_1+\sqrt{\alpha_1^2+4\alpha_2})$. We conclude that in this case there exists 1, 2 or 3 strictly negative eigenvalues. $\square$

Now we consider the Cauchy problem (3.1). We look for the weak solutions with the *Ansatz*

$$\psi(t,z)=\psi_r(t,z)+v_0(t)\chi(z)z^{\frac{5}{2}}\log z+v_1(t)\chi(z)z^{\frac{1}{2}}+\phi_2(t)z^{-\frac{3}{2}},\tag{3.22}$$

$$\begin{aligned}&v_1,\ \phi_2\in C^2(\mathbb{R}),\ \psi_r(t,z)+v_0(t)\chi(z)z^{\frac{5}{2}}\log z\in C^2(\mathbb{R}_t;\mathbf{h}^1)\cap C^1(\mathbb{R}_t;\mathbf{h}^2),\\&v_0\in C^0(\mathbb{R}),\ \psi_r\in C^0(\mathbb{R}_t;\mathbf{h}^3),\end{aligned}$$

and we want to construct the strong solutions that satisfy

$$\psi_r(t,z)=\psi_R(t,z)+v_{-1}(t)\chi(z)z^{\frac{5}{2}},\ v_{-1}\in C^0(\mathbb{R}),\ \psi_R\in C^0(\mathbb{R}_t;\mathbf{h}^4),$$

and the boundary condition

$$v_{-1}(t)+\alpha_0 v_0(t)+\alpha_1 v_1(t)+\alpha_2\phi_2(t)=0,\quad t\in\mathbb{R}.$$

We can state the main result of this part:

Theorem 3.3. *Let $\alpha=(\alpha_0,\alpha_1,\alpha_2)$ be in $\mathbb{R}^3$ satisfying the constraints (2.56) and (2.57). Then for all $m\geq 0$ and any $f\in\mathfrak{h}_1$, $g\in\mathfrak{h}_0$, the Cauchy problem (3.1) has a unique solution*

$$\psi_\alpha\in C^0\left(\mathbb{R}_t;\mathfrak{h}_1\right)\cap C^1\left(\mathbb{R}_t;\mathfrak{h}_0\right)\cap C^2\left(\mathbb{R}_t;\mathfrak{h}_{-1}\right)\cap\mathcal{D}'\left(\mathbb{R}_t;\mathfrak{d}_\alpha\right),\tag{3.23}$$

moreover there exists $C,K>0$ independent of $m\geq 0$ such that

$$\begin{aligned}&\|\partial_t\psi_\alpha(t)\|_{\mathfrak{h}_0}+\|\psi_\alpha(t)\|_{\mathfrak{h}_1}+m\|\psi_\alpha(t)\|_{\mathfrak{h}_0}\\&\quad\leq C\left(\|g\|_{\mathfrak{h}_0}+\|f\|_{\mathfrak{h}_1}+m\|f\|_{\mathfrak{h}_0}\right)e^{(K-m^2)_+|t|},\end{aligned}\tag{3.24}$$

and for all $\Theta\in C_0^\infty(\mathbb{R}_t)$ we have:

$$\begin{aligned}&\|\int\Theta(t)\psi_\alpha(t)dt\|_{\mathfrak{h}_2}\\&\quad\leq C\left(\|g\|_{\mathfrak{h}_0}+\|f\|_{\mathfrak{h}_1}+m\|f\|_{\mathfrak{h}_0}\right)\int\left(|\Theta(t)|+|\Theta''(t)|\right)e^{(K-m^2)_+|t|}dt.\end{aligned}\tag{3.25}$$

There exists a conserved energy, i.e. a non-trivial, continuous quadratic form $\mathcal{E}_\alpha$ defined on $\mathfrak{h}_1\oplus\mathfrak{h}_0$, that satisfies:

$$\forall t\in\mathbb{R},\quad\mathcal{E}_\alpha\left(\psi_\alpha(t),\partial_t\psi_\alpha(t)\right)=\mathcal{E}_\alpha(f,g).\tag{3.26}$$

This energy is not positive definite in general but $\mathcal{E}_\alpha$ is equivalent to $\|f\|^2_{\mathfrak{h}_1}+\|g\|^2_{\mathfrak{h}_0}$ on $C_0^\infty(]0,\infty[)\oplus C_0^\infty(]0,\infty[)$ and given for $f,g\in C_0^\infty(]0,\infty[)$ by

$$\begin{aligned}\mathcal{E}_\alpha(f,g)&=\|P_1P_2f\|^2_{L^2}-(\mu_1+\mu_2)\|P_2f\|^2_{L^2}+\mu_1\mu_2\|P_1f\|^2_{L^2}\\&\quad+m^2\left(\|P_2f\|^2_{L^2}-(\mu_1+\mu_2)\|P_1f\|^2_{L^2}+\mu_1\mu_2\|f\|^2_{L^2}\right)\\&\quad+\|P_2g\|^2_{L^2}-(\mu_1+\mu_2)\|P_1g\|^2_{L^2}+\mu_1\mu_2\|g\|^2_{L^2}\end{aligned}\tag{3.27}$$

for some $\mu_1<\mu_2<0$. When α satisfies (3.17) or (3.18), $\mathcal{E}_\alpha$ is positive on $\mathfrak{h}_1\oplus\mathfrak{h}_0$.

The propagation is causal, i.e.

$$supp(\psi_\alpha(t,.))\subset\{z;\ |z|\leq|t|\}+\left[supp(f)\cup supp(g)\right].\tag{3.28}$$

For all $(f,g)\in\mathfrak{h}_1\times\mathfrak{h}_0$, $(f,g)\neq(0,0)$, we have $\phi_2\neq 0$. In particular, when $f,g\in C_0^\infty(]0,\infty[)$, $(f,g)\neq(0,0)$, then ψ_α is not the Friedrichs solution.

If $f,g\in C_0^\infty(]0,\infty[)$, $(f,g)\neq(0,0)$, then $\psi_\alpha\neq\psi_{\alpha'}$ if $\alpha\neq\alpha'$.

When $f\in\mathfrak{d}_\alpha$, $g\in\mathfrak{h}_1$ then ψ_α satisfies:

$$\psi_\alpha\in C^0\left(\mathbb{R}_t;\mathfrak{d}_\alpha\right)\cap C^1\left(\mathbb{R}_t;\mathfrak{h}_1\right)\cap C^2\left(\mathbb{R}_t;\mathfrak{h}_0\right),\tag{3.29}$$

$$\|\partial_t\psi_\alpha(t)\|_{\mathfrak{h}_1}+\|\psi_\alpha(t)\|_{\mathfrak{h}_2}+m\|\psi_\alpha(t)\|_{\mathfrak{h}_1}\leq C\left(\|g\|_{\mathfrak{h}_1}+\|f\|_{\mathfrak{h}_2}+m\|f\|_{\mathfrak{h}_1}\right)e^{(K-m^2)_+|t|}.\tag{3.30}$$

It is interesting to remark that the leading term $\phi_2(t)z^{-\frac{3}{2}}$ completely characterizes the whole solution ψ_α since the continuous linear map

$$(f,g) \in \mathfrak{h}_1 \times \mathfrak{h}_0 \longmapsto \phi_2 \in C^2(\mathbb{R}) \tag{3.31}$$

is one-to-one.

Proof of Theorem 3.3. We consider $\mu_1, \mu_2 < 0$, $\mu_1 \neq \mu_2$ and $\lambda \in \mathbb{R}^3$ that are associated with α by Theorem 2.5. We introduce $u_\lambda(t,Z) := |Z|^{\frac{5}{2}}\ \psi_\alpha(|Z|)$. Then Lemma 3.1 assures that $u_\lambda \in C^m(\mathbb{R}_t;\mathbb{H}_k)$ iff $\psi_\alpha \in C^m(\mathbb{R}_t;\mathfrak{h}_k)$ and $u_\lambda \in \mathcal{D}'(\mathbb{R}_t;\mathbb{D}(q_\lambda))$ iff $\psi_\alpha \in \mathcal{D}'(\mathbb{R}_t;\mathfrak{d}_\alpha)$. Moreover since $\mathbf{A}_\alpha$ is unitarily equivalent to $R\mathbb{A}_\lambda$, we can see that ψ_α is the wanted solution iff u_λ is the solution of (2.10) and (2.11) with the corresponding initial data. Therefore Theorem 2.1 gives the existence, the uniqueness, the estimates of the solution of the Cauchy problem (3.1). Theorem 2.4 provides the finite velocity result (3.28) and the fact that if $(f,g) \neq (0,0)$, then ψ_α is not a Friedrichs solution since the dynamics for u_λ is not trivial. Moreover Theorem 2.5 implies that different α yield to different solutions when $f, g \in C_0^\infty(]0,\infty[)$, $(f,g) \neq (0,0)$. Finally the energy is given for the strong solutions by

$$\mathcal{E}_\alpha(\psi_\alpha, \partial_t\psi_\alpha) := \langle \mathbf{A}_\alpha\psi_\alpha; \psi_\alpha\rangle_0 + m^2\,\|\psi_\alpha\|_0^2 + \|\partial_t\psi_\alpha\|_0^2 = \pi^3\mathcal{E}_\lambda(u_\lambda, \partial_t u_\lambda), \tag{3.32}$$

where the norm $\|.\|_0$ is defined by (3.20), and this energy is positive when α satisfies (3.17) or (3.18) since the spectrum of $\mathbf{A}_\alpha$ is $[0,\infty[$ in this case by Proposition 3.2. At last, the expression (3.27) is obtained by a direct computation by using the facts that $P_1^*P_1 = P_2$ and for $u_\lambda \in C_0^\infty(\mathbb{R}^6 \setminus \{0\})$ we have:

$$\|u_\lambda\|_{H^2}^2 = \pi^3\,\langle (P_2-\mu_1)\psi_\alpha; (P_2-\mu_2)\psi_\alpha\rangle_{L^2(0,\infty)},$$
$$\|\nabla_Z u_\lambda\|_{H^2}^2 = \pi^3\,\langle P_1(P_2-\mu_1)\psi_\alpha; P_1(P_2-\mu_2)\psi_\alpha\rangle_{L^2(0,\infty)}.$$

□

We end this part by some remarks. Firstly, we note that when α satisfies (3.17) or (3.18), the operator $\mathbf{A}_\alpha$ is a positive self-adjoint operator in $(\mathfrak{h}_0, \|.\|_0)$. Then, in this case, the solution is just given by the spectral functional calculus:

$$\psi_\alpha(t,.) = \cos\left(t\sqrt{\mathbf{A}_\alpha + m^2}\right) f + \frac{\sin\left(t\sqrt{\mathbf{A}_\alpha + m^2}\right)}{\sqrt{\mathbf{A}_\alpha + m^2}}\, g,$$

and we can solve the Cauchy problem in the scale of the Hilbert spaces associated with the powers of $\mathbf{A}_\alpha$. More precisely, when $m > 0$ or when α satisfies (3.17), the Cauchy problem is well-posed for $f \in \left[Dom\left(\left(\mathbf{A}_\alpha + m^2\right)^{\frac{s+1}{2}}\right)\right]$, $g \in \left[Dom\left(\left(\mathbf{A}_\alpha + m^2\right)^{\frac{s}{2}}\right)\right]$, where $[Dom(B)]$ denotes the completion of $Dom(B)$ for the norm $\|B.\|_0$. Secondly, when α satisfies (3.18), the kernel of $\mathbf{A}_\alpha$ is $\mathbb{C}z^{-\frac{3}{2}}$ and the time-periodic solutions $e^{\pm imt}z^{-\frac{3}{2}}$ belong to $C^0(\mathbb{R}_t;\mathfrak{d}_\alpha)$. We can express ψ_α in terms of the graviton part supported by $z^{-\frac{3}{2}}$:

$$\psi_\alpha(t,z) = \psi_\alpha^0(t)z^{-\frac{3}{2}} + \psi_\alpha^\perp(t,z), \quad \left\langle \psi_\alpha^\perp(t,.); z^{-\frac{3}{2}}\right\rangle_0 = 0, \tag{3.33}$$

where the amplitude of the graviton is given by

$$\psi_\alpha^0(t) = \|z^{-\frac{3}{2}}\|_0^{-2}\left(\cos(mt) < f; z^{-\frac{3}{2}} >_0 + \frac{\sin(mt)}{m} < g; z^{-\frac{3}{2}} >_0\right), \tag{3.34}$$

and we have to replace $\frac{\sin(mt)}{m}$ by t when $m = 0$. Finally, if we could establish the absence of singular continuous spectrum of $\mathbf{A}_\alpha$ in $\mathfrak{h}_0$, then $\psi_\alpha^\perp(t, .)$ would tend weakly to zero as $| t |\to \infty$. As a consequence, if we expand $\psi_\alpha^\perp(t, .)$ as

$$\psi_\alpha^\perp(t, z) == \psi_r^\perp(t, z) + v_0^\perp(t)\chi(z)z^{\frac{5}{2}}\log z + v_1^\perp(t)\chi(z)z^{\frac{1}{2}} + \phi_2^\perp(t)z^{-\frac{3}{2}},$$

we have

$$v_0^\perp(t) \to 0, \quad v_1^\perp(t) \to 0, \quad \phi_2^\perp(t) \to 0 \text{ as } |t| \to \infty. \tag{3.35}$$

An interesting consequence would be

$$\phi_2(t) - \psi_\alpha^0(t) \to 0, \quad |t| \to \infty, \tag{3.36}$$

i.e. the more singular part in the expansion (3.22) would be asymptotically given by the graviton.

4. New Dynamics in AdS^5

In this section we construct new unitary dynamics for the gravitational waves in the Anti-de Sitter universe. We consider the Cauchy problem

$$\left(\partial_t^2 - \Delta_{\mathbf{x}} - \partial_z^2 + \frac{15}{4z^2}\right)\Phi = 0, \quad t \in \mathbb{R}, \quad \mathbf{x} \in \mathbb{R}^3, \quad z \in]0, \infty[, \tag{4.1}$$

$$\Phi(0, \mathbf{x}, z) = \Phi_0(\mathbf{x}, z), \quad \partial_t\Phi(0, \mathbf{x}, z) = \Phi_1(\mathbf{x}, z). \tag{4.2}$$

We look for the solutions that have an expansion of the following form

$$\begin{aligned}\Phi(t, \mathbf{x}, z) = \Phi_r(t, \mathbf{x}, z)z^{\frac{5}{2}} + \phi_{-1}(t, \mathbf{x})\chi(z)z^{\frac{5}{2}} + \phi_0(t, \mathbf{x})\chi(z)z^{\frac{5}{2}}\log z\\ +\phi_1(t, \mathbf{x})\chi(z)z^{\frac{1}{2}} + \phi_2(t, \mathbf{x})z^{-\frac{3}{2}},\end{aligned} \tag{4.3}$$

where $\chi \in C_0^\infty(\mathbb{R})$, $\chi(z) = 1$ in a neighborhood of 0 and

$$\Phi_r(t, \mathbf{x}, 0) = 0. \tag{4.4}$$

The term $\phi_2(t, \mathbf{x})z^{-\frac{3}{2}}$ is the part of the wave in the sector of the massless graviton. The behaviour of the field on the boundary of the universe is assumed to be for some $(\alpha_0, \alpha_1, \alpha_2) \in \mathbb{R}^3$:

$$\phi_{-1}(t, \mathbf{x}) + \alpha_0\phi_0(t, \mathbf{x}) + \alpha_1\phi_1(t, \mathbf{x}) + \alpha_2\phi_2(t, \mathbf{x}) = 0, \quad t \in \mathbb{R}, \quad \mathbf{x} \in \mathbb{R}^3. \tag{4.5}$$

We introduce the following Hilbert spaces endowed with the natural norms ($\mathfrak{h}_0$ being provided with the norm (3.20)):

$$\mathfrak{H}_0 := L^2\left(\mathbb{R}^3_{\mathbf{x}}; \mathfrak{h}_0\right) = \Big\{\Phi(\mathbf{x}, z) = \phi_r(\mathbf{x}, z) + \phi_1(\mathbf{x})\chi(z)z^{\frac{1}{2}} + \phi_2(\mathbf{x})z^{-\frac{3}{2}}, \quad \phi_r \in L^2(\mathbb{R}^3_{\mathbf{x}}; \mathbf{h}^2),\ \phi_j \in L^2(\mathbb{R}^3_{\mathbf{x}})\Big\}, \tag{4.6}$$

$$\begin{aligned}\mathfrak{H}_1 := &\left\{\Phi \in L^2\left(\mathbb{R}^3_{\mathbf{x}}; \mathfrak{h}_1\right);\ \nabla_{\mathbf{x}}\Phi \in \mathfrak{H}_0\right\}\\ = &\Big\{\Phi(\mathbf{x}, z) = \phi_r(\mathbf{x}, z) + \phi_0(\mathbf{x})\chi(z)z^{\frac{5}{2}}\log z + \phi_1(\mathbf{x})\chi(z)z^{\frac{1}{2}} + \phi_2(\mathbf{x})z^{-\frac{3}{2}},\\ &\phi_r \in L^2(\mathbb{R}^3_{\mathbf{x}}; \mathbf{h}^3),\quad \phi_0 \in L^2(\mathbb{R}^3_{\mathbf{x}}),\ \phi_1, \phi_2 \in H^1(\mathbb{R}^3_{\mathbf{x}}),\\ &\nabla_{\mathbf{x}}\left(\phi_r + \phi_0\chi z^{\frac{5}{2}}\log z\right) \in L^2(\mathbb{R}^3_{\mathbf{x}}; \mathbf{h}^2)\Big\},\end{aligned} \tag{4.7}$$

$$\mathfrak{H}_2 := \left\{\Phi \in L^2\left(\mathbb{R}^3_{\mathbf{x}}; \mathfrak{h}_2\right);\ \nabla_{\mathbf{x}}\Phi \in \mathfrak{H}_1\right\}. \tag{4.8}$$

In particular, $\Phi \in \mathfrak{H}_2$ iff

$$\Phi(\mathbf{x}, z) = \Phi_r(\mathbf{x}, z)z^{\frac{5}{2}} + \phi_{-1}(\mathbf{x})\chi(z)z^{\frac{5}{2}} + \phi_0(\mathbf{x})\chi(z)z^{\frac{5}{2}}\log z + \phi_1(\mathbf{x})\chi(z)z^{\frac{1}{2}} + \phi_2(\mathbf{x})z^{-\frac{3}{2}} \tag{4.9}$$

with

$$\begin{cases}\phi_{-1} \in L^2(\mathbb{R}^3_{\mathbf{x}}),\ \phi_0 \in H^1(\mathbb{R}^3_{\mathbf{x}}),\ \phi_1, \phi_2 \in H^2(\mathbb{R}^3_{\mathbf{x}}),\quad \Phi_r(\mathbf{x}, z)z^{\frac{5}{2}} \in L^2(\mathbb{R}^3_{\mathbf{x}}; \mathbf{h}^4),\\ \nabla_{\mathbf{x}}\left(\Phi_r(\mathbf{x}, z)z^{\frac{5}{2}} + \phi_{-1}(\mathbf{x})\chi(z)z^{\frac{5}{2}}\right) \in L^2(\mathbb{R}^3_{\mathbf{x}}; \mathbf{h}^3),\\ \nabla^2_{\mathbf{x}}\left(\Phi_r(\mathbf{x}, z)z^{\frac{5}{2}} + \phi_{-1}(\mathbf{x})\chi(z)z^{\frac{5}{2}} + \phi_0(\mathbf{x})\chi(z)z^{\frac{5}{2}}\log z\right) \in L^2(\mathbb{R}^3_{\mathbf{x}}; \mathbf{h}^2).\end{cases} \tag{4.10}$$

For convenience and to make more clear the role of the massless graviton, we have omitted the cut-off function $\chi(z)$ in front of $\phi_2(\mathbf{x})z^{-\frac{3}{2}}$. It is clear that this minor change does not affect the definition of the spaces since $(1-\chi(z))\phi_2(\mathbf{x})z^{-\frac{3}{2}}$ belongs to $H^m(\mathbb{R}^3_{\mathbf{x}}; \mathbf{h}^4)$ when $\phi_2 \in H^m(\mathbb{R}^3_{\mathbf{x}})$.

To take account of the constraint (4.5), we introduce the subspace:

$$\mathfrak{D}_\alpha := \{\Phi \in \mathfrak{H}_2;\quad \phi_{-1}(\mathbf{x}) + \alpha_0\phi_0(\mathbf{x}) + \alpha_1\phi_1(\mathbf{x}) + \alpha_2\phi_2(\mathbf{x}) = 0\}. \tag{4.11}$$

The main result of this paper is the following:

Theorem 4.1. *Let $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ be in $\mathbb{R}^3$ satisfying the constraints (2.56) and (2.57). Then for any $\Phi_0 \in \mathfrak{H}_1$, $\Phi_1 \in \mathfrak{H}_0$, the Cauchy problem (4.1), (4.2) has a unique solution*

$$\Phi_\alpha \in C^0(\mathbb{R}_t; \mathfrak{H}_1) \cap C^1(\mathbb{R}_t; \mathfrak{H}_0) \cap \mathcal{D}'(\mathbb{R}_t; \mathfrak{D}_\alpha). \tag{4.12}$$

Moreover there exists $C, \kappa > 0$ independent of Φ_j such that:

$$\|\partial_t\Phi_\alpha(t)\|_{\mathfrak{H}_0} + \|\Phi_\alpha(t)\|_{\mathfrak{H}_1} \leq C\left(\|\Phi_1\|_{\mathfrak{H}_0} + \|\Phi_0\|_{\mathfrak{H}_1}\right)e^{\kappa|t|}, \tag{4.13}$$

and for all $\Theta \in C^\infty_0(\mathbb{R}_t)$ we have:

$$\left\|\int\Theta(t)\Phi_\alpha(t)dt\right\|_{\mathfrak{H}_2} \leq C\left(\|\Phi_1\|_{\mathfrak{H}_0} + \|\Phi_0\|_{\mathfrak{H}_1}\right)\int\left(|\Theta(t)| + |\Theta''(t)|\right)e^{\kappa|t|}dt. \tag{4.14}$$

When $\Phi_0, \Phi_1 \in C_0^\infty\left(\mathbb{R}^3_{\mathbf{x}}\times]0,\infty[\right)$, $(\Phi_0,\Phi_1)\neq(0,0)$, *then* $\phi_2\neq 0$ *hence* Φ_α *is not the Friedrichs solution. Moreover the map*

$$\Gamma_\alpha : (\Phi_0,\Phi_1)\in\mathfrak{H}_1\times\mathfrak{H}_0 \longmapsto \phi_2\in C^0\left(\mathbb{R}_t; H^1\left(\mathbb{R}^3_{\mathbf{x}}\right)\right)\cap C^1\left(\mathbb{R}_t; L^2\left(\mathbb{R}^3_{\mathbf{x}}\right)\right) \quad (4.15)$$

is linear continuous and one-to-one.

For any $\Phi_0, \Phi_1 \in C_0^\infty\left(\mathbb{R}^3_{\mathbf{x}}\times]0,\infty[\right)$, $(\Phi_0,\Phi_1)\neq(0,0)$, *we have* $\Phi_\alpha\neq\Phi_{\alpha'}$ *if* $\alpha\neq\alpha'$.

There exists a conserved energy, i.e. a non-trivial, continuous quadratic form $\mathbb{E}_\alpha$ *defined on* $\mathfrak{H}_1\oplus\mathfrak{H}_0$, *that satisfies:*

$$\forall t\in\mathbb{R},\quad \mathbb{E}_\alpha\left(\Phi_\alpha(t),\partial_t\Phi_\alpha(t)\right)=\mathcal{E}_\alpha(\Phi_0,\Phi_1). \quad (4.16)$$

This energy is not positive definite in general but $\mathbb{E}_\alpha$ *is equivalent to* $\|\Phi_0\|^2_{\mathfrak{H}_1}+\|\Phi_1\|^2_{\mathfrak{H}_0}$ *on* $C_0^\infty(\mathbb{R}^3_{\mathbf{x}}\times]0,\infty[_z)\oplus C_0^\infty(\mathbb{R}^3_{\mathbf{x}}\times]0,\infty[_z)$ *and given for* $\Phi_0,\Phi_1\in C_0^\infty(\mathbb{R}^3_{\mathbf{x}}\times]0,\infty[_z)$ *by*

$$\begin{aligned}\mathbb{E}_\alpha(\Phi_0,\Phi_1) &= \|P_1P_2\Phi_0\|^2_{L^2}-(\mu_1+\mu_2)\|P_2\Phi_0\|^2_{L^2}+\mu_1\mu_2\|P_1\Phi_0\|^2_{L^2}\\ &\quad+\|\nabla_{\mathbf{x}}P_2\Phi_0\|^2_{L^2}-(\mu_1+\mu_2)\|\nabla_{\mathbf{x}}P_1\Phi_0\|^2_{L^2}+\mu_1\mu_2\|\nabla_{\mathbf{x}}\Phi_0\|^2_{L^2}\\ &\quad+\|P_2\Phi_1\|^2_{L^2}-(\mu_1+\mu_2)\|P_1\Phi_1\|^2_{L^2}+\mu_1\mu_2\|\Phi_1\|^2_{L^2}\end{aligned} \quad (4.17)$$

for some $\mu_1<\mu_2<0$. *When* α *satisfies (3.17) or (3.18),* $\mathbb{E}_\alpha$ *is positive on* $\mathfrak{H}_1\oplus\mathfrak{H}_0$.

When $\Phi_0\in\mathfrak{D}_\alpha$, $\Phi_1\in\mathfrak{H}_1$ *then* Φ_α *satisfies:*

$$\Phi_\alpha\in C^0\left(\mathbb{R}_t;\mathfrak{D}_\alpha\right)\cap C^1\left(\mathbb{R}_t;\mathfrak{H}_1\right)\cap C^2\left(\mathbb{R}_t;\mathfrak{H}_0\right), \quad (4.18)$$

$$\|\partial_t\Phi_\alpha(t)\|_{\mathfrak{H}_1}+\|\Phi_\alpha(t)\|_{\mathfrak{H}_2}\leq C\left(\|\Phi_1\|_{\mathfrak{H}_1}+\|\Phi_0\|_{\mathfrak{H}_2}\right)e^{\kappa|t|}. \quad (4.19)$$

There exists $M>0$ *such that if* $\hat{\Phi}_j(\boldsymbol{\xi},z)=0$ *for all* $|\boldsymbol{\xi}|\leq M$, *then we can take* $\kappa=0$ *in the estimates (4.13), (4.14) and (4.19) and* $\mathbb{E}_\alpha(\Phi_0,\Phi_1)>0$.

When the equation

$$\log x+2\alpha_0+\frac{8\alpha_1}{x}-\frac{32\alpha_2}{x^2}=0, \quad (4.20)$$

has a solution $x=m^2$, $m>0$, *then* $\phi_{[m]}(t,\mathbf{x})z^{\frac12}K_2(mz)$, *where* $\phi_{[m]}\in C^0(\mathbb{R}_t;H^2(\mathbb{R}^3_{\mathbf{x}}))\cap C^1(\mathbb{R}_t;H^1(\mathbb{R}^3_{\mathbf{x}}))$ *is a solution of* $\partial_t^2\phi_{[m]}-\Delta_{\mathbf{x}}\phi_{[m]}-m^2\phi_{[m]}=0$, *is a solution that satisfies (4.18).*

When α *satisfies (3.18), the massless graviton* $\Phi_G(t,\mathbf{x},z):=\phi_{[0]}(t,\mathbf{x})z^{-\frac32}$ *where,* $\phi_{[0]}\in C^0(\mathbb{R}_t;H^2(\mathbb{R}^3_{\mathbf{x}}))$ *is solution of* $\partial_t^2\phi_{[0]}-\Delta_{\mathbf{x}}\phi_{[0]}=0$, *is a solution of (4.1) that satisfies (4.18), and its energy is given by*

$$\mathbb{E}_\alpha(\Phi_G,\partial_t\Phi_G)=\|z^{-\frac32}\|_0^2\int_{\mathbb{R}^3_{\mathbf{x}}}|\nabla_{t,\mathbf{x}}\phi_{[0]}(t,\mathbf{x})|^2d\mathbf{x}. \quad (4.21)$$

Proof of Theorem 4.1. We shall use the partial Fourier transform with respect to $\mathbf{x}$ that is denoted $\mathcal{F}_{\mathbf{x}}$. Let Φ_α be a solution of (4.1), (4.2), (4.18). Given $T>0$, $\Phi_\alpha\in H^1(]-T,T[;\mathfrak{H}_1)\subset L^2(\mathbb{R}^3_{\mathbf{x}};H^1(]-T,T[;\mathfrak{h}_1))$. Then $\mathcal{F}_{\mathbf{x}}\Phi_\alpha\in L^2(\mathbb{R}^3_{\boldsymbol{\xi}};H^1(]-T,T[;\mathfrak{h}_1))\subset L^2(\mathbb{R}^3_{\boldsymbol{\xi}};C^0([-T,T];\mathfrak{h}_1))$. We have also $\partial_t\Phi_\alpha\in H^1(]-T,T[;\mathfrak{H}_0)\subset L^2(\mathbb{R}^3_{\mathbf{x}};H^1(]-T,T[;\mathfrak{h}_0))$. Then $\partial_t\mathcal{F}_{\mathbf{x}}\Phi_\alpha\in L^2(\mathbb{R}^3_{\boldsymbol{\xi}};H^1(]-T,T[;\mathfrak{h}_0))\subset L^2(\mathbb{R}^3_{\boldsymbol{\xi}};C^0([-T,T];\mathfrak{h}_0))$.

Moreover $\Phi_\alpha \in L^2(]-T,T[;\mathfrak{D}_\alpha) \subset L^2(\mathbb{R}^3_{\mathbf{x}}; L^2(]-T,T[;\mathfrak{d}_\alpha))$. Then $\mathcal{F}_{\mathbf{x}}\Phi_\alpha \in L^2(\mathbb{R}^3_{\boldsymbol{\xi}}; L^2(]-T,T[;\mathfrak{d}_\alpha))$.

We deduce that for almost all $\boldsymbol{\xi} \in \mathbb{R}^3$, $\mathcal{F}_{\mathbf{x}}\Phi_\alpha(t,\boldsymbol{\xi},z)$ is the unique solution $\psi_{\boldsymbol{\xi}}$, satisfying (3.23), of (3.1) with

$$m = \mid\boldsymbol{\xi}\mid, \quad f(z) = \mathcal{F}_{\mathbf{x}}\Phi_0(\boldsymbol{\xi},z), \quad g(z) = \mathcal{F}_{\mathbf{x}}\Phi_1(\boldsymbol{\xi},z). \tag{4.22}$$

Hence we conclude that

$$\Phi_\alpha(t,\mathbf{x},z) = \mathcal{F}^{-1}_{\boldsymbol{\xi}}\left(\psi_{\boldsymbol{\xi}}(t,z)\right)(\mathbf{x}), \tag{4.23}$$

and we get the uniqueness of the solution.

More generally, when Φ_α is a solution of (4.1), (4.2), (4.12), we take $\theta \in C^\infty_0(\mathbb{R})$ such that $0 \le \theta$, $\int \theta(t)dt = 1$, and we consider $\Phi_{\alpha,n}(t,\mathbf{x},z) = n\int \theta(nt-ns)\Phi_\alpha(s,\mathbf{x},z)ds$. We can easily prove that $\Phi_{\alpha,n}$ tends to Φ_α in $C^0(\mathbb{R}_t;\mathfrak{H}_1)\cap C^1(\mathbb{R}_t;\mathfrak{H}_0)\cap C^2(\mathbb{R}_t;\mathfrak{H}_{-1})\cap \mathcal{D}'(\mathbb{R}_t;\mathfrak{D}_\alpha)$ as $n\to\infty$, and $\Phi_{\alpha,n}$ is a solution of (4.1), (4.18). The previous result shows that

$$\Phi_{\alpha,n}(t,\mathbf{x},z) = \mathcal{F}^{-1}_{\boldsymbol{\xi}}\left(\psi_{\boldsymbol{\xi},n}(t,z)\right)(\mathbf{x}),$$

where $\psi_{\boldsymbol{\xi},n}$ is solution of (3.1) with $m = \mid\boldsymbol{\xi}\mid$, $f(z) = \mathcal{F}_{\mathbf{x}}\Phi_{\alpha,n}(0,\boldsymbol{\xi},z)$, $g(z) = \mathcal{F}_{\mathbf{x}}\partial_t\Phi_{\alpha,n}(0,\boldsymbol{\xi},z)$ satisfying (3.23). Since $\Phi_{\alpha,n}(0,\mathbf{x},z)$ and $\partial_t\Phi_{\alpha,n}(0,\mathbf{x},z)$ tend respectively to Φ_0 and Φ_1 in $\mathfrak{H}_1$ and $\mathfrak{H}_0$, then $\mathcal{F}_{\mathbf{x}}\Phi_{\alpha,n}(0,\boldsymbol{\xi},z)$ and $\mathcal{F}_{\mathbf{x}}\partial_t\Phi_{\alpha,n}(0,\boldsymbol{\xi},z)$ tend respectively to $\mathcal{F}_{\mathbf{x}}\Phi_0(\boldsymbol{\xi},z)$ and $\mathcal{F}_{\mathbf{x}}\Phi_1(\boldsymbol{\xi},z)$ in $L^2(\mathbb{R}^3_{\boldsymbol{\xi}};\mathfrak{h}_1)$ and $L^2(\mathbb{R}^3_{\boldsymbol{\xi}};\mathfrak{h}_0)$. We deduce by (3.24) that $\psi_{\boldsymbol{\xi},n}$ tends in $L^2(\mathbb{R}^3_{\boldsymbol{\xi}}; L^2([-T,T];\mathfrak{h}_1)$ to the solution $\psi_{\boldsymbol{\xi}}$ of (3.1), (3.24) with the data (3.24). We conclude that (4.23) is true again and the proof of the uniqueness is complete.

To establish the existence of the solution, it is sufficient to solve the Cauchy problem and to get estimates (4.13), (4.13), (4.19) for a dense subspace of initial data. Hence we consider the case where there exists $R>0$ such that $\mathcal{F}_{\mathbf{x}}\Phi_j(\boldsymbol{\xi},z) = 0$ for any $|\boldsymbol{\xi}|> R$. Then we get by the Lebesgue theorems, the Parseval equality and Theorem 3.3, that

$$\Phi_\alpha(t,\mathbf{x},z) := \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{|\boldsymbol{\xi}|\le R} e^{i\mathbf{x}.\boldsymbol{\xi}}\psi_{\boldsymbol{\xi}}(t,z)d\boldsymbol{\xi}$$

is the wanted solution, moreover estimates (4.13), (4.14), (4.19) directly follow from the integration of (3.24), (3.25), (3.30) with respect to $\boldsymbol{\xi}$, and we can take $\kappa = 0$ when $\hat{\Phi}_j(\boldsymbol{\xi},z) = 0$ for all $|\boldsymbol{\xi}|\le M$ where $M = \sqrt{K}$.

The continuity of Γ_α is deduced from (4.6), (4.7) and (4.12). To prove the injectivity, we suppose that $\phi_2 = 0$ for some $\Phi_0 \in \mathfrak{H}_1$, $\Phi_1 \in \mathfrak{H}_0$, then we have $\mathcal{F}_{\mathbf{x}}\phi_2 = 0$ and Theorem 3.3 implies that $\psi_{\boldsymbol{\xi}}(t,z) = 0$. We conclude that $\Phi_0 = \Phi_1 = 0$. Now if $\Phi_\alpha = \Phi_{\alpha'}$, then $\mathcal{F}_{\mathbf{x}}\Phi_\alpha = \mathcal{F}_{\mathbf{x}}\Phi_{\alpha'}$ and this theorem assures that $\alpha = \alpha'$.

The properties of the energy are obtained by the same way from (3.26) and (3.27) with the Parseval equality and the formula

$$\mathbb{E}_\alpha\left(\Phi_\alpha(t),\partial_t\Phi_\alpha(t)\right) = \int \mathcal{E}_\alpha\left(\psi_{\boldsymbol{\xi}}(t),\partial_t\psi_{\boldsymbol{\xi}}(t)\right)d\boldsymbol{\xi}.$$

We also have with (3.32):

$$\mathbb{E}_\alpha\left(\Phi_0,\Phi_1\right) = \int \langle \mathbf{A}_\alpha\mathcal{F}_{\mathbf{x}}\Phi_0(\boldsymbol{\xi},.);\mathcal{F}_{\mathbf{x}}\Phi_0(\boldsymbol{\xi},.)\rangle_0\, d\boldsymbol{\xi} + \|\nabla_{\mathbf{x}}\Phi_0\|^2_{\mathfrak{H}_0} + \|\Phi_1\|^2_{\mathfrak{H}_0},$$

that proves (4.21). Finally, since Proposition 3.2 assures that for $m > 0$, $\psi_m(z) := z^{\frac{1}{2}} K_2(mz)$, and for $m = 0$ $\psi_0(z) := z^{-\frac{3}{2}}$, satisfy $(P_2 + m^2)\psi_m = 0$, and belong to $\mathfrak{d}_\alpha$ when $x = m^2 > 0$ is a solution of (4.20), or α satisfies (3.18) for $m = 0$. We conclude that $\Phi_\alpha(t, \mathbf{x}, z) = \phi_{[m]}(t, \mathbf{x})\psi_m(z)$ are solutions of (4.1) satisfying (4.18). □

We conclude this paper with some comments. If we expand the strong solution as

$$\Phi_\alpha(t, \mathbf{x}, z) = \phi_r(t, \mathbf{x}, z) + \phi_0(t, \mathbf{x})\chi(z)z^{\frac{5}{2}} \log z + \phi_1(t, \mathbf{x})\chi(z)z^{\frac{1}{2}} + \phi_2(t, \mathbf{x})z^{-\frac{3}{2}}, \tag{4.24}$$

then we can see with (4.9) and (4.10) that Eq. (4.1) is equivalent to a system of coupled PDEs (we denote $\square := \partial_t^2 - \Delta_{\mathbf{x}}$):

$$\square\phi_2 + 4\phi_1 = 0, \tag{4.25}$$

$$\square\phi_1 - 4\phi_0 = 0, \tag{4.26}$$

$$\left[\square - \partial_z^2 + \frac{15}{4z^2}\right]\left(\phi_r + \chi(z)z^{\frac{5}{2}} \log(z)\phi_0\right) = -4\chi(z)z^{\frac{1}{2}}\phi_0 + \left(\chi''(z)z^{\frac{1}{2}} + \chi'(z)z^{-\frac{1}{2}} + 4(1 - \chi(z))z^{-\frac{3}{2}}\right)\phi_1, \tag{4.27}$$

supplemented by the boundary constraint at the time-like horizon:

$$\lim_{z\to 0} z^{-\frac{5}{2}}\phi_r(t, \mathbf{x}, z) + \alpha_0\phi_0(t, \mathbf{x}) + \alpha_1\phi_1(t, \mathbf{x}) + \alpha_2\phi_2(t, \mathbf{x}) = 0. \tag{4.28}$$

The leading term of Φ_α in (4.24) is $\phi_2(t, \mathbf{x})z^{-\frac{3}{2}}$, and ϕ_2 has to be considered as the regularized boundary value of the field,

$$\phi_2(t, \mathbf{x}) := \lim_{z\to 0} z^{\frac{3}{2}}\Phi_\alpha(t, \mathbf{x}, z). \tag{4.29}$$

This renormalization can be expressed by the operator Γ_α defined by (4.15), which associates the field ϕ_2 on the conformal boundary, to the field into the bulk. In the context of the AdS/CFT conjecture, the fundamental question of the injectivity of this operator arises. Our crucial result states that Γ_α is one-to-one, therefore we have established the validity of a kind of holographic principle: the boundary value ϕ_2 entirely characterizes the whole field in the bulk, and consequently we may regard ϕ_2 as the hologram of Φ_α.

As regards the asymptotic dynamics on the boundary, we note by (4.25) that ϕ_2 is not a free wave in the Minkowski space-time (see below for a link with the massless graviton), and $\Phi_F := \phi_r + \chi(z)z^{\frac{5}{2}} \log(z)\phi_0$ is a Friedrichs solution of the inhomogeneous wave equation of the gravitational fluctuations, *i.e.* Φ_F satisfies (1.3). The part of the field given by $\phi_1(t, \mathbf{x})\chi(z)z^{\frac{1}{2}}$ is rather peculiar and specific to our functional framework: it satisfies the Dirichlet condition on the boundary without being a Friedrichs field since its H^1-norm is infinite.

A particularly significant family of constraints on the boundary of the Anti-de Sitter universe is given by the condition (3.18) that corresponds to

$$\phi_{-1}(t, \mathbf{x}) + \alpha_0\phi_0(t, \mathbf{x}) + \alpha_1\phi_1(t, \mathbf{x}) = 0$$

with

$$0 < \alpha_1, \quad -\frac{1}{2} - \frac{3}{2}\log 2 < \alpha_0 + \frac{1}{2}\log\alpha_1 < \frac{1}{4} - \frac{1}{2}\log 2 - \gamma.$$

In this case the energy is positive and $\sqrt{\mathbb{E}_\alpha(\Phi_0,\Phi_1)}$ is a norm on $\mathfrak{H}_1\times\mathfrak{H}_0$. Hence we can consider the Hilbert space $\mathfrak{K}_1\times\mathfrak{H}_0$ defined as the completion of this space for this norm. We remark that $\mathfrak{H}_1\neq\mathfrak{K}_1$ since

$$\|\phi(\mathbf{x})z^{-\frac{3}{2}}\|^2_{\mathfrak{K}_1}=\|z^{-\frac{3}{2}}\|_0^2\int_{\mathbb{R}^3_{\mathbf{x}}}|\nabla_{\mathbf{x}}\phi(\mathbf{x})|^2d\mathbf{x},$$

$$\|\phi(\mathbf{x})z^{-\frac{3}{2}}\|^2_{\mathfrak{H}_1}=\|\phi(\mathbf{x})z^{-\frac{3}{2}}\|^2_{\mathfrak{K}_1}+\|z^{-\frac{3}{2}}\|^2_{\mathfrak{h}_1}\int_{\mathbb{R}^3_{\mathbf{x}}}|\phi(\mathbf{x})|^2d\mathbf{x}.$$

Then the Cauchy problem is well posed in $\mathfrak{K}_1\times\mathfrak{H}_0$ and the solution is given by a unitary group. Finally (3.33), (3.34) and (4.23) allow to split the solution Φ_α into a massless graviton Φ_G and an orthogonal part $\Phi^\perp$, solutions of (4.1) satisfying:

$$\Phi_\alpha=\Phi_G+\Phi^\perp,\quad \Phi_G(t,\mathbf{x},z)=\phi_{[0]}(t,\mathbf{x})z^{-\frac{3}{2}},$$

where

$$\partial_t^2\phi_{[0]}-\Delta_{\mathbf{x}}\phi_{[0]}=0,\quad \phi_{[0]}(0,\mathbf{x})=\|z^{-\frac{3}{2}}\|_0^{-2}\left\langle\Phi_0(\mathbf{x},.);z^{-\frac{3}{2}}\right\rangle_0,$$
$$\partial_t\phi_{[0]}(0,\mathbf{x})=\|z^{-\frac{3}{2}}\|_0^{-2}\left\langle\Phi_1(\mathbf{x},.);z^{-\frac{3}{2}}\right\rangle_0,$$

and for all $t\in\mathbb{R}$ and almost $\mathbf{x}\in\mathbb{R}^3$,

$$\left\langle\Phi^\perp(t,\mathbf{x},.);z^{-\frac{3}{2}}\right\rangle_0=0.$$

If we expand $\Phi^\perp$ as

$$\Phi^\perp(t,\mathbf{x},z)=\phi_r^\perp(t,\mathbf{x},z)+\phi_0^\perp(t,\mathbf{x})\chi(z)z^{\frac{5}{2}}\log z+\phi_1^\perp(t,\mathbf{x})\chi(z)z^{\frac{1}{2}}+\phi_2^\perp(t,\mathbf{x})z^{-\frac{3}{2}},$$

we conjecture in the spirit of (3.35) and (3.36), that

$$\|\nabla_{t,\mathbf{x}}\phi_0^\perp(t,.)\|_{L^2(\mathbb{R}^3_{\mathbf{x}})},\quad \|\nabla_{t,\mathbf{x}}\phi_1^\perp(t,.)\|_{L^2(\mathbb{R}^3_{\mathbf{x}})},\quad \|\nabla_{t,\mathbf{x}}\phi_2^\perp(t,.)\|_{L^2(\mathbb{R}^3_{\mathbf{x}})}\to 0,\quad |\,t\,|\to\infty,$$

hence

$$\lim_{|t|\to\infty}\|\nabla_{t,\mathbf{x}}\phi_{[0]}(t,.)-\nabla_{t,\mathbf{x}}\phi_2(t,.)\|_{L^2(\mathbb{R}^3_{\mathbf{x}})}=0, \tag{4.30}$$

that is to say, the more singular part of the gravitational wave is asymptotically given by the massless graviton. Taking account of (4.27), we also expect that $\phi_r^\perp$ is asymptotically equal to Friedrichs solutions $\phi_r^\pm$ of the homogenous equation as $t\to\pm\infty$. To summarize, we conjecture that the field in the AdS^5 bulk could be asymptotically split into a massless graviton localized near the conformal boundary, and a Friedrichs solution, that is equal to zero on this boundary:

$$\Phi_\alpha(t,\mathbf{x},z)\approx\phi_{[0]}(t,\mathbf{x})z^{-\frac{3}{2}}+\phi_r^\pm(t,\mathbf{x},z),\quad t\to\pm\infty,\quad \phi_r^\pm(t,\mathbf{x},0)=0.$$

The proof of this result will need a sharp analysis of the spectral properties of the hamiltonian, mainly we would have to establish that its singular continuous spectrum is empty. Then it would be tempting to develop a complete scattering theory based on the sequence of operators

$$\left(\phi_{[0]},\phi_r^-\right)\longmapsto\Phi_\alpha\longmapsto\left(\phi_{[0]},\phi_r^+\right).$$

A related issue is the characterization of the range of the holographic operator Γ_α. We know that this space contains all the massless gravitons, and we have conjectured with (4.30) that it is contained in a set of fields that are asympotically free waves. A precise characterization will provide a complete description of the dynamics on the conformal boundary.

Last but not least, we leave open the deep question on the privileged constraints on the boundary on the Anti-de Sitter universe, among the large family of the boundary conditions that we have introduced in this work. From the mathematical point of view, these boundary conditions should yield an empty singular continuous spectrum, and a range of the holographic operator as large as possible, and accurately characterized. Another challenging question is the choice of the Hilbert space. It would be interesting to investigate alternative choices of hilbertian structures, nay to use Pontryagin spaces, that lead to other boundary conditions. For instance we could use the "cascade model" introduced in [7]. For the physical point of view, since $\frac{1}{z}$ has to be considered as an energy scale E, our boundary constraints intertwine the behaviours of the field at several energy scales as $E \to \infty$. Our physical framework is very poor since we deal just with one linear scalar classical field. We could expect that a rigorous asymptotic analysis of a more complete model involving N gauge quantum fields, would make more clear the link between these different scales and bring out a privileged condition.

5. Appendix

This Appendix is devoted to the proof of Lemma 2.2. We use the Bessel formula that gives the Fourier transform $\hat{f}$ of a spherically symmetric function $f \in L^1(\mathbb{R}^N)$,

$$\hat{f}(\zeta) := \int_{\mathbb{R}^N} e^{-iX.\zeta} f(X)dX = \frac{(2\pi)^{\frac{N}{2}}}{|\zeta|^{\frac{N}{2}-1}} \int_0^\infty J_{\frac{N}{2}-1}(|\zeta|\, r)F(r)r^{\frac{N}{2}}dr, \quad F(|X|) := f(X),$$

to get:

$$\Phi_0(Z) = \frac{1}{8\pi^3\, |Z|^2} \int_0^\infty J_2(z\, |Z|) \frac{z^3}{(z^2-\mu_0)(z^2-\mu_1)(z^2-\mu_2)} dz.$$

We write

$$\frac{1}{z^2-\mu_j} = 2\int_0^\infty e^{-(z^2-\mu_j)t_j^2} t_j dt_j,$$

to obtain

$$\Phi_0(Z) = \frac{1}{\pi^3\, |Z|^2} \int_0^\infty \int_0^\infty \int_0^\infty e^{\mu_0 t_0^2+\mu_1 t_1^2+\mu_2 t_2^2}$$
$$\times \left(\int_0^\infty J_2(z\, |Z|) e^{-z^2(t_0^2+t_1^2+t_2^2)} z^3 dz \right) t_0 t_1 t_2 dt_0 dt_1 dt_2.$$

We recall formula (10.22.51) of [16]:

$$\int_0^\infty J_2(z\, |Z|) e^{-z^2p^2} z^3 dz = \frac{|Z|^2}{8p^6} e^{-\frac{\cancel{Z}^2}{4p^2}},$$

and by replacing it in the previous expression, we deduce that

$$\Phi_0(Z) = \frac{1}{8\pi^3}\int_0^\infty\int_0^\infty\int_0^\infty e^{\mu_0 t_0^2+\mu_1 t_1^2+\mu_2 t_2^2-\frac{|Z|^2}{4(t_0^2+t_1^2+t_2^2)}}\frac{t_0 t_1 t_2}{(t_0^2+t_1^2+t_2^2)^3}dt_0dt_1dt_2.$$

We use the spherical coordinates of $\mathbb{R}^3$, $t_0 = \rho\cos\varphi\sin\theta$, $t_1 = \rho\sin\varphi\sin\theta$, $t_2 = \rho\cos\theta$ to get

$$\begin{aligned}
&\Phi_0(Z)\\
&= \frac{1}{8\pi^3}\int_0^\infty\left(\int_0^{\frac{\pi}{2}}\left(\int_0^{\frac{\pi}{2}} e^{\rho^2\sin^2\theta(\mu_0\cos^2\varphi+\mu_1\sin^2\varphi)}\cos\varphi\sin\varphi d\varphi\right)e^{\mu_2\rho^2\cos^2\theta}\sin^3\theta\cos\theta d\theta\right)e^{-\frac{|Z|^2}{4\rho^2}}\frac{d\rho}{\rho}\\
&= \frac{1}{16\pi^3(\mu_0-\mu_1)}\int_0^\infty\left(\int_0^{\frac{\pi}{2}}\left[e^{\rho^2(\mu_0\sin^2\theta+\mu_2\cos^2\theta)}-e^{\rho^2(\mu_1\sin^2\theta+\mu_2\cos^2\theta)}\right]\cos\theta\sin\theta d\theta\right)e^{-\frac{|Z|^2}{4\rho^2}}\frac{d\rho}{\rho^3}\\
&= \frac{1}{32\pi^3(\mu_0-\mu_1)(\mu_1-\mu_2)}\int_0^\infty e^{\mu_1\rho^2-\frac{|Z|^2}{4\rho^2}}\frac{d\rho}{\rho^5}+\frac{1}{32\pi^3(\mu_1-\mu_2)(\mu_2-\mu_0)}\int_0^\infty e^{\mu_2\rho^2-\frac{|Z|^2}{4\rho^2}}\frac{d\rho}{\rho^5}\\
&\quad+\frac{1}{32\pi^3(\mu_2-\mu_0)(\mu_0-\mu_1)}\int_0^\infty e^{\mu_0\rho^2-\frac{|Z|^2}{4\rho^2}}\frac{d\rho}{\rho^5}.
\end{aligned}$$

We can express the modified Bessel function K_2 by formula (10.32.10) of [16] to get

$$\int_0^\infty e^{\mu_j\rho^2-\frac{|Z|^2}{4\rho^2}}\frac{d\rho}{\rho^5} = -\frac{8\mu_j}{|Z|^2}K_2(\sqrt{-\mu_j}\,|Z|),$$

therefore we obtain the expression of Φ_0:

$$\begin{aligned}
\Phi_0(Z) = -\frac{1}{4\pi^3\,|Z|^2}\Bigg[&\frac{\mu_1}{(\mu_0-\mu_1)(\mu_1-\mu_2)}K_2(\sqrt{-\mu_1}\,|Z|)\\
&+\frac{\mu_2}{(\mu_1-\mu_2)(\mu_2-\mu_0)}K_2(\sqrt{-\mu_2}\,|Z|)\\
&+\frac{\mu_0}{(\mu_2-\mu_0)(\mu_0-\mu_1)}K_2(\sqrt{-\mu_0}\,|Z|)\Bigg].
\end{aligned}$$

We directly obtain the expression of φ_j with a change of variable in formula (II, 3 ; 20) of [18]:

$$\varphi_j(Z) = -\frac{\mu_j}{8\pi^3\,|Z|^2}K_2(\sqrt{-\mu_j}\,|Z|). \tag{5.1}$$

We know that $K_2(z)$ is an analytic function on the surface of the logarithm, and for $z>0$ we have the following asymptotics (see [16], formulae (10.25.3):

$$K_2(z)\sim\sqrt{\frac{\pi}{2z}}e^{-z},\quad z\to\infty,\quad K_2(z)\sim\frac{2}{z^2},\quad z\to 0^+.$$

We deduce that Φ_0 and φ_j are in $L^1(\mathbb{R}^6)$. To derive the asymptotic forms near zero, we use formula (10.31.1) of [16] that allows to establish that for $z>0$:

$$K_2(z) = \frac{2}{z^2}-\frac{1}{2}-\frac{z^2}{8}\log z+z^2F(z^2)+z^4G(z^2)\log z, \tag{5.2}$$

where F and G are entire and if γ denotes Euler's constant, we have:

$$F(0) = \frac{4\log 2 + 3 - 4\gamma}{32}. \tag{5.3}$$

Equation (2.20) follows from (5.1) and (5.2) with

$$\begin{aligned} F_j(Z) = (1-\chi(Z))\varphi_j(Z) \\ + \chi(Z)\Bigg(-\frac{\mu_j^2}{128\pi^3}\log(-\mu_j) + \frac{\mu_j^2}{8\pi^3}F(-\mu_j|Z|^2) \\ -\frac{\mu_j^3|Z|^2}{8\pi^3}G(-\mu_j|Z|^2)\log(-\mu_j|Z|)\Bigg). \end{aligned}$$

Since $(1-\chi)\varphi_j \in H^\infty(\mathbb{R}^6)$ by elliptic regularity and $|Z|^2\log(|Z|) \in H^4_{loc}(\mathbb{R}^6)$ we conclude that $F_j \in H^4(\mathbb{R}^6)$, and (5.3) gives (2.22). Finally we have

$$\Phi_0 = 2\left(\frac{\varphi_1}{(\mu_0-\mu_1)(\mu_1-\mu_2)} + \frac{\varphi_2}{(\mu_1-\mu_2)(\mu_2-\mu_0)} + \frac{\varphi_0}{(\mu_2-\mu_0)(\mu_0-\mu_1)}\right),$$

hence (2.21) follows from (2.20) with

$$G_0 = 2\left(\frac{F_1}{(\mu_0-\mu_1)(\mu_1-\mu_2)} + \frac{F_2}{(\mu_1-\mu_2)(\mu_2-\mu_0)} + \frac{F_0}{(\mu_2-\mu_0)(\mu_0-\mu_1)}\right),$$

and with this expression of G_0, (2.23) follows from (2.22). At last the link between (U_r, u_0, u_1, u_2) and (V_r, v_0, v_1, v_2) is easily deduced from (2.20) and (2.21) *via* some tedious computations:

$$\begin{aligned} v_2 = \frac{1}{4\pi^3}(u_1+u_2), \quad v_1 = \frac{1}{16\pi^3}(\mu_1u_1+\mu_2u_2), \\ v_0 = \frac{1}{64\pi^3}(2u_0 - \mu_1^2u_1 - \mu_2^2u_2), \quad V_r = U_r + u_0G_0 + u_1F_1 + u_2F_2, \end{aligned} \tag{5.4}$$

$$\begin{aligned} u_1 = \frac{16\pi^3v_1 - 4\pi^3\mu_2v_2}{\mu_1-\mu_2}, \quad u_2 = \frac{16\pi^3v_1 - 4\pi^3\mu_1v_2}{\mu_2-\mu_1}, \\ u_0 = 32\pi^3v_0 + 8\pi^3(\mu_1+\mu_2)v_1 - 2\pi^3\mu_1\mu_2v_2, \\ u_r = v_r + \frac{4v_1(\mu_1+\mu_2)-v_2\mu_1\mu_2}{16}\chi(Z)\log(|Z|) - \frac{16\pi^3v_1-4\pi^3\mu_2v_2}{\mu_1-\mu_2}F_1 \\ -\frac{16\pi^3v_1-4\pi^3\mu_1v_2}{\mu_2-\mu_1}F_2, \end{aligned} \tag{5.5}$$

$$\begin{aligned} U_r = V_r - \left(32\pi^3v_0 + 8\pi^3(\mu_1+\mu_2)v_1 - 2\pi^3\mu_1\mu_2v_2\right)G_0 \\ -\frac{16\pi^3v_1-4\pi^3\mu_2v_2}{\mu_1-\mu_2}F_1 - \frac{16\pi^3v_1-4\pi^3\mu_1v_2}{\mu_2-\mu_1}F_2. \end{aligned}$$

These expressions show that the coordinates u_1, u_2, u_0 depend on u, μ_1, μ_2, but are independent of the choice of μ_0. Furthermore, since $\chi(Z)\log(|Z|) \in H^{3-\epsilon}(\mathbb{R}^6)$ and $F_j,\ G_0 \in H^4(\mathbb{R}^6)$, we see that the $\|.\|_{\mathbb{H}_k}$-norms and the $|.|_k$-norms are equivalent.

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