Commun. Math. Phys. 283, 127–167 (2008)
Digital Object Identifier (DOI) 10.1007/s00220-008-0564-8

Communications in
Mathematical Physics

The Dirac System on the Anti-de Sitter Universe

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Received: 20 July 2007 / Accepted: 20 March 2008
Published online: 18 July 2008 –

Abstract: We investigate the global solutions of the Dirac equation on the Anti-de-Sitter Universe. Since this space is not globally hyperbolic, the Cauchy problem is not, *a priori*, well-posed. Nevertheless we can prove that there exists unitary dynamics, but its uniqueness crucially depends on the ratio beween the mass M of the field and the cosmological constant $\Lambda > 0$: it appears a critical value, $\Lambda/12$, which plays a role similar to the Breitenlohner-Freedman bound for the scalar fields. When $M^2 \geq \Lambda/12$ there exists a unique unitary dynamics. On the contrary, for the light fermions satisfying $M^2 < \Lambda/12$, we construct several asymptotic conditions at infinity, such that the problem becomes well-posed. In all the cases, the spectrum of the hamiltonian is discrete. We also prove a result of equipartition of the energy.

I. Introduction

There has been much recent interest in the field theory in the covering space of the Anti-de-Sitter space-time $CAdS$, that appears as the ground state of the gauged supergravity group [15]. This lorentzian manifold is the maximally symmetric solution of the Einstein equations with cosmological constant $-\Lambda < 0$ included. Its topology is $\mathbb{R}_t \times \mathbb{R}^3_X$, but its causality is non-trivial because it is non-globally hyperbolic: the Cauchy data on $\{t = 0\} \times \mathbb{R}^3$ determines the evolution of the fields only in a region D, bounded by a null hypersurface, called a Cauchy horizon. More precisely D is defined by $\mid t \mid < \sqrt{\frac{3}{\Lambda}}\left(\frac{\pi}{2} - \arctan\left(\sqrt{\frac{\Lambda}{3}} \mid X \mid\right)\right)$. Thus we can think that to specify the physics apart from D, we have to impose some asymptotic constraint at infinity as $\mid X \mid \to \infty$. Since the conformal boundary of $CAdS$ is timelike, this condition can be considered as a boundary condition. It is exactly the case for the massless, conformally coupled scalar fields that are conformally invariant in $CAdS$, and these fields have been studied in this spirit by Avis, Isham and Storey in [1]. For the massive fields the situation is different because the gravitational potential relative to any origin increases at large spatial distances from the origin. It causes confinement of massive particles and prevents them from escaping to infinity. In fact, the situation is rather subtle and depends on the ratio between the mass from the field and the cosmological constant. This phenomenon has been discovered by Breitenlohner and Freedman [7,8], who have showed the existence of two critical values, the B-F bounds, for the scalar fields; the first one assures the positivity of the energy, and the second one assures the uniqueness of the dynamics. In this paper, we establish a similar result for the Dirac fields.[1] The square of the mass of the spinors is compared with a unique B-F bound that is equal to $\Lambda/12$. We shall see that the physics of the heavy fermions ($M^2 \geq \Lambda/12$) is uniquely determined, but there exists a lot of possible dynamics for the light fermions ($M^2 < \Lambda/12$), involving the asymptotic forms, at the $CAdS$ infinity, of classical boundary conditions, local or non-local: *MIT-bag, Chiral, APS* conditions, etc. From the mathematical point of view, the solutions of the initial value problem are given in D by the Leray-Hadamard theorem for the hyperbolic equations $\partial_t \Psi = \mathbf{H}(X, \partial_X)\Psi$, and on the whole space-time, we solve the Cauchy problem by a spectral approach, i.e. we look for the solutions formally given by $\Psi(t) = e^{it\mathbf{H}}\Psi(0)$. Therefore we have to construct self-adjoint extensions of the Dirac hamiltonian $\mathbf{H}(X, \partial_X)$. This method was used by A. Ishibashi and R.M. Wald [21,22], for the integer spin fields.

The paper is organized as follows. In Part II, we briefly describe the Anti-de-Sitter manifold, mainly the different systems of coordinates and the properties of the null and time-like geodesics. The explicit forms of the Dirac equation on $CAdS$ are described in Sect. III, and we state the main result, Theorem III.4. We perform the spinoidal spherical harmonics decomposition in the following part. The asymptotic conditions and the self-adjoint extensions are discussed in the final section. In a short appendix, we present a new proof of the B-F bounds for the Klein-Gordon equation.

We end this introduction with some bibliographic information. Above all, we have to mention the works treating the scalar fields on $CAdS$, [1,7,8,22]. We refer to [15,19,29] for a presentation of the Anti-de-Sitter universe. There are many mathematical works on the one-half spin field on curved space-time, in particular [4,17,18,25–28]. The gravitational potential plays the role of a variable mass that tends to the infinity at the space infinity; the rather similar Dirac equation on Minkowski space with increasing potential has been considered in [23,34,38]. The literature on the boundary value problems for the Dirac system is huge ; among important contributions, we can cite [5,6,9,10,16,20]. There are few papers concerning the deep problem of the global existence of fields on the non-globally hyperbolic lorentzian manifolds, in particular [2,11,13,21,32,37] .

II. The Anti-de-Sitter Space Time

Given $\Lambda > 0$, the anti-de-Sitter space AdS is defined as the quadric

$$(X^1)^2 + (X^2)^2 + (X^3)^2 - U^2 - V^2 = -\frac{3}{\Lambda}$$

embedded in the flat 5-dimensional space $\mathbb{R}^5$ with the metric

$$ds^2 = dU^2 + dV^2 - (dX^1)^2 - (dX^2)^2 - (dX^3)^2.$$

[1] The author thanks the anonymous referee for his valuable comments on the B-F bounds.

AdS is the maximally symmetric solution of Einstein's equations with an attractive cosmological constant $-\Lambda < 0$. To describe AdS it is convenient to set

$$U = R\cos\left(\sqrt{\frac{\Lambda}{3}}T\right), \quad V = R\sin\left(\sqrt{\frac{\Lambda}{3}}T\right),$$

then we can see that

$$AdS = S^1_T \times \mathbb{R}^3_{(X^1,X^2,X^3)},$$

$$ds^2_{AdS} = \frac{\Lambda}{3}R^2dT^2 + dR^2 - (dX^1)^2 - (dX^2)^2 - (dX^3)^2,$$
$$R = \sqrt{(X^1)^2 + (X^2)^2 + (X^3)^2 + \frac{3}{\Lambda}}.$$

For constant T, the slice $\{T\} \times \mathbb{R}^3$ is exactly the 3-dimensional hyperbolic space $\mathbb{H}^3$ that is the upper sheet of the hyperboloid $(X^1)^2 + (X^2)^2 + (X^3)^2 - W^2 = -\frac{3}{\Lambda}$ in the Minkowski space $\mathbb{R}^4_{(X^1,X^2,X^3,W)}$ with the metric $(dX^1)^2 + (dX^2)^2 + (dX^3)^2 - dW^2$. It is useful to use the spherical coordinates

$$r = \sqrt{(X^1)^2 + (X^2)^2 + (X^3)^2} \in [0, \infty[, \quad \textit{and if} \quad 0 < r, \quad \omega = \frac{1}{r}(X^1, X^2, X^3) \in S^2,$$

for which the hyperbolic metric becomes

$$ds^2_{\mathbb{H}^3} = \left(1 + \frac{\Lambda}{3}r^2\right)^{-1} dr^2 + r^2 d\omega^2,$$

where $d\omega^2$ is the euclidean metric on the unit two-sphere S^2,

$$d\omega^2 = d\theta^2 + \sin^2\theta d\varphi^2, \quad 0 \le \theta \le \pi, \quad 0 \le \varphi < 2\pi.$$

We shall use the nice picture of the hyperbolic space, the so called Poincaré ball. We introduce

$$1 \le j \le 3, \quad x^j = \sqrt{\frac{\Lambda}{3}}\frac{1}{1 + \sqrt{1 + \frac{\Lambda}{3}r^2}}X^j,$$

$$\varrho = \sqrt{\frac{\Lambda}{3}}\frac{r}{1 + \sqrt{1 + \frac{\Lambda}{3}r^2}} \in [0, 1[,$$

then $\mathbb{H}^3$ can be seen as the unit ball

$$\mathbb{B} = \{\mathbf{x} := (x^1, x^2, x^2) \in \mathbb{R}^3; \quad \varrho^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 < 1\}$$

endowed with the metric

$$ds^2_{\mathbb{H}^3} = \frac{3}{\Lambda}\frac{4}{\left(1 - \varrho^2\right)}\left(d\varrho^2 + \varrho^2 d\omega^2\right), \quad 0 \le \varrho < 1, \quad \omega \in S^2.$$

We note that the time coordinate T is periodic, and this property implies the existence of closed timelike curves. To avoid this unpleasant fact, we replace $T \in S^1$ by $t \in \mathbb{R}$, i.e. we change the topology, and we consider in this paper the Universal Covering Space of the anti-de-Sitter space-time, that is the lorentzian manifold $CAdS := (\mathcal{M}, g)$ defined by

$$\mathcal{M} = \mathbb{R}_t \times \mathbb{R}^3_{(X^1,X^2,X^3)} = \mathbb{R}_t \times \mathbb{B}_{(x^1,x^2,x^3)},$$

with the metric,

$$g_{\mu\nu}dx^\mu dx^\nu = \left(1 + \frac{\Lambda}{3}r^2\right) dt^2 - \left(1 + \frac{\Lambda}{3}r^2\right)^{-1} dr^2 - r^2 d\omega^2, \quad 0 \le r < \infty, \quad \omega \in S^2,$$
$$= \left(\frac{1+\varrho^2}{1-\varrho^2}\right)^2 dt^2 - \frac{3}{\Lambda}\frac{4}{(1-\varrho^2)}\left(d\varrho^2 + \varrho^2 d\omega^2\right), \quad 0 \le \varrho < 1, \quad \omega \in S^2.$$

It will be useful to introduce a third radial coordinate,

$$x = \arctan\left(\sqrt{\frac{\Lambda}{3}}r\right) = 2\arctan\varrho. \tag{II.1}$$

Then the Anti-de-Sitter manifold can be described by:

$$(t, x, \theta, \varphi) \in \mathbb{R} \times [0, \frac{\pi}{2}[\times[0, \pi] \times [0, 2\pi[,$$

$$g_{\mu\nu}dx^\mu dx^\nu = \left(1 + \tan^2 x\right) \tilde{g}_{\mu\nu}dx^\mu dx^\nu,$$

where $\tilde{g}$ is given by

$$\tilde{g}_{\mu\nu}dx^\mu dx^\nu = dt^2 - \frac{3}{\Lambda}\left(dx^2 + \sin^2 x d\theta^2 + \sin^2 x \sin^2\theta d\varphi^2\right).$$

Therefore, if the 3-sphere S^3 is parametrized by $(x, \theta, \varphi) \in [0, \pi] \times [0, \pi] \times [0, 2\pi[$, and S^3_+ is the upper hemisphere $[0, \frac{\pi}{2}[_x \times [0, \pi]_\theta \times [0, 2\pi[_\varphi$, $CAdS$ can be considered as conformally equivalent to the submanifold $\widetilde{\mathcal{M}} = \mathbb{R}_t \times S^3_+$ of the Einstein cylinder $(\mathcal{E}, \tilde{g})$,

$$\mathcal{E} := \mathbb{R}_t \times S^3, \tag{II.2}$$

and the crucial point is that the boundary $\partial\widetilde{\mathcal{M}} = \mathbb{R}_t \times \left\{x = \frac{\pi}{2}\right\} \times S^2_{\theta,\varphi}$ is time-like. Nevertheless, we should note that, unlike the black-hole horizon of the Schwarzschild metric (that is a characteristic submanifold of the Kruskal space-time), the time-like infinity of $CAdS$, like the cosmological horizon of the De Sitter universe, (or a rainbow) is seen in the same way by any observer: since $CAdS$ is *frame-homogeneous* (i.e. any Lorentz frame on $CAdS$ can be carried to any other by the differential map of an isometry of $CAdS$), no point is privileged.

Finally we recall that the null geodesics of AdS are straight lines in $\mathbb{R}^5_{(X^1,X^2,X^3,U,V)}$ and the timelike geodesics are ellipses, intersection of AdS with the 2-planes of $\mathbb{R}^5$ passing through the origin 0. As a consequence, $CAdS$ is geodesically complete, and time oriented by the Killing vector field ∂_t, but its causality is not at all trivial: (1)

given a point P on the slice $t=0$, the future-pointing null geodesics starting from P form a curving cone of which the boundary approaches but does not reach the slice $t=\frac{\pi}{2}\sqrt{\frac{3}{\Lambda}}$, hence $CAdS$ is not globally hyperbolic; (2) the future-pointing timelike geodesics on $CAdS$ starting from P, all meet a conjugate point Q at $t=\pi\sqrt{\frac{3}{\Lambda}}$, P and Q project on antipodal points of AdS. Therefore the time-like geodesics on $CAdS$ can be parametrized by $(t,\mathbf{x}(t))_{t\in\mathbb{R}}$, where the function $t\mapsto\mathbf{x}(t)$ is t-periodic, with period $2\pi\sqrt{\frac{3}{\Lambda}}$. These unusual properties yield important consequences for the propagation of the fields: (1) suggests that we could have to add some condition at the "infinity" $S^2=\partial\mathbb{B}$ to solve an initial value problem, at least for the massless fermions; nevertheless, since the massive particles propagate along the time-like geodesics, (2) seems to imply that such a condition is not necessary for the massive fields. In fact, the situation is rather subtle and depends on the ratio between the square of the mass of the fermion, and the cosmological constant. We shall see that no asymptotic constraint at infinity is necessary for the heavy spinors, but there are many possible physical constraints for the light masses. In all the cases, the spectrum of the hamiltonian of the massive fields is discrete.

III. The Dirac Equation on $CAdS$

We consider the Dirac equation with mass $M\in\mathbb{R}$ on a 3+1 dimensional lorentzian manifold $(\mathcal{M},g)$:

$$i\gamma^{\mu}_{(g)}\nabla_{\mu}\psi-M\psi=0. \tag{III.1}$$

The notations are the following. ∇_{μ} are the covariant derivatives, $\gamma^{\mu}_{(g)}$, $0\leq\mu\leq3$, are the Dirac matrices, unique up to a unitary transform, satisfying:

$$\gamma^{0*}_{(g)}=\gamma^{0}_{(g)},\quad \gamma^{j*}_{(g)}=-\gamma^{j}_{(g)},\ \ 1\leq j\leq3,\quad \gamma^{\mu}_{(g)}\gamma^{\nu}_{(g)}+\gamma^{\nu}_{(g)}\gamma^{\mu}_{(g)}=2g^{\mu\nu}\mathbf{1}. \tag{III.2}$$

Here A^* denotes the conjugate transpose of any complex matrix A. We make the following choices for the Dirac matrices on the Minkowski space time $\mathbb{R}^{1+3}$: γ^{μ} are the 4×4 matrices of the Pauli-Dirac representation given for $\mu=0,1,2,3$ by:

$$\gamma^0=\begin{pmatrix}I&0\\0&-I\end{pmatrix},\quad \gamma^j=\begin{pmatrix}0&\sigma^j\\-\sigma^j&0\end{pmatrix},$$

where

$$I=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \sigma^1=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \sigma^2=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma^3=\begin{pmatrix}0&-i\\i&0\end{pmatrix}.$$

We also introduce another Dirac matrix that plays an important role in the boundary problems:

$$\gamma^5:=-i\gamma^0\gamma^1\gamma^2\gamma^3=\begin{pmatrix}0&I\\I&0\end{pmatrix}, \tag{III.3}$$

that satisfies

$$\gamma^5\gamma^{\mu}+\gamma^{\mu}\gamma^5=0,\quad 0\leq\mu\leq3.$$

We know that when the metric is spherically symmetric,

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = F(r)dt^2 - \frac{1}{F(r)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

then, if we choose the local orthonormal Lorentz frame $\{e_a^{\mu}, a = 0, 1, 2, 3\}$ defined by

$$e_a{}^{\mu} = \left|g^{\mu\mu}\right|^{\frac{1}{2}}, \quad if \quad \mu = a, \quad e_a{}^{\mu} = 0 \ \ if \quad \mu \neq a,$$

the Dirac equation has the following form in (t, r, θ, φ) coordinates (see e.g. [26–28]):

$$\left\{iF^{-\frac{1}{2}}\gamma^0\frac{\partial}{\partial t} + iF^{\frac{1}{2}}\gamma^1\left(\frac{\partial}{\partial r} + \frac{1}{r} + \frac{F'}{4F}\right) + \frac{i}{r}\gamma^2\left(\frac{\partial}{\partial\theta} + \frac{1}{2\tan\theta}\right)\right.$$
$$\left. + \frac{i}{r\sin\theta}\gamma^3\frac{\partial}{\partial\varphi} - M\right\}\psi = 0.$$

For the Anti-de-Sitter manifold we have

$$F(r) = \left(1 + \frac{\Lambda}{3}r^2\right),$$

and it is convenient to make a first change of spinor ; we use the radial coordinate (II.1), and we put

$$\Phi(t, x, \theta, \varphi) := r\left(1 + \frac{\Lambda}{3}r^2\right)^{\frac{1}{4}}\psi(t, r, \theta, \varphi). \tag{III.4}$$

Then we obtain the Dirac equation on the Anti-de-Sitter universe with the coordinates $t \in \mathbb{R}$, $x \in [0, \frac{\pi}{2}[$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi[$:

$$\sqrt{\frac{3}{\Lambda}}\frac{\partial}{\partial t}\Phi + \gamma^0\gamma^1\frac{\partial}{\partial x}\Phi + \frac{1}{\sin x}\left[\gamma^0\gamma^2\left(\frac{\partial}{\partial\theta} + \frac{1}{2\tan\theta}\right) + \frac{1}{\sin\theta}\gamma^0\gamma^3\frac{\partial}{\partial\varphi}\right]\Phi$$
$$+\frac{i}{\cos x}M\sqrt{\frac{3}{\Lambda}}\gamma^0\Phi = 0. \tag{III.5}$$

Since the part of this differential operator involving ∂_t, ∂_x is with constant coefficients, the form of this equation is convenient to make a separation of variables by using the generalized spin spherical harmonics. But this decomposition has an inconvenience: since the one-half spin harmonics are not smooth functions on S^2, the functional framework involves spaces that are different from the usual Sobolev spaces on S^2 as we shall see in the following part. It will also be useful to write the Dirac equation with the coordinates $(t, \varrho, \theta, \varphi) \in \mathbb{R} \times [0, 1[\times [0, \pi] \times [0, 2\pi[$. We put

$$\underset{\sim}{\Phi}(t, \varrho, \theta, \varphi) := \Phi(t, x, \theta, \varphi),$$

and the Dirac equation becomes:

$$\sqrt{\frac{3}{\Lambda}}\frac{\partial}{\partial t}\underset{\sim}{\Phi} + \left(\frac{1+\varrho^2}{2}\right)\gamma^0$$
$$\times\left[\gamma^1\frac{\partial}{\partial\varrho} + \frac{1}{\varrho}\gamma^2\left(\frac{\partial}{\partial\theta} + \frac{1}{2\tan\theta}\right) + \frac{1}{\varrho\sin\theta}\gamma^3\frac{\partial}{\partial\varphi} + \frac{2iM}{1-\varrho^2}\sqrt{\frac{3}{\Lambda}}\right]\underset{\sim}{\Phi} = 0.$$

In the part involving γ^1, γ^2, γ^3, we recognize the usual Dirac operator in spherical coordinates on $\mathbb{R}^3$ with the euclidean metric. This is a nice way to get the Dirac operator on $CAdS$ in cartesian coordinates. Adapting an approach of [33], we introduce

$$a := \frac{1}{2}\left(I - \gamma^1\gamma^2 - \gamma^2\gamma^3 - \gamma^3\gamma^1\right),$$

$$S(\theta,\varphi) := e^{\frac{\varphi}{2}\gamma^1\gamma^2} e^{\frac{\theta}{2}\gamma^3\gamma^1} a. \tag{III.6}$$

We easily check that

$$aa^* = I, \quad SS^* = I,$$

$$\gamma^1 a = a\gamma^2, \quad \gamma^2 a = a\gamma^3, \quad \gamma^3 a = a\gamma^1.$$

We put

$$\underset{\sim}{\gamma}^1(\varrho,\theta) := \gamma^1, \quad \underset{\sim}{\gamma}^2(\varrho,\theta) := \frac{1}{\varrho}\gamma^2, \quad \underset{\sim}{\gamma}^3(\varrho,\theta) := \frac{1}{\varrho\sin\theta}\gamma^3,$$

$$\begin{cases} \tilde{\gamma}^1(\varrho,\theta,\varphi) := \cos\varphi\sin\theta\gamma^1 + \sin\varphi\sin\theta\gamma^2 + \cos\theta\gamma^3, \\ \tilde{\gamma}^2(\varrho,\theta,\varphi) := \frac{1}{\varrho}\left(\cos\varphi\cos\theta\gamma^1 + \sin\varphi\cos\theta\gamma^2 - \sin\theta\gamma^3\right), \\ \tilde{\gamma}^3(\varrho,\theta,\varphi) := \frac{1}{\varrho\sin\theta}\left(-\sin\varphi\gamma^1 + \cos\varphi\gamma^2\right). \end{cases}$$

Tedious calculations give:

$$1 \le j \le 3, \quad S(\theta,\varphi)\underset{\sim}{\gamma}^j(\varrho,\theta) = \tilde{\gamma}^j(\varrho,\theta,\varphi)S(\theta,\varphi).$$

The cartesian coordinates $\mathbf{x} := (x^1, x^2, x^3)$ on $\mathbb{B}$ being

$$x^1 = \varrho\cos\varphi\sin\theta, \quad x^2 = \varrho\sin\varphi\sin\theta, \quad x^3 = \varrho\cos\theta, \tag{III.7}$$

we define the spinors Ψ, $\tilde{\Phi}$ on $\mathbb{B}$ by the relations

$$\Psi(x^1,x^2,x^3) = \tilde{\Phi}(\varrho,\theta,\varphi) := \frac{1}{\varrho}S(\theta,\varphi)\underset{\sim}{\Phi}(\varrho,\theta,\varphi),$$

and the Dirac operators

$$\begin{cases} \mathbb{D} := \gamma^1\frac{\partial}{\partial x^1} + \gamma^2\frac{\partial}{\partial x^2} + \gamma^3\frac{\partial}{\partial x^3}, \\ \tilde{\mathbb{D}} := \tilde{\gamma}^1\frac{\partial}{\partial\varrho} + \tilde{\gamma}^2\frac{\partial}{\partial\theta} + \tilde{\gamma}^3\frac{\partial}{\partial\varphi}, \\ \underset{\sim}{\mathbb{D}} := \underset{\sim}{\gamma}^1\frac{\partial}{\partial\varrho} + \underset{\sim}{\gamma}^2\left(\frac{\partial}{\partial\theta} + \frac{1}{2\tan\theta}\right) + \underset{\sim}{\gamma}^3\frac{\partial}{\partial\varphi}. \end{cases}$$

We omit the direct calculus that gives the links between these operators:

Lemma III.1.

$$\left(\mathbb{D}\Psi\right)(x^1,x^2,x^3)=\left(\tilde{\mathbb{D}}\tilde{\Phi}\right)(\varrho,\theta,\varphi)=\frac{1}{\varrho}S(\theta,\varphi)\left(\underset{\sim}{\mathbb{D}}\underset{\sim}{\Phi}\right)(\varrho,\theta,\varphi).$$

We denote $\mathbf{S}$ the operator that relates the spinors in cartesian and spherical coordinates:

$$\mathbf{S}:\Phi\mapsto\mathbf{S}\Phi=\Psi,\quad \Psi(x^1,x^2,x^3):=\frac{1}{\tan\left(\frac{x}{2}\right)}S(\theta,\varphi)\Phi(x,\theta,\varphi). \tag{III.8}$$

Then, if $\Phi(t,.)$ is a solution of (III.5), the Dirac equation satisfied by $\Psi(t,.):=\mathbf{S}\Phi(t,.)$ for $t\in\mathbb{R}$, $\mathbf{x}=(x^1,x^2,x^3)\in\mathbb{B}$ has the form:

$$\sqrt{\frac{3}{\Lambda}}\gamma^0\frac{\partial}{\partial t}\Psi+\left(\frac{1+\varrho^2}{2}\right)\left[\gamma^1\frac{\partial}{\partial x^1}+\gamma^2\frac{\partial}{\partial x^2}+\gamma^3\frac{\partial}{\partial x^3}+\frac{2iM}{1-\varrho^2}\sqrt{\frac{3}{\Lambda}}\right]\Psi=0. \tag{III.9}$$

Since the charge of the spinor is the formally conserved L^2 norm, it is natural to introduce the Hilbert space:

$$\mathbf{L}^2:=\left[L^2\left(\mathbb{B},\frac{2}{1+\varrho^2}\mathbf{dx}\right)\right]^4, \tag{III.10}$$

and given $\Psi_0\in\mathbf{L}^2$ we want to solve the initial problem, *i.e.* to find a unique

$$\Psi\in C^0(\mathbb{R}_t;\mathbf{L}^2) \tag{III.11}$$

solution of (III.9) satisfying:

$$\Psi(t=0,.)=\Psi_0(.), \tag{III.12}$$

and the conservation law:

$$\forall t\in\mathbb{R},\quad \parallel\Psi(t)\parallel_{\mathbf{L}^2}=\parallel\Psi_0\parallel_{\mathbf{L}^2}. \tag{III.13}$$

Moreover, since $\frac{\partial}{\partial t}$ is a Killing vector field on $CAdS$, it is natural to assume that

$$t\in\mathbb{R}\longmapsto(\Psi_0\mapsto\Psi(t)), \tag{III.14}$$

is a group acting on $\mathbf{L}^2$. Therefore we look for strongly continuous unitary groups $U(t)$ on $\mathbf{L}^2$ that solve (III.5). According to the Stone theorem, the problem consists in finding self-adjoint realizations on $\mathbf{L}^2$ of the differential operator

$$\mathbf{H}_M:=i\left(\frac{1+\varrho^2}{2}\right)\gamma^0\left[\gamma^1\frac{\partial}{\partial x^1}+\gamma^2\frac{\partial}{\partial x^2}+\gamma^3\frac{\partial}{\partial x^3}+\frac{2iM}{1-\varrho^2}\sqrt{\frac{3}{\Lambda}}\right], \tag{III.15}$$

with domain

$$D(\mathbf{H}_M)=\left\{\Psi\in\mathbf{L}^2;\ \mathbf{H}_M\Psi\in\mathbf{L}^2\right\}, \tag{III.16}$$

by adding suitable constraints at the *CAdS* infinity $\varrho=1$. The answer crucially depends on the mass of the spinor.

First we discuss the massless case. When $M=0$, the Dirac system is conformal invariant, and it is equivalent to solving the Cauchy problem in the half of the Einstein cylinder, $\mathbb{R}_t \times S^3_+$. Therefore we can extend the initial data from the hemisphere S^3_+ to the whole sphere S^3, and solve the Cauchy problem on the Einstein cylinder $\mathbb{R}_t \times S^3$. This is tantamount to solving Eq. (III.9) on $\mathbb{R}_t \times \mathbb{R}^3_{\mathbf{x}}$, instead of $\mathbb{R}_t \times \mathbb{B}_{\mathbf{x}}$. This approach was used by S.J. Avis, C.J. Isham, D. Storey [1] for the scalar field, and later, by Y. Choquet-Bruhat for the Yang-Mills-Higgs equations [11]. By this way, we impose no boundary condition at the $CAdS$ infinity, or, in other words, a "perfectly transparent" boundary condition, and we easily obtain global solutions on $CAdS$. We have to remark that since there exists a lot of ways to extend the initial data, such a solution is not uniquely determined by the Cauchy data on S^3_+. Moreover the effect of this "perfectly transparent" condition is to recirculate the energy: the conserved charge is the L^2-norm on S^3 while the L^2-norm on S^3_+ is changing in time, and so (III.13) is not satisfied. In order to assure the conservation (III.13), we can take another route, and impose some "reflecting" boundary conditions on $\{x = \frac{\pi}{2}\} \times S^2$. In [1], several conditions are discussed for the scalar massless field. For the Dirac equation, we note that when $M = 0$, Eq. (III.9) has smooth coefficients up to the boundary $\mid \mathbf{x} \mid = 1$. Therefore, in the massless case, we deal with a classical mixed hyperbolic problem, and different boundary conditions for the Dirac system with regular potential are well known (see e.g. [5,6,9,10,16,20]). We recall an important local boundary condition for the Dirac spinors defined on some open domain Ω of the space-time, the so called generalized *MIT-bag* condition:

$$n_\mu \gamma^\mu \Psi(t, x^1, x^2, x^3) = i e^{i\alpha\gamma^5} \Psi(t, x^1, x^2, x^3), \quad (t, x^1, x^2, x^3) \in \partial\Omega,$$

where n^μ is the outgoing normal quadrivector at $\partial\Omega$ and $\alpha \in \mathbb{R}$ is the chiral angle. When $\alpha = 0$ this is the *MIT-bag* condition for the hadrons and when $\alpha = \pi$ this is the *Chiral* condition. Another fundamental boundary condition is the non-local *APS* condition introduced by M.F. Atiyah, V. K. Patodi, and I. M. Singer (see e.g. [6]) and defined by

$$\mathbf{1}_{]0,\infty[}\left(D_{\partial\Omega}\right) \Psi = 0 \;\; on \;\; \partial\Omega,$$

where $D_{\partial\Omega}$ is the Dirac operator on $\partial\Omega$. More recently, O. Hijazi, S. Montiel, A. Roldan [20] have introduced the *mAPS* condition:

$$\mathbf{1}_{]0,\infty[}\left(D_{\partial\Omega}\right) \left(Id - n_\mu \gamma^\mu\right) \Psi = 0 \;\; on \;\; \partial\Omega.$$

For $\Omega = \mathbb{R}_t \times \mathbb{B}$, these boundary conditions become

$$\mathcal{B}\Psi(t, \omega) = 0, \quad (t, \omega) \in \mathbb{R} \times S^2, \tag{III.17}$$

where

$$MIT - bag\ condition: \quad \mathcal{B}_{MIT} = \tilde{\gamma}^1 + iId, \tag{III.18}$$

$$Chiral\ condition: \quad \mathcal{B}_{CHI} = \tilde{\gamma}^1 - iId, \tag{III.19}$$

$$APS\ condition: \quad \mathcal{B}_{APS} = \mathbf{1}_{]0,\infty[}\left(D_{S^2}\right), \tag{III.20}$$

$$mAPS\ condition: \quad \mathcal{B}_{mAPS} = \mathbf{1}_{]0,\infty[}\left(D_{S^2}\right)\left(\tilde{\gamma}^1 + Id\right), \tag{III.21}$$

where D_{S^2} is the intrinsic Dirac operator on the two-sphere:

$$\widetilde{D_{S^2}\Psi} = i\gamma^0 \left(\tilde{\gamma}^2 \frac{\partial}{\partial\theta} + \tilde{\gamma}^3 \frac{\partial}{\partial\varphi} \right) \tilde{\Phi}.$$

We conclude that there exists many unitary dynamics for the massless spin-$\frac{1}{2}$ field on $CAdS$, that we can easily construct by solving (III.9) with $M = 0$, (III.12), (III.17), by invoking the classical theorems on the mixed hyperbolic problems. In consequence, our work is mainly concerned with the massive field, and in the sequel, we consider only this case.

When $M \neq 0$ the situation is very different because the potential blows up as $\varrho \to 1$. The analogous situation of the infinite mass at the infinity of the Minkowski space has been investigated in [23,34]. In our case, the key result is the asymptotic behaviour, near the boundary, of the spinors of $D(\mathbf{H}_M)$. We note that it is sufficient to consider only the case of the positive mass, because the chiral transform

$$\Psi \longrightarrow \gamma^5 \Psi$$

changes M into $-M$ since we have

$$\gamma^5 \mathbf{H}_M \gamma^5 = \mathbf{H}_{-M}.$$

We remark that the *MIT-bag* and the *Chiral* conditions are exchanged by the chiral transform, and the *APS* condition is chiral invariant.

Theorem III.2. *Let Ψ be in $D(\mathbf{H}_M)$ with $M \in \mathbb{R}^*$. Then*

$$\Psi \in \left[C^0 \left( ]0,1[_\varrho ; H^{\frac{1}{2}}(S^2_\omega) \right) \right]^4, \tag{III.22}$$

$$\int_0^1 \| \Psi(\varrho\omega) \|^2_{H^1(S^2_\omega)} \varrho d\varrho \leq \| \mathbf{H}_M \Psi \|^2_{\mathbf{L}^2} . \tag{III.23}$$

When $M^2 > \frac{\Lambda}{12}$, we have

$$\| \Psi(\varrho\omega) \|_{L^2(S^2_\omega)} = O\left(\sqrt{1-\varrho} \right), \quad \varrho \to 1. \tag{III.24}$$

When $M^2 = \frac{\Lambda}{12}$, we have

$$\| \Psi(\varrho\omega) \|_{L^2(S^2_\omega)} = O\left(\sqrt{(\varrho - 1)\ln(1-\varrho)} \right), \quad \varrho \to 1. \tag{III.25}$$

When $0 < M^2 < \frac{\Lambda}{12}$, we put $m := M\sqrt{\frac{3}{\Lambda}}$, and there exists $\Psi_- \in \left[H^{\frac{1}{2}}(S^2) \right]^4$, $\Psi_+ \in \left[L^2(S^2) \right]^4$, and $\psi \in \left[C^0 \left([0,1]_\varrho ; L^2(S^2_\omega) \right) \right]^4$ satisfying

$$\Psi(\varrho\omega) = (1-\varrho)^{-m}\, \Psi_-(\omega) + (1-\varrho)^m\, \Psi_+(\omega) + \psi(\varrho\omega), \tag{III.26}$$

$$\tilde{\gamma}^1 \Psi_- + i\Psi_- = 0, \quad \tilde{\gamma}^1 \Psi_+ - i\Psi_+ = 0, \tag{III.27}$$

$$\| \psi(\varrho\omega) \|_{L^2(S^2_\omega)} = o\left(\sqrt{1-\varrho} \right), \quad \varrho \to 1. \tag{III.28}$$

Conversely, for any $\Psi_- \in \left[H^{\frac{1}{2}+m}(S^2) \right]^4$, $\Psi_+ \in \left[H^{\frac{1}{2}-m}(S^2) \right]^4$ satisfying (III.27), there exists $\Psi \in D(\mathbf{H}_M)$ satisfying (III.26) and (III.28).

Remark III.3. We shall see that (III.24) can be improved and when $M^2 > \frac{\Lambda}{12}$ the elliptic estimate below (III.32) implies that $\int_0^1 \parallel \Psi(\varrho\omega) \parallel^2_{L^2(S^2_\omega)} \frac{d\varrho}{(1-\varrho)^2} < \infty$. When $M^2 \geq \frac{\Lambda}{12}$, then $\Psi \in \left[C^0(]0,1]_\varrho; L^2(S^2_\omega)\right]^4$, but the trace of Ψ on $\partial\mathbb{B}$ does not exist for $M^2 < \frac{\Lambda}{12}$. Moreover we see with (III.23) that when $M \neq 0$, $\mathbf{H}_M\Psi = 0$ implies $\Psi = 0$.The situation is different when $M = 0$: we have $\Psi \in \left[C^0\left(]0,1]_\varrho; H^{-\frac{1}{2}}(S^2_\omega)\right)\right]^4$ for $\Psi \in D(\mathbf{H}_0)$, and this result is optimal: there exists $\Psi \in \mathbf{L}^2$, $\Psi \neq 0$, with $\mathbf{H}_0\Psi = 0$ and $\Psi(\omega) \in \left[H^{-\frac{1}{2}}(S^2_\omega)\right]^4 \setminus \cup_{s>-\frac{1}{2}} \left[H^s(S^2_\omega)\right]^4$.

We note that when $M^2 \geq \frac{\Lambda}{12}$, the elements of the domain of $\mathbf{H}_M$ satisfy the homogeneous Dirichlet Condition on $\partial\mathbb{B}$. We shall see that $\mathbf{H}_M$ is self-adjoint. On the contrary, when $0 < M < \sqrt{\frac{\Lambda}{12}}$, the trace of Ψ on $\partial\mathbb{B}$ is not defined, the leading term $(1-\varrho)^{-m}\Psi_-$ satisfies the *MIT-bag* Condition and the next term $(1-\varrho)^m\Psi_+$ satisfies the *Chiral* Condition (and the converse for $-\sqrt{\frac{\Lambda}{12}} < M < 0$). We introduce natural generalizations of the classic boundary conditions in terms of asymptotic behaviours near S^2:

$$\parallel \mathcal{B}\Psi(\varrho\omega) \parallel_{L^2(S^2_\omega)} = o\left(\sqrt{1-\varrho}\right), \tag{III.29}$$

and we consider the operators $\mathbb{H}_\mathcal{B}$, $\mathcal{B} = \mathcal{B}_{MIT},\ \mathcal{B}_{CHI},\ \mathcal{B}_{APS},\ \mathcal{B}_{mAPS}$, defined as the differential operator $\mathbf{H}_M$ endowed with the domain

$$D\left(\mathbb{H}_\mathcal{B}\right) := \left\{\Psi \in D(\mathbf{H}_M);\ \parallel \mathcal{B}\Psi(\varrho\omega) \parallel_{L^2(S^2_\omega)} = o\left(\sqrt{1-\varrho}\right)\right\}.$$

We remark that (III.26), (III.27) and (III.28) imply:

$$D\left(\mathbb{H}_{\mathcal{B}_{MIT}}\right) := \{\Psi \in D(\mathbf{H}_M);\ \Psi_+ = 0\ \ if\ M > 0,\ \Psi_- = 0\ \ if\ M < 0\},$$
$$D\left(\mathbb{H}_{\mathcal{B}_{CHI}}\right) := \{\Psi \in D(\mathbf{H}_M);\ \Psi_- = 0\ \ if\ M > 0,\ \Psi_+ = 0\ \ if\ M < 0\},$$

$$D\left(\mathbb{H}_{\mathcal{B}_{APS}}\right) = D\left(\mathbb{H}_{\mathcal{B}_{mAPS}}\right) = \left\{\Psi \in D(\mathbf{H}_M);\ \mathbf{1}_{]0,\infty[}\left(D_{S^2}\right)\Psi_+ = \mathbf{1}_{]0,\infty[}\left(D_{S^2}\right)\Psi_- = 0\right\}.$$

We now construct a large family of asymptotic conditions, generalizing the previous one, by imposing a linear relation between Ψ_- and Ψ_+. If we denote $\Psi_\pm = {}^t\left(\psi^1_\pm, \psi^2_\pm, \psi^3_\pm, \psi^4_\pm\right)$, the constraints of polarization (III.27) allow to express $\psi^{3,4}_\pm$ by using $\psi^{1,2}_\pm$:

$$\begin{pmatrix}\psi^3_\pm(\omega)\\ \psi^4_\pm(\omega)\end{pmatrix} = \pm\, i\boldsymbol{\omega}.\boldsymbol{\sigma}\begin{pmatrix}\psi^1_\pm(\omega)\\ \psi^2_\pm(\omega)\end{pmatrix},\quad \boldsymbol{\omega}.\boldsymbol{\sigma} := \sum_1^3 \omega^j\sigma^j.$$

We consider two densely defined self-adjoint operators $(\mathbf{A}^\pm, D(\mathbf{A}^\pm))$ on $L^2(S^2) \times L^2(S^2)$, satisfying

$$D(\mathbf{A}^+) = L^2(S^2) \times L^2(S^2),\quad D(\mathbf{A}^-) \supset H^{\frac{1}{2}}(S^2) \times H^{\frac{1}{2}}(S^2), \tag{III.30}$$

$$\mathbf{A}^\pm\left(C^\infty(S^2) \times C^\infty(S^2)\right) \subset H^{\frac{1}{2}\pm m}(S^2) \times H^{\frac{1}{2}\pm m}(S^2). \tag{III.31}$$

As an example, we can choose $\mathbf{A}^-$ any hermitian matrix of $H^{\frac{1}{2}}(S^2;\mathbb{C}^{2\times 2})$, and $\mathbf{A}^+$ any hermitian matrix of $H^{\frac{1}{2}+m}\cap L^\infty(S^2;\mathbb{C}^{2\times 2})$. We define the operators $(\mathbb{H}_{\mathbf{A}^+}, D(\mathbb{H}_{\mathbf{A}^+}))$, $(\mathbb{H}_{\mathbf{A}^-}, D(\mathbb{H}_{\mathbf{A}^-}))$, where

$$D\left(\mathbb{H}_{\mathbf{A}^\pm}\right) := \left\{\Psi\in D(\mathbf{H}_M);\ \begin{pmatrix}\psi^1_\mp\\ \psi^2_\mp\end{pmatrix} = \mathbf{A}^\pm\begin{pmatrix}\psi^1_\pm\\ \psi^2_\pm\end{pmatrix}\right\}.$$

For $\mathbf{A}^- = \mathbf{A}^+ = 0$, we obviously have $\mathbb{H}_{\mathbf{A}^\mp} = \mathbb{H}_{\mathcal{B}_{MIT}}$, $\mathbb{H}_{\mathbf{A}^\pm} = \mathbb{H}_{\mathcal{B}_{CHI}}$ if $\pm\ M > 0$. Furthermore, the chiral transform $\Psi \to \gamma^5\Psi$ leads to the exchanges $M \to -M$, $\mathbb{H}_{\mathcal{B}_{MIT}} \to \mathbb{H}_{\mathcal{B}_{CHI}}$, $\mathbb{H}_{\mathcal{B}_{CHI}} \to \mathbb{H}_{\mathcal{B}_{MIT}}$, $\mathbb{H}_{\mathcal{B}_{APS}} \to \mathbb{H}_{\mathcal{B}_{APS}}$, $\mathbb{H}_{\mathcal{B}_{\mathbf{A}^\pm}} \to \mathbb{H}_{\mathcal{B}_{\boldsymbol{\omega}.\boldsymbol{\sigma}\mathbf{A}^\pm\boldsymbol{\omega}.\boldsymbol{\sigma}}}$.

We now state the main theorem of this paper.

Theorem III.4 (Main result). *Given $M\in\mathbb{R}^*$, we consider the massive Dirac hamiltonian $\mathbf{H}_M$ defined by (III.15), (III.16). When $M^2\geq\frac{\Lambda}{12}$, $\mathbf{H}_M$ is essentially self-adjoint on $\left[C_0^\infty(\mathbb{B})\right]^4$, and if $M^2>\frac{\Lambda}{12}$, then $D\left(\mathbf{H}_M\right) = \left[H_0^1(\mathbb{B})\right]^4$, and for all $\Psi\in D\left(\mathbf{H}_M\right)$, we have the following elliptic estimate:*

$$\sqrt{\frac{\Lambda}{12}}\parallel \mathbf{H}_M\Psi\parallel_{\mathbf{L}^2}\geq\left(\mid M\mid-\sqrt{\frac{\Lambda}{12}}\right)\parallel\nabla_{\mathbf{x}}\Psi\parallel_{\mathbf{L}^2}. \tag{III.32}$$

When $M^2<\frac{\Lambda}{12}$, $\mathbb{H}_{\mathbf{A}^+}$, $\mathbb{H}_{\mathbf{A}^-}$, $\mathbb{H}_{\mathcal{B}_{APS}}$, $\mathbb{H}_{\mathcal{B}_{mAPS}}$ are self-adjoint on $\mathbf{L}^2$, and $\mathbb{H}_{\mathcal{B}_{APS}} = \mathbb{H}_{\mathcal{B}_{mAPS}}$.

The resolvent of any self-adjoint realization of $\mathbf{H}_M$, $M\in\mathbb{R}^*$, is compact on $\mathbf{L}^2$, and so, the spectrum of these operators is discrete.

We see that $\frac{\Lambda}{12}$ is an important critical value. It plays exactly the same role that the bounds that Breitenlohner and Freedman have discovered for the scalar massive fields [7,8]. We recall that these authors have considered the Klein-Gordon equation $\mid g\mid^{-\frac{1}{2}}\partial_\mu\left(\mid g\mid^{\frac{1}{2}}g^{\mu\nu}\partial_\nu u\right)-\alpha\frac{\Lambda}{3}u=0$, for which $\alpha=2$ corresponds to the massless case. By a sharp analysis of the modes, they have established, among other results, that: (i) the natural energy is positive when $\alpha\leq 9/4$, in particular for the light tachyons associated with $2<\alpha<9/4$; (ii) the dynamics is unique when $\alpha\leq 5/4$; (iii) there exists a lot of unitary dynamics when $5/4<\alpha<9/4$. For the sake of completeness we give in the one-page Appendix, a new and very simple proof of these results, based on a Hardy estimate and on the Kato-Rellich theorem. For the spin-$\frac{1}{2}$ field with real mass, the most important conserved quantity is the L^2-norm that is always positive, hence one bound will suffice to distinguish the different cases: it is $\frac{\Lambda}{12}$. We have to emphasize that this value was already presented in the discussion of the massive $OSp(1,4)$ scalar multiplet in [7,8]. This multiplet consists of a Dirac spinor with mass M, and two Klein-Gordon fields for which $\alpha=2\pm M\sqrt{3/\Lambda}-3M^2/\Lambda$. We can easily check that $\alpha\leq 9/4$ for any $M\in\mathbb{R}$, and $\alpha\leq 5/4$ iff $M^2\geq\Lambda/12$. Therefore our own result is coherent with this particular model of Anti-de Sitter supersymmetry: the constraints for the uniqueness of the dynamics are simultaneously satisfied for the spin field and the scalar fields. The case $\alpha>9/4$ describes the heavy tachyons in $CAdS$, and corresponds to the case of an *imaginary* mass for the Dirac field. This regime seems to be unphysical since the energy of a scalar tachyon is not positive, and the L^2-norm of a spin-$\frac{1}{2}$ field with an imaginary mass is not conserved. Of the mathematical point of view, it is doubtful that the global Cauchy problem with these parameters is well posed, and of the physical point of view, we could suspect that the AdS background is not stable with respect to the fluctuations of such fields. We do not address this situation in this paper.

We now turn over to the Cauchy problem.

Theorem III.5. *Given $\Psi_0 \in \mathbf{L}^2$, there exist solutions of* (III.9), (III.11), (III.12), *and all the solutions are equal for*

$$(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{B}, \quad | t | < \sqrt{\frac{3}{\Lambda}} \left(\frac{\pi}{2} - 2 \arctan \varrho \right). \tag{III.33}$$

When $M^2 \geq \frac{\Lambda}{12}$, the Cauchy problem (III.9), (III.11), (III.12) *has a unique solution. This solution satisfies (III.13).*

We achieve this part with a result of equipartition of the energy. We know, [3], that the solutions $\Psi \in C^0(\mathbb{R}_t; L^2(\mathbb{R}^3; \mathbb{C}^4))$ of the massive Dirac equation on the Minkowski space-time, satisfy

$$\lim_{|t| \to \infty} \int_{\mathbb{R}^3} \Psi^* \gamma^0 \gamma^5 \Psi(t, \mathbf{x}) d\mathbf{x} = 0.$$

Since the spectrum of the possible hamiltonians for the massive fermions on $CAdS$ is discrete, we cannot expect such an asymptotic behaviour. Nevertheless, we establish the existence of a similar limit, in the weaker sense of Cesaro:

Theorem III.6. *Let $\Psi \in C^0(\mathbb{R}_t; \mathbf{L}^2)$ be a solution of* (III.9), *given by $\Psi(t) = e^{it\sqrt{\frac{\Lambda}{3}}\mathbb{H}}\Psi(0)$, where $\mathbb{H}$ is a self-adjoint realization of $\mathbf{H}_M$, $M \in \mathbb{R}^*$. Then we have:*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \int_{\mathbb{B}} \Psi^* \gamma^0 \gamma^5 \Psi(t, \mathbf{x}) d\mathbf{x} dt = 0. \tag{III.34}$$

The proofs of these results are presented in Parts V and VI. They are made much easier by the use of the spherical coordinates. Operator $\mathbf{S}$, defined by (III.8), that relates the spinors within the two systems of coordinates, is an isometry from

$$\mathcal{L}^2 := \left[L^2 \left([0, \frac{\pi}{2}[_x \times [0, \pi]_\theta \times [0, 2\pi[_\varphi, \sin\theta dx d\theta d\varphi \right) \right]^4, \tag{III.35}$$

onto $\mathbf{L}^2$, and satisfies the intertwining relation

$$\mathbf{H}_M \mathbf{S} = \mathbf{S} H_m, \tag{III.36}$$

where H_m is the differential operator

$$H_m := i\gamma^0\gamma^1 \frac{\partial}{\partial x} + \frac{i}{\sin x} \left[\gamma^0\gamma^2 \left(\frac{\partial}{\partial\theta} + \frac{1}{2\tan\theta} \right) + \frac{1}{\sin\theta} \gamma^0\gamma^3 \frac{\partial}{\partial\varphi} \right]$$
$$- \frac{m}{\cos x} \gamma^0, \quad m = M\sqrt{\frac{3}{\Lambda}}. \tag{III.37}$$

The problem essentially consists in finding self-adjoint realizations of H_m in $\mathcal{L}^2$. The difficulty comes from the blow-up of the gravitational interaction on the boundary. We see that H_0 is just the Dirac operator on the 3-sphere $S^3 \leftrightarrow [0, \pi]_x \times [0, \pi]_\theta \times [0, 2\pi[_\varphi$, restricted to the upper hemisphere $S^3_+ \leftrightarrow [0, \frac{\pi}{2}[_x \times [0, \pi]_\theta \times [0, 2\pi[_\varphi$. The key result, Theorem V.1, deals with the asymptotic behaviour of $\Phi \in D(H_m)$ at the equatorial 2-sphere $S^2 = \partial S^3_+$, as $x \to \frac{\pi}{2}$. The tool is a careful analysis based on the diagonalization of D_{S^2} by the spinoidal spherical harmonics.

IV. The Spinoidal Spherical Harmonics

We start by introducing several tools based on the spinor representation of the rotation group (see [14,26,35]). It is well known that there exists two Hilbert bases of $L^2(S^2)$, given by: $\left(T^l_{\frac{1}{2},n}(\theta,\varphi)\right)_{(l,n)\in I}$, $\left(T^l_{-\frac{1}{2},n}(\theta,\varphi)\right)_{(l,n)\in I}$,

$$\begin{aligned} I &:= \left\{(l,n);\ l\in\mathbb{N}+\frac{1}{2},\ \ n\in\mathbb{Z}+\frac{1}{2},\ \ l-\mid n\mid\in\mathbb{N}\right\} \\ &= \left\{(l,n);\ l\in\mathbb{N}+\frac{1}{2},\ \ n=-l,-l+1,\dots,l\right\}, \end{aligned} \tag{IV.1}$$

$$T^l_{\pm\frac{1}{2},n}(\theta,\varphi)=e^{-in\varphi}P^l_{\pm\frac{1}{2},n}(\cos\theta),$$

where $P^l_{\pm\frac{1}{2},n}$ can be expressed in terms of generalized Jacobi functions:

$$P^l_{\pm\frac{1}{2},n}(X)=A^l_{\pm,n}(1-X)^{\frac{\pm 1-2n}{4}}(1+X)^{\frac{\mp 1-2n}{4}}\frac{d^{l-n}}{dX^{l-n}}\left[(1-X)^{l\mp\frac{1}{2}}(1+X)^{l\pm\frac{1}{2}}\right],$$

and the constant

$$A^l_{\pm,n}=\frac{(-1)^{l\mp\frac{1}{2}}i^{n\mp\frac{1}{2}}}{2^l\left(l\mp\frac{1}{2}\right)!}\sqrt{\frac{\left(l\mp\frac{1}{2}\right)!(l+n)!}{\left(l\pm\frac{1}{2}\right)!(l-n)!}}\sqrt{\frac{2l+1}{4\pi}}$$

is chosen to normalize the basis functions (in comparison with the notations adopted in the book [14], the functions $P^l_{m,n}$ are multiplied by $\sqrt{(2l+1)/4\pi}$):

$$\int_0^{2\pi}\int_0^{\pi}T^l_{\pm\frac{1}{2},n}(\theta,\varphi)\overline{T^{l'}_{\pm\frac{1}{2},n'}(\theta,\varphi)}\sin\theta d\theta d\varphi=\delta_{l,l'}\delta_{n,n'}.$$

Therefore we can expand any function $f\in L^2(S^2)$ on both these bases bases

$$f(\theta,\varphi)=\sum_{(l,n)\in I}u^l_{\pm,n}(f)T^l_{\pm\frac{1}{2},n}(\theta,\varphi),\ \ u^l_{\pm,n}(f)\in\mathbb{C},$$

and by the Plancherel formula:

$$\parallel f\parallel^2_{L^2}=\sum_{(l,n)\in I}\mid u^l_{+,n}(f)\mid^2=\sum_{(l,n)\in I}\mid u^l_{-,n}(f)\mid^2.$$

More generally, for $s\in\mathbb{R}$, we introduce the Hilbert spaces $W^s_\pm$ defined as the closure of the space

$$W^{\pm}_f:=\left\{\sum_{finite}u^l_{\pm,n}T^l_{\pm\frac{1}{2},n};\ \ u^l_{\pm,n}\in\mathbb{C}\right\} \tag{IV.2}$$

for the norm

$$\| f \|^2_{W^s_\pm} := \sum_{(l,n)\in I} \left(l+\frac{1}{2}\right)^{2s} \mid u^l_{\pm,n}(f) \mid^2 .$$

We note that the basis functions are *not* smooth on S^2 since $T^l_{\pm\frac{1}{2},n}(\theta, 2\pi) = -T^l_{\pm\frac{1}{2},n}(\theta, 0) \# 0$. Hence $W^s_\pm$ is not a classical Sobolev space on S^2. We state some properties of these spaces. Firstly it is easy to prove that for

$$s \geq 0 \Longrightarrow W^s_\pm = \left\{ f \in L^2\left(S^2\right);\ \ \| f \|_{W^s_\pm} < \infty \right\},$$

and the topological dual of $W^s_\pm$ can be isometrically identified with $W^{-s}_\pm$:

$$s \in \mathbb{R}, \quad \left(W^s_\pm\right)' = W^{-s}_\pm.$$

Secondly we show that $W^s_\pm$ contains the test functions on $]0,\pi[_\theta \times]0, 2\pi[_\varphi$. To see that, we recall the differential equations satisfied by the basis functions:

$$\left(\frac{\partial}{\partial\theta} + \frac{1}{2\tan\theta}\right) T^l_{\pm\frac{1}{2},n} = \pm\frac{n}{\sin\theta} T^l_{\pm\frac{1}{2},n} - i\left(l+\frac{1}{2}\right) T^l_{\mp\frac{1}{2},n}, \tag{IV.3}$$

$$\frac{\partial}{\partial\varphi} T^l_{\pm\frac{1}{2},n} = -inT^l_{\pm\frac{1}{2},n}. \tag{IV.4}$$

If $f \in C^\infty_0(]0,\pi[_\theta \times]0, 2\pi[_\varphi)$ then $(\partial_\theta + \frac{1}{2}\cot\theta \mp \frac{i}{\sin\theta}\partial_\varphi) f \in C^\infty_0(]0,\pi[_\theta \times]0, 2\pi[_\varphi)$ and for any integer N, the differential equation (IV.3) assures that

$$\left(l+\frac{1}{2}\right)^{2N} u^l_{\pm,n}(f) = (-1)^N u^l_{\pm,n}\left(\left[\frac{\partial}{\partial\theta} + \frac{1}{2\tan\theta} \mp \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right]^{2N} f\right) \in l^2(I).$$

We conclude that any test function belongs to $W^s_\pm$ for any real s, and the series $\Sigma_I u^l_{\pm,n} T^l_{\pm\frac{1}{2},n} \in W^s_\pm$ converges in the sense of the distributions on $]0,\pi[_\theta \times]0, 2\pi[_\varphi$, in particular for all $s < 0$. We deduce that $(\partial_\theta + \frac{1}{2}\cot\theta \mp \frac{i}{\sin\theta}\partial_\varphi)$, acting in the sense of the distributions, is an isometry from $W^s_\pm$ onto $W^{s-1}_\mp$. But we have to be careful since the set of the test functions is not dense in general in $W^s_\pm$, $s > 0$: we cannot identify W^{-s} with a subspace of distributions, and there can exist $f \in W^{-s}_\pm \setminus \{0\}$ which is null in the sense of the distributions on $]0,\pi[_\theta \times]0, 2\pi[_\varphi$. For instance, since $(\sin\theta)^{-\frac{1}{2}} \in L^2(S^2)$, we have

$$f_\pm := \sum_{(l,n)\in I} \left(l+\frac{1}{2}\right) u^l_{\mp,n}\left(\frac{1}{\sqrt{\sin\theta}}\right) T^l_{\pm\frac{1}{2},n} \in W^{-1}_\pm, \quad \| f_\pm \|_{W^{-1}_\pm} = \sqrt{2}\pi,$$

but its restriction on the test functions is the null distribution because

$$\begin{aligned} f_{\pm | C^\infty_0(]0,\pi[_\theta\times]0,2\pi[_\varphi)} &= i\left[\frac{\partial}{\partial\theta} + \frac{1}{2\tan\theta} \mp \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right]\left(\frac{1}{\sqrt{\sin\theta}}\right) \\ &= 0 \ \ in \ \ \mathcal{D}'(]0,\pi[\times]0,2\pi[). \end{aligned}$$

Finally we investigate the links between W_+^s and W_-^s. We know that

$$P^l_{\frac{1}{2},n} = P^l_{-\frac{1}{2},-n},$$

and

$$\overline{P^l_{\pm\frac{1}{2},n}} = (-1)^{n\mp\frac{1}{2}} P^l_{\pm\frac{1}{2},n},$$

hence

$$\overline{u^l_{\pm,n}(f)} = (-1)^{n\mp\frac{1}{2}} u^l_{\mp,-n}(\overline{f}),$$

and we have

$$s \in \mathbb{R}, \quad f \in W^s_\pm \Longleftrightarrow \overline{f} \in W^s_\mp, \quad \| f \|_{W^s_\pm} = \| \overline{f} \|_{W^s_\mp} .$$

We warn that in general $W_+^s \neq W_-^s$. Indeed, given $f_\pm \in W^1_\pm$, we have by (IV.3):

$$\left(\frac{\partial}{\partial\theta} + \frac{1}{2\tan\theta} \mp \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right) f_\pm = \sum_{(l,n)\in I} -i\left(l+\frac{1}{2}\right) u^l_{\pm,n}(f_\pm) T^l_{\mp\frac{1}{2},n} \in L^2(S^2).$$

We deduce that

$$f \in W^1_+ \cap W^1_- \Longrightarrow \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} f \in L^2(S^2).$$

Then if we consider

$$T^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}} e^{-i\frac{\varphi}{2}} \cos\frac{\theta}{2} \in W^1_+,$$

we see that $\frac{1}{\sin\theta}\partial_\varphi T^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}} \notin L^2(S^2)$, and we conclude that

$$W^1_+ \neq W^1_-.$$

Therefore it is convenient to introduce the isometry $\mathcal{J}$ on $L^2(S^2)$ defined by

$$\mathcal{J}\left(T^l_{+\frac{1}{2},n}\right) = T^l_{-\frac{1}{2},n}.$$

Then we have

$$\mathcal{J}^*\left(T^l_{-\frac{1}{2},n}\right) = T^l_{+\frac{1}{2},n},$$

and $\mathcal{J}$ is an isometry from W_+^s onto W_-^s.

We now return to the Dirac field. In: the same way, we can expand any spinor defined on S^2, $\Phi(\theta,\varphi) \in L^2(S^2;\mathbb{C}^4)$:

$$\Phi(\theta,\varphi) = \sum_{(l,n)\in I} \begin{pmatrix} u^l_{1,n} T^l_{-\frac{1}{2},n}(\theta,\varphi) \\ u^l_{2,n} T^l_{+\frac{1}{2},n}(\theta,\varphi) \\ u^l_{3,n} T^l_{-\frac{1}{2},n}(\theta,\varphi) \\ u^l_{4,n} T^l_{+\frac{1}{2},n}(\theta,\varphi) \end{pmatrix}, \quad u^l_{j,n} \in \mathbb{C}.$$

The main interest of this expansion is the following: if we consider the angular part of the hamiltonian H_m,

$$\mathbf{D} := i\gamma^0\gamma^2\left(\frac{\partial}{\partial\theta}+\frac{1}{2\tan\theta}\right)+\frac{i}{\sin\theta}\gamma^0\gamma^3\frac{\partial}{\partial\varphi},$$

an elementary but tedious computation shows that:

$$\mathbf{D}\Phi(\theta,\varphi)=\sum_{(l,n)\in I}\left(l+\frac{1}{2}\right)\begin{pmatrix} u^l_{4,n}T^l_{-\frac{1}{2},n}(\theta,\varphi)\\ u^l_{3,n}T^l_{+\frac{1}{2},n}(\theta,\varphi)\\ u^l_{2,n}T^l_{-\frac{1}{2},n}(\theta,\varphi)\\ u^l_{1,n}T^l_{+\frac{1}{2},n}(\theta,\varphi)\end{pmatrix}. \tag{IV.5}$$

Hence it is natural to introduce the Hilbert spaces

$$\mathcal{W}^s := W^s_- \times W^s_+ \times W^s_- \times W^s_+$$

endowed with the norm:

$$\parallel \Phi \parallel^2_{\mathcal{W}^s} := \sum_{j=1}^{4}\sum_{(l,n)\in I}\left(l+\frac{1}{2}\right)^{2s} \mid u^l_{j,n}\mid^2 . \tag{IV.6}$$

$\mathcal{W}^s$ is also the closure for this norm, of the subspace

$$\mathcal{W}_f := W^-_f \times W^+_f \times W^-_f \times W^+_f.$$

As a differential operator, $\mathbf{D}$ acts from $\mathcal{W}^s$ to $\mathcal{W}^{s-1}$ and $\mathbf{D}$ endowed with the domain $\mathcal{W}^1$ is self-adjoint on $\mathcal{W}^0$. We see that the spectrum of $(\mathbf{D},\mathcal{W}^1)$ is $\left\{\pm\left(l+\frac{1}{2}\right),\quad l\in\mathbb{N}\right\}$, its positive subspace $L^2_+\left(S^2;\mathbb{C}^4\right)$ is spanned by the eigenvectors $\left(T^l_{-\frac{1}{2},n},0,0,T^l_{+\frac{1}{2},n}\right)$, $\left(0,T^l_{+\frac{1}{2},n},T^l_{-\frac{1}{2},n},0\right)$, $(l,n)\in I$, and the negative subspace $L^2_-\left(S^2;\mathbb{C}^4\right)$ is spanned by the eigenvectors $\left(T^l_{-\frac{1}{2},n},0,0,-T^l_{+\frac{1}{2},n}\right)$, $\left(0,T^l_{+\frac{1}{2},n},-T^l_{-\frac{1}{2},n},0\right)$, $(l,n)\in I$. We can characterize these spaces by using the operator $\mathcal{J}$:

$$L^2_\pm\left(S^2;\mathbb{C}^4\right)=\left\{\begin{pmatrix}\psi\\ \chi\\ \pm\mathcal{J}\chi\\ \pm\mathcal{J}^*\psi\end{pmatrix},\quad \psi,\chi\in L^2\left(S^2\right)\right\}.$$

We easily obtain the orthogonal projectors $\mathbf{K}_\pm$ on $L^2_\pm\left(S^2;\mathbb{C}^4\right)$:

$$\mathbf{K}_\pm=\frac{1}{2}\begin{pmatrix}1&0&0&\pm\mathcal{J}\\ 0&1&\pm\mathcal{J}^*&0\\ 0&\pm\mathcal{J}&1&0\\ \pm\mathcal{J}^*&0&0&1\end{pmatrix}. \tag{IV.7}$$

$\mathbf{K}_{\pm}$ can be extended into bounded operators on $\mathcal{W}^s$, $s \in \mathbb{R}$. These operators are used to define the global boundary conditions of M.F. Atiyah, V. K. Patodi, and I. M. Singer (see e.g. [6]):

$$\mathbf{K}_{\pm}\Phi = 0, \tag{IV.8}$$

and the boundary condition introduced by O. Hijazi, S. Montiel, A. Roldan [20],

$$\mathbf{K}_{+}\left(Id + \gamma^1\right)\Phi = 0.$$

$\mathcal{W}^s$ is also invariant by the operator

$$\mathbf{B}_{\alpha} := \gamma^1 + ie^{i\alpha\gamma^5}, \quad \alpha \in \mathbb{R},$$

involved in the local MIT-bag boundary condition:

$$\mathbf{B}_0\Phi = 0,$$

and the chiral condition:

$$\mathbf{B}_{\pi}\Phi = 0.$$

If we consider the operator $\mathbf{\Lambda} := \gamma^0\gamma^2\mathbf{D}$ as a positive, unbounded, selfadjoint operator on $\mathcal{W}^0$ with domain $\mathcal{W}^1$, then for $0 \le s \le 1$, $\mathcal{W}^s$ is the domain of $\mathbf{\Lambda}^s$, that is to say, these spaces are spaces of interpolation (see e.g. [24]):

$$\mathcal{W}^s = \left[\mathcal{W}^1, \mathcal{W}^0\right]_{1-s}, \quad 0 \le s \le 1.$$

The link between this space and the usual Sobolev spaces on S^2 is given by the following:

Proposition IV.1. *For any $s \in \mathbb{R}$, the linear map*

$$\Phi(\theta,\varphi) \longmapsto \Psi(x^1,x^2,x^3) = S(\theta,\varphi)\Phi(\theta,\varphi), \quad (x^1,x^2,x^3) \in S^2,$$

defined from $\mathcal{W}_f$ to $\left[L^2\left(S^2\right)\right]^4$, where S is given by (III.6) *and x^j, θ, φ are related to* (III.7)*, can be extended into a bounded isomorphism from $\mathcal{W}^s$ onto $\left[H^s(S^2)\right]^4$.*

Proof of Proposition IV.1. A tedious but elementary calculation shows that:

$$S(\theta,\varphi) = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

where

$$S_{11} = S_{22} = \frac{1}{2}\begin{pmatrix} (1+i)\left(e^{-i\frac{\varphi}{2}}\cos\frac{\theta}{2} + e^{i\frac{\varphi}{2}}\sin\frac{\theta}{2}\right) & (1+i)\left(e^{i\frac{\varphi}{2}}\cos\frac{\theta}{2} - e^{-i\frac{\varphi}{2}}\sin\frac{\theta}{2}\right) \\ (1-i)\left(-e^{-i\frac{\varphi}{2}}\cos\frac{\theta}{2} + e^{i\frac{\varphi}{2}}\sin\frac{\theta}{2}\right) & (1-i)\left(e^{i\frac{\varphi}{2}}\cos\frac{\theta}{2} + e^{-i\frac{\varphi}{2}}\sin\frac{\theta}{2}\right) \end{pmatrix}, \tag{IV.9}$$

$$S_{12} = S_{21} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Following [36], p.337, formula (3) with $n = 0$, we have

$$\sqrt{l+1}\,P^{l+\frac{1}{2}}_{m-\frac{1}{2},-\frac{1}{2}}(\cos\theta) = \sqrt{l-m+1}\cos\left(\frac{\theta}{2}\right)P^l_{m,0}(\cos\theta) + \sqrt{l+m}\sin\left(\frac{\theta}{2}\right)P^l_{m-1,0}(\cos\theta),$$

then since

$$P^l_{m,n} = (-1)^{m+n}P^l_{n,m},$$

we get for $l \in \mathbb{N}$, $m \in \mathbb{Z}$, $-l \le m \le l+1$:

$$e^{-i\frac{\varphi}{2}}\cos\left(\frac{\theta}{2}\right)T^{l+\frac{1}{2}}_{-\frac{1}{2},m-\frac{1}{2}}(\theta,\varphi) = (-1)^{m-1}\sqrt{\frac{l-m+1}{l+1}}\frac{x^3+1}{2}Y^l_m(\theta,\varphi) + (-1)^{m-1}\sqrt{\frac{l+m}{l+1}}\frac{x^1-ix^2}{2}Y^l_{m-1}(\theta,\varphi),$$

$$e^{i\frac{\varphi}{2}}\sin\left(\frac{\theta}{2}\right)T^{l+\frac{1}{2}}_{-\frac{1}{2},m-\frac{1}{2}}(\theta,\varphi) = (-1)^{m-1}\sqrt{\frac{l-m+1}{l+1}}\frac{x^1+ix^2}{2}Y^l_m(\theta,\varphi) + (-1)^{m-1}\sqrt{\frac{l+m}{l+1}}\frac{1-x^3}{2}Y^l_{m-1}(\theta,\varphi).$$

In the same way, with [36], p.337, formula (4) with $n = 0$, we have

$$\sqrt{l+1}\,P^{l+\frac{1}{2}}_{m-\frac{1}{2},\frac{1}{2}}(\cos\theta) = -\sqrt{l-m+1}\sin\left(\frac{\theta}{2}\right)P^l_{m,0}(\cos\theta) + \sqrt{l+m}\cos\left(\frac{\theta}{2}\right)P^l_{m-1,0}(\cos\theta),$$

then we get for $l \in \mathbb{N}$, $m \in \mathbb{Z}$, $-l+1 \le m \le l$:

$$e^{i\frac{\varphi}{2}}\cos\left(\frac{\theta}{2}\right)T^{l+\frac{1}{2}}_{\frac{1}{2},m-\frac{1}{2}}(\theta,\varphi) = (-1)^{m+1}\sqrt{\frac{l-m+1}{l+1}}\frac{x^1+ix^2}{2}Y^l_m(\theta,\varphi) + (-1)^{m}\sqrt{\frac{l+m}{l+1}}\frac{x^3+1}{2}Y^l_{m-1}(\theta,\varphi),$$

$$e^{-i\frac{\varphi}{2}}\sin\left(\frac{\theta}{2}\right)T^{l+\frac{1}{2}}_{\frac{1}{2},m-\frac{1}{2}}(\theta,\varphi) = (-1)^{m+1}\sqrt{\frac{l-m+1}{l+1}}\frac{1-x^3}{2}Y^l_m(\theta,\varphi) + (-1)^{m}\sqrt{\frac{l+m}{l+1}}\frac{x^1-ix^2}{2}Y^l_{m-1}(\theta,\varphi).$$

Since $f \mapsto x^j f$ is bounded on $H^s(S^2)$ and f belongs to $H^s(S^2)$ iff

$$f = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\alpha_{l,m}Y^l_m, \quad \sum_{l,m} l^{2s} \mid \alpha_{l,m} \mid^2 < \infty,$$

we conclude that the linear map $\Phi \mapsto S(\theta,\varphi)\Phi = \Psi$ is bounded from $\mathcal{W}_f$ endowed with the norm of $\mathcal{W}^s$ to $\left[H^s(S^2)\right]^4$, hence it can be extended into a continuous linear map $\mathbb{S} : \Phi \mapsto \Psi$ from $\mathcal{W}^s$ to $\left[H^s(S^2)\right]^4$. Then $\mathbb{S}^*$ is a bounded linear map from $\left[H^{-s}(S^2)\right]^4$ to $\mathcal{W}^{-s}$ for any $s \in \mathbb{R}$. Since $\mathbb{S}^*\Psi = S^*(\theta,\varphi)\Psi$ for $\Psi \in \left[C_0^\infty(S^2)\right]^4$, and $S^*(\theta,\varphi) = S^{-1}(\theta,\varphi)$, we conclude that $\mathbb{S}\mathbb{S}^* = Id_{H^s}$, $\mathbb{S}^*\mathbb{S} = Id_{\mathcal{W}^s}$. $\square$

V. Asymptotic Behaviour at the Boundary

In this part we investigate the properties of the spinors that belong to the natural domain of the hamiltonian, especially the asymptotic behaviours near the boundary. We begin with its form in spherical coordinates, H_m given by (III.37), and

$$D\left(H_m\right) := \left\{\Phi \in \mathcal{L}^2;\quad H_m\Phi \in \mathcal{L}^2\right\}. \tag{V.1}$$

Theorem V.1. *For any $\Phi \in D\left(H_m\right)$ we have*

$$\Phi \in C^0\left([0, \frac{\pi}{2}[_x; \mathcal{W}^{\frac{1}{2}}\right), \tag{V.2}$$

$$\| \Phi(x,.) \|_{\mathcal{W}^{\frac{1}{2}}} = O\left(\sqrt{x}\right), \quad x \to 0, \tag{V.3}$$

and when $0 < m$ we have

$$\int_0^{\frac{\pi}{2}} \| \Phi(x,.) \|_{\mathcal{W}^1}^2 \frac{dx}{\sin x} \leq \| H_m\Phi \|_{\mathcal{L}^2}^2 . \tag{V.4}$$

When $\frac{1}{2} < m$, we have

$$\| \Phi(x,.) \|_{L^2(S^2)} = O\left(\sqrt{\frac{\pi}{2} - x}\right), \quad x \to \frac{\pi}{2}. \tag{V.5}$$

When $m = \frac{1}{2}$, we have:

$$\| \Phi(x,.) \|_{L^2(S^2)} = O\left(\sqrt{\left(x - \frac{\pi}{2}\right)\ln\left(\frac{\pi}{2} - x\right)}\right), \quad x \to \frac{\pi}{2}. \tag{V.6}$$

When $0 < m < \frac{1}{2}$, there exists $\psi_- \in W_-^{\frac{1}{2}}$, $\chi_- \in W_+^{\frac{1}{2}}$, $\psi_+, \chi_+ \in L^2(S^2)$, and $\phi \in C^0\left([0, \frac{\pi}{2}]_x; L^2(S^2;\mathbb{C}^4)\right)$ satisfying

$$\Phi(x,\theta,\varphi) = \left(\frac{\pi}{2} - x\right)^{-m} \begin{pmatrix} \psi_-(\theta,\varphi) \\ \chi_-(\theta,\varphi) \\ -i\psi_-(\theta,\varphi) \\ i\chi_-(\theta,\varphi) \end{pmatrix}$$

$$+\left(\frac{\pi}{2} - x\right)^{m} \begin{pmatrix} \psi_+(\theta,\varphi) \\ \chi_+(\theta,\varphi) \\ i\psi_+(\theta,\varphi) \\ -i\chi_+(\theta,\varphi) \end{pmatrix} + \phi(x,\theta,\varphi), \tag{V.7}$$

$$\| \phi(x,.) \|_{L^2(S^2)} = o\left(\sqrt{\frac{\pi}{2}-x}\right), \quad x \to \frac{\pi}{2}. \tag{V.8}$$

Conversely, for any $\psi_- \in W_-^{\frac{1}{2}+m}$, $\chi_- \in W_+^{\frac{1}{2}+m}$, $\psi_+ \in W_-^{\frac{1}{2}-m}$, $\chi_+ \in W_+^{\frac{1}{2}-m}$ *there exists* $\Phi \in D(H_m)$ *satisfying* (V.7) *and* (V.8).

When $m = 0$, *then*

$$\Phi \in C^0\left([0, \frac{\pi}{2}]_x; \mathcal{W}^{-\frac{1}{2}}\right). \tag{V.9}$$

Remark V.2. Equation (V.4) shows that when $m > 0$, $H_m\Phi = 0$ implies $\Phi = 0$. On the contrary, when $m = 0$, the left member of (V.4) can be infinite even if $H_0\Phi = 0$. Furthermore the space $\mathcal{W}^{-\frac{1}{2}}$ is optimal for the traces on $x = \frac{\pi}{2}$: there exists $\Phi \in D(H_0)$ such that $\Phi(\frac{\pi}{2}) \notin \cup_{s>-\frac{1}{2}} \mathcal{W}^s$. As an example, we consider a sequence $\left(C_{l,n}\right)_{(l,n)\in I} \subset \mathbb{C}$ such that

$$\sum_{(l,n)\in I} \left(l+\frac{1}{2}\right)^{-1} \left|C_{l,n}\right|^2 < \infty, \quad -1 < s \Rightarrow \sum_{(l,n)\in I} \left(l+\frac{1}{2}\right)^{s} \left|C_{l,n}\right|^2 = \infty,$$

we can take for instance $C_{l,n} = \frac{1}{\sqrt{l}\log(l+1)}$, and we put

$$\Phi(x,\theta,\varphi) = \sum_{(l,n)\in I} C_{l,n} \tan\left(\frac{x}{2}\right)^{l+\frac{1}{2}} \begin{pmatrix} T^l_{-\frac{1}{2},n}(\theta,\varphi) \\ -iT^l_{\frac{1}{2},n}(\theta,\varphi) \\ 0 \\ 0 \end{pmatrix}.$$

Then we easily check that

$$\Phi \in \mathcal{L}^2, \quad H_0\Phi = 0, \quad 0 < s \Rightarrow \int_0^{\frac{\pi}{2}} \| \Phi(x,.) \|^2_{\mathcal{W}^s} \, dx = \infty,$$
$$\Phi(\frac{\pi}{2},.) \in \mathcal{W}^{-\frac{1}{2}} \setminus \cup_{s>-\frac{1}{2}} \mathcal{W}^s.$$

Remark V.3. For $0 < m < \frac{1}{2}$, the leading terms of Φ satisfy the MIT-bag or the Chiral boundary condition since:

$$\mathbf{B}_0\Phi(x) = 2i\left(\frac{\pi}{2}-x\right)^m \begin{pmatrix} \psi_+ \\ \chi_+ \\ i\psi_+ \\ -i\chi_+ \end{pmatrix} + \mathbf{B}_0\phi(x),$$

$$\mathbf{B}_\pi\Phi(x) = -2i\left(\frac{\pi}{2}-x\right)^{-m} \begin{pmatrix} \psi_- \\ \chi_- \\ -i\psi_- \\ i\chi_- \end{pmatrix} + \mathbf{B}_\pi\phi(x).$$

Proof of Theorem V.1. We expand any spinor $\Phi(x,\theta,\varphi)$ in the previous way:

$$\Phi(x,\theta,\varphi)=\sum_{(l,n)\in I}\begin{pmatrix} u^l_{1,n}(x)T^l_{-\frac{1}{2},n}(\theta,\varphi)\\ u^l_{2,n}(x)T^l_{+\frac{1}{2},n}(\theta,\varphi)\\ u^l_{3,n}(x)T^l_{-\frac{1}{2},n}(\theta,\varphi)\\ u^l_{4,n}(x)T^l_{+\frac{1}{2},n}(\theta,\varphi)\end{pmatrix},$$

and we have:

$$\parallel \Phi \parallel^2_{\mathcal{L}^2}=\sum_{j=1}^{4}\sum_{(l,n)\in I}\parallel u^l_{j,n}\parallel^2_{L^2(0,\frac{\pi}{2})}.$$

Furthermore, for $\Phi\in D(H_m)$, (IV.5) gives:

$$H_m\Phi(x,\theta,\varphi)=\sum_{(l,n)\in I}\begin{pmatrix} f^l_{1,n}(x)T^l_{-\frac{1}{2},n}(\theta,\varphi)\\ f^l_{2,n}(x)T^l_{+\frac{1}{2},n}(\theta,\varphi)\\ f^l_{3,n}(x)T^l_{-\frac{1}{2},n}(\theta,\varphi)\\ f^l_{4,n}(x)T^l_{+\frac{1}{2},n}(\theta,\varphi)\end{pmatrix},$$

with

$$\begin{cases} i\left(u^l_{3,n}\right)'+\frac{\left(l+\frac{1}{2}\right)}{\sin x}u^l_{4,n}-\frac{m}{\cos x}u^l_{1,n}=f^l_{1,n},\\ -i\left(u^l_{4,n}\right)'+\frac{\left(l+\frac{1}{2}\right)}{\sin x}u^l_{3,n}-\frac{m}{\cos x}u^l_{2,n}=f^l_{2,n},\\ i\left(u^l_{1,n}\right)'+\frac{\left(l+\frac{1}{2}\right)}{\sin x}u^l_{2,n}+\frac{m}{\cos x}u^l_{3,n}=f^l_{3,n},\\ -i\left(u^l_{2,n}\right)'+\frac{\left(l+\frac{1}{2}\right)}{\sin x}u^l_{1,n}+\frac{m}{\cos x}u^l_{4,n}=f^l_{4,n},\end{cases}\tag{V.10}$$

and

$$\parallel H_m\Phi \parallel^2_{\mathcal{L}^2}=\sum_{j=1}^{4}\sum_{(l,n)\in I}\parallel f^l_{j,n}\parallel^2_{L^2(0,\frac{\pi}{2})}.$$

For $1\leq h,k\leq 4$, we put

$$u^{l,\pm}_{hk,n}=u^l_{h,n}\pm iu^l_{k,n},\quad f^{l,\pm}_{hk,n}=f^l_{h,n}\pm if^l_{k,n}.$$

We have

$$\left(u^{l,\pm}_{12,n}\right)'\mp\frac{l+\frac{1}{2}}{\sin x}u^{l,\pm}_{12,n}=\frac{im}{\cos x}u^{l,\mp}_{34,n}-if^{l,\mp}_{34,n},$$

$$\left(u^{l,\pm}_{34,n}\right)'\mp\frac{l+\frac{1}{2}}{\sin x}u^{l,\pm}_{34,n}=-\frac{im}{\cos x}u^{l,\mp}_{12,n}-if^{l,\mp}_{12,n}.$$

Given $w_+^l \in L^2(0, \frac{\pi}{2})$, any solution v_+^l of

$$\frac{d}{dx}v_+^l - \frac{l+\frac{1}{2}}{\sin x}v_+^l = w_+^l, \quad 0 < x < \frac{\pi}{2},$$

belongs to $H^1_{loc}(]0, \frac{\pi}{2}]) \subset C^0(]0, \frac{\pi}{2}])$ and v_+^l can be written:

$$v_+^l(x) = v_+^l(\frac{\pi}{2})\left(\tan\left(\frac{x}{2}\right)\right)^{l+\frac{1}{2}} - \int_x^{\frac{\pi}{2}} \left(\frac{\tan\left(\frac{x}{2}\right)}{\tan\left(\frac{y}{2}\right)}\right)^{l+\frac{1}{2}} w_+^l(y)dy. \tag{V.11}$$

On the one hand, by integrating we get:

$$\mid v_+^l(\frac{\pi}{2}) \mid^2 \leq C(l+1)(\| v_+^l \|^2_{L^2} + \| w_+^l \|^2_{L^2}). \tag{V.12}$$

On the other hand, we easily show that for $0 < x \leq \frac{\pi}{2}$

$$\int_x^{\frac{\pi}{2}} \left(\tan\left(\frac{y}{2}\right)\right)^{-2l-1} dy \leq \frac{1}{2l}\left(\tan\left(\frac{x}{2}\right)\right)^{-2l}\left(1 - \left(\tan\left(\frac{x}{2}\right)\right)^{2l}\right),$$

therefore since $\tan(x/2) \leq x$ on $[0, \frac{\pi}{2}]$, we obtain that:

$$2l\left|v_+^l(x) - v_+^l\left(\frac{\pi}{2}\right)\left(\tan\left(\frac{x}{2}\right)\right)^{l+\frac{1}{2}}\right|^2 \leq \mid x \mid \| w_+^l \|^2_{L^2(x,\frac{\pi}{2})}\left(1 - \left(\tan\left(\frac{x}{2}\right)\right)^{2l}\right),$$

and we conclude that

$$v_+^l(\frac{\pi}{2}) = 0 \Longrightarrow l \mid v_+^l(x) \mid^2 \leq \mid x \mid \| w_+^l \|^2_{L^2}. \tag{V.13}$$

Now the solutions v_-^l of

$$\frac{d}{dx}v_-^l + \frac{l+\frac{1}{2}}{\sin x}v_-^l = w_-^l \in L^2(0, \frac{\pi}{2}), \tag{V.14}$$

have the form

$$v_-^l(x) = C\left(\tan\left(\frac{x}{2}\right)\right)^{-l-\frac{1}{2}} + \int_0^x \left(\frac{\tan\left(\frac{y}{2}\right)}{\tan\left(\frac{x}{2}\right)}\right)^{l+\frac{1}{2}} w_-^l(y)dy. \tag{V.15}$$

Then, when $v_- \in L^2(0, \frac{\pi}{2})$ and $l \geq 0$, we have $C = 0$. Since for $0 \leq x \leq \frac{\pi}{2}$ we have

$$\int_0^x \left(\tan\left(\frac{y}{2}\right)\right)^{2l+1} dy \leq \frac{1}{l+1}\left(\tan\left(\frac{x}{2}\right)\right)^{2l+2},$$

we obtain that the L^2 solutions of (V.14) satisfy:

$$(l+1) \mid v_-^l(x) \mid^2 \leq \mid x \mid \| w_-^l \|^2_{L^2}. \tag{V.16}$$

For any $\chi \in C_0^\infty([0, \frac{\pi}{2}[)$, we apply the previous estimates to

$$v_\pm^l = \chi u^{l,\pm}_{12(34),n}, \quad w_\pm^l = +(-)\frac{im}{\cos x}\chi u^{l,\mp}_{34(12),n} - i\chi f^{l,\mp}_{34(12),n} - \chi' u^{l,\pm}_{12(34),n}.$$

From (V.13) and (V.16), we deduce

$$l \sum_{hk=12,34} \mid \chi(x) u^{l,\pm}_{hk,n}(x) \mid^2 \leq C(\chi) \mid x \mid \sum_{j=1}^{4} \parallel u^l_{j,n} \parallel^2_{L^2} + \parallel f^l_{j,n} \parallel^2_{L^2}, \tag{V.17}$$

where $C(\chi) > 0$ depends only on χ. We get (V.2) and (V.3) that are consequences of (V.17). When $m = 0$, we can take

$$v^l_{\pm} = u^{l,\pm}_{12(34),n}, \quad w^l_{\pm} = -i f^{l,\mp}_{34(12),n},$$

and we get from (V.12) and (V.16) that

$$(l+1)^{-1} \sum_{hk=12,34} \mid u^{l,\pm}_{hk,n}(x) \mid^2 \leq C \mid x \mid \sum_{j=1}^{4} \parallel u^l_{j,n} \parallel^2_{L^2} + \parallel f^l_{j,n} \parallel^2_{L^2}. \tag{V.18}$$

This estimate yields (V.9).

Now we have

$$\left(u^{l,\pm}_{13,n}\right)' \mp \frac{m}{\cos x} u^{l,\pm}_{13,n} = \pm f^{l,\mp}_{13,n} + i \frac{l+\frac{1}{2}}{\sin x} u^{l,\pm}_{24,n},$$

$$\left(u^{l,\pm}_{24,n}\right)' \pm \frac{m}{\cos x} u^{l,\pm}_{24,n} = \mp f^{l,\mp}_{24,n} - i \frac{l+\frac{1}{2}}{\sin x} u^{l,\pm}_{13,n}.$$

Given $m \geq 0$, $w^l_+ \in L^2(0, \frac{\pi}{2})$, any solution v^l_+ of

$$\frac{d}{dx} v^l_+ + \frac{m}{\cos x} v^l_+ = w^l_+, \quad 0 < x < \frac{\pi}{2},$$

belongs to $H^1_{loc}([0, \frac{\pi}{2}[) \subset C^0([0, \frac{\pi}{2}[)$ and when

$$v^l_+(0) = 0,$$

v^l_+ can be written:

$$v^l_+(x) = \int_0^x \left(\frac{\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)}{\tan\left(\frac{\pi}{4} - \frac{y}{2}\right)} \right)^m w^l_+(y) dy. \tag{V.19}$$

Therefore the Cauchy-Schwarz estimate yields

$$\frac{1}{2} < m \Longrightarrow \mid v^l_+(x) \mid \leq C \parallel w^l_+ \parallel_{L^2} \sqrt{\frac{\pi}{2} - x}, \tag{V.20}$$

$$m = \frac{1}{2} \Longrightarrow \mid v^l_+(x) \mid \leq C \parallel w^l_+ \parallel_{L^2} \sqrt{\left(\frac{\pi}{2} - x\right) \ln\left(\frac{\pi}{2} - x\right)}, \tag{V.21}$$

$$0 \leq m < \frac{1}{2} \Longrightarrow \mid v^l_+(x) \mid \leq C \parallel w^l_+ \parallel_{L^2} \left(\frac{\pi}{2} - x\right)^m.$$

We make this last estimate precise for $0 \leq m < \frac{1}{2}$:

$$\begin{aligned} &\left| v_+^l(x) - 2^{-m}\left(\frac{\pi}{2}-x\right)^m \int_0^{\frac{\pi}{2}} \left[\tan\left(\frac{\pi}{4}-\frac{y}{2}\right)\right]^{-m} w_+^l(y)dy \right| \\ &\leq C\left(\| w_+^l \|_{L^2} \left(\frac{\pi}{2}-x\right)^{m+2} + \| w_+^l \|_{L^2(x,\frac{\pi}{2})} \sqrt{\frac{\pi}{2}-x}\right), \end{aligned} \tag{V.22}$$

in particular we have

$$0 < m \Longrightarrow \lim_{x\to\frac{\pi}{2}} v_+^l(x) = 0. \tag{V.23}$$

On the other hand the solutions v_-^l of

$$\frac{d}{dx}v_-^l - \frac{m}{\cos x}v_-^l = w_-^l, \quad 0 < x < \frac{\pi}{2},$$

have the form

$$v_-^l(x) = C_l\left[\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]^{-m} - \int_x^{\frac{\pi}{2}} \left(\frac{\tan\left(\frac{\pi}{4}-\frac{y}{2}\right)}{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right)^m w_-^l(y)dy, \tag{V.24}$$

thus,

$$\left| v_-^l(x) - C_l\left[\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]^{-m}\right| \leq \| w_-^l \|_{L^2(x,\frac{\pi}{2})} \sqrt{\frac{\pi}{2}-x}, \tag{V.25}$$

and

$$v_-^l \in L^2(0,\frac{\pi}{2}), \quad \frac{1}{2} \leq m \Longrightarrow C_l = 0, \tag{V.26}$$

$$0 \leq m < \frac{1}{2} \Longrightarrow C_l = v_-^l(0) + \int_0^{\frac{\pi}{2}} \left(\tan\left(\frac{\pi}{4}-\frac{y}{2}\right)\right)^m w_-^l(y)dy.$$

We pick $\chi \in C_0^\infty(]0, \frac{\pi}{2}]$ such that $\chi(\frac{\pi}{2}) = 1$, and we apply the previous estimates to

$$v_\pm^l = \chi u_{13(24),n}^{l,\mp(\pm)}, \quad w_\pm^l = w_{13(24),n}^{l,\mp(\pm)} := \mp(\pm)\chi f_{13(24),n}^{l,\pm(\mp)} + (-)i\frac{l+\frac{1}{2}}{\sin x}\chi u_{24(13),n}^{l,\mp(\pm)} - \chi' u_{13(24),n}^{l,\mp(\pm)}.$$

From (V.23) we deduce that when $m > 0$:

$$\lim_{x\to\frac{\pi}{2}} u_{1,n}^l(x) - iu_{3,n}^l(x) = \lim_{x\to\frac{\pi}{2}} u_{2,n}^l(x) + iu_{4,n}^l(x) = 0,$$

hence

$$\lim_{x\to\frac{\pi}{2}} \Im\left(u_{1,n}^l(x)\overline{u_{2,n}^l(x)} + (u_{3,n}^l(x)\overline{u_{4,n}^l(x)}\right) = 0. \tag{V.27}$$

Now multiplying (V.10) by $\overline{u^l_{j,n}}$ and taking the real part we get:

$$\frac{d}{dx}\Im\left(u^l_{1,n}\overline{u^l_{2,n}}+(u^l_{3,n}\overline{u^l_{4,n}}\right)+\frac{\left(l+\frac{1}{2}\right)}{\sin x}\sum_1^4 \mid u^l_{j,n}\mid^2$$
$$=\Re\left(f^l_{1,n}\overline{u^l_{4,n}}+f^l_{2,n}\overline{u^l_{3,n}}+f^l_{3,n}\overline{u^l_{2,n}}+f^l_{4,n}\overline{u^l_{1,n}}\right),$$

and thanks to (V.17) and (V.27) we obtain

$$\int_0^{\frac{\pi}{2}}\frac{\left(l+\frac{1}{2}\right)^2}{\sin x}\sum_{j=1}^4 \mid u^l_{j,n}(x)\mid^2 dx\leq\sum_{j=1}^4\parallel f^l_{j,n}\parallel^2_{L^2},$$

that proves (V.4). We also see that:

$$\parallel w^{l,\mp(\pm)}_{13(24),n}\parallel_{L^2}\leq C(\chi)\sum_{j=1}^4\parallel f^l_{j,n}\parallel_{L^2}. \tag{V.28}$$

Therefore when $m\geq\frac{1}{2}$, (V.5) and (V.6) follow from (V.20), (V.21), (V.25) and (V.26). On the other hand, when $0<m<\frac{1}{2}$, (V.22), (V.25) and (V.28) assure there exists $\varphi^{l,\mp(\pm}_{13(24),n}\in C^0([0,\frac{\pi}{2}])$ such that:

$$u^{l,-(+)}_{13(24),n}(x)=\left(\frac{\pi}{2}-x\right)^m\int_0^{\frac{\pi}{2}}\left(2\tan\left(\frac{\pi}{4}-\frac{y}{2}\right)\right)^{-m}w^{l,-(+)}_{13(24),n}(y)dy$$
$$+\varphi^{l,-(+)}_{13(24),n}(x)\sqrt{\frac{\pi}{2}-x},$$

$$u^{l,+(-)}_{13(24),n}(x)=\left(\frac{\pi}{2}-x\right)^{-m}\int_0^{\frac{\pi}{2}}\left(\frac{\frac{\pi}{2}-x}{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)}\tan\left(\frac{\pi}{4}-\frac{y}{2}\right)\right)^m w^{l,+(-)}_{13(24),n}(y)dy$$
$$+\varphi^{l,+(-)}_{13(24),n}(x)\sqrt{\frac{\pi}{2}-x},$$

$$\lim_{x\to\frac{x}{2}}\sum_{(l,n)\in I}\left|\varphi^{l,\mp(\pm)}_{13(24),n}(x)\right|^2=0.$$

We deduce that there exists $\psi_\pm,\chi_\pm\in L^2(S^2)$ such that Φ can be expressed according to (V.7), (V.8). It remains to prove the regularity of ψ_- and χ_-. We consider

$$\Psi(x,\theta,\varphi):=\left(\frac{\cos x}{1+\sin x}\right)^m\left(1+i\gamma^1\right)\Phi.$$

Equations (V.7), (V.8) assure that

$$\Psi(x,.)\longrightarrow\begin{pmatrix}\psi_-\\ \chi_-\\ -i\psi_-\\ i\chi_-\end{pmatrix}\quad in\ \mathcal{W}^0\ \ as\ \ x\to\frac{\pi}{2}. \tag{V.29}$$

We calculate

$$\frac{\partial}{\partial x}\Psi(x,.) = \left(\frac{\cos x}{1+\sin x}\right)^m \left(1+i\gamma^1\right)\gamma^0\left(H_m\Phi - \frac{1}{\sin x}\mathbf{D}\Phi\right).$$

Since $\Phi \in L^2\left([0,\frac{\pi}{2}[_x;\mathcal{W}^1\right)$ by (V.4), we deduce that

$$\Psi \in L^2\left([1,\frac{\pi}{2}[_x;\mathcal{W}^1\right),$$

$$\frac{\partial}{\partial x}\Psi \in L^2\left([1,\frac{\pi}{2}[_x;\mathcal{W}^0\right).$$

The theorem of the intermediate derivative ([24], p. 23) shows that

$$\Psi \in C^0\left([1,\frac{\pi}{2}]_x;\left[\mathcal{W}^1,\mathcal{W}^0\right]_{\frac{1}{2}}\right).$$

Recalling that $\left[\mathcal{W}^1,\mathcal{W}^0\right]_{\frac{1}{2}} = \mathcal{W}^{\frac{1}{2}}$, we conclude by (V.29) that $\psi_- \in W_-^{\frac{1}{2}}$, $\chi_- \in W_+^{\frac{1}{2}}$.

Finally we consider $\psi_\pm \in W_-^{\frac{1}{2}\mp m}$, $\chi_\pm \in W_+^{\frac{1}{2}\mp m}$, and we want to construct $\Phi \in D(H_m)$ satisfying (V.7) and (V.8). We choose $f \in C_0^\infty([0,1[)$ such that $f(0)=1$, and we put

$$\Phi(x) = \left(\frac{\pi}{2}-x\right)^{-m}\begin{pmatrix}\psi_-\\ \chi_-\\ -i\psi_-\\ i\chi_-\end{pmatrix} + \left(\frac{\pi}{2}-x\right)^{m}\begin{pmatrix}\psi_+\\ \chi_+\\ i\psi_+\\ -i\chi_+\end{pmatrix} + \phi(x),$$

where

$$\phi(x) = \left(\frac{\pi}{2}-x\right)^{-m}\sum_{(l,n)\in I}\left[f\left(l\left(\frac{\pi}{2}-x\right)\right)-1\right]\begin{pmatrix}u^l_{-,n}(\psi_-)T^l_{-\frac{1}{2},n}\\ u^l_{+,n}(\chi_-)T^l_{+\frac{1}{2},n}\\ -iu^l_{-,n}(\psi_-)T^l_{-\frac{1}{2},n}\\ iu^l_{+,n}(\chi_-)T^l_{+\frac{1}{2},n}\end{pmatrix}$$
$$+\left(\frac{\pi}{2}-x\right)^{m}\sum_{(l,n)\in I}\left[f\left(l\left(\frac{\pi}{2}-x\right)\right)-1\right]\begin{pmatrix}u^l_{-,n}(\psi_+)T^l_{-\frac{1}{2},n}\\ u^l_{+,n}(\chi_+)T^l_{+\frac{1}{2},n}\\ iu^l_{-,n}(\psi_+)T^l_{-\frac{1}{2},n}\\ -iu^l_{+,n}(\chi_+)T^l_{+\frac{1}{2},n}\end{pmatrix}.$$

We use the fact that

$$\left|f\left(l\left(\frac{\pi}{2}-x\right)\right)-1\right|^2 \leq \left(\frac{\pi}{2}-x\right)^{1\pm 2m}\left(\int_0^{l\left(\frac{\pi}{2}-x\right)} \mid f'(t)\mid^{\frac{2}{1\mp 2m}}dt\right)^{1\mp 2m} l^{1\pm 2m},$$

to get

$$
\begin{aligned}
&\| \phi(x,.) \|^2_{L^2(S^2)} \leq \\
&2\left(\frac{\pi}{2}-x\right)\sum_{(l,n)\in I}\left(\int_0^{l\left(\frac{\pi}{2}-x\right)} \mid f'(t)\mid^{\frac{2}{1-2m}}\,dt\right)^{1-2m} \\
&\quad l^{1+2m}\left(\mid u^l_{-,n}(\psi_-)\mid^2+\mid u^l_{+,n}(\chi_-)\mid^2\right) \\
&+2\left(\frac{\pi}{2}-x\right)\sum_{(l,n)\in I}\left(\int_0^{l\left(\frac{\pi}{2}-x\right)} \mid f'(t)\mid^{\frac{2}{1+2m}}\,dt\right)^{1+2m} \\
&\quad l^{1-2m}\left(\mid u^l_{-,n}(\psi_+)\mid^2+\mid u^l_{+,n}(\chi_+)\mid^2\right).
\end{aligned}
$$

The dominated convergence theorem assures that ϕ satisfies (V.8), and so $\Phi \in \mathcal{L}^2$. To achieve the proof, we have to show that $H_m\Phi \in \mathcal{L}^2$. We calculate:

$$
\begin{aligned}
H_m\Phi(x) = {} & m\left(\frac{\pi}{2}-x\right)^{-m-1}\left(1-\frac{\frac{\pi}{2}-x}{\cos x}\right)\sum_{(l,n)\in I} f\left(l\left(\frac{\pi}{2}-x\right)\right)\begin{pmatrix} u^l_{-,n}(\psi_-)T^l_{-\frac{1}{2},n} \\ u^l_{+,n}(\chi_-)T^l_{+\frac{1}{2},n} \\ -iu^l_{-,n}(\psi_-)T^l_{-\frac{1}{2},n} \\ iu^l_{+,n}(\chi_-)T^l_{+\frac{1}{2},n} \end{pmatrix} \\
& -\left(\frac{\pi}{2}-x\right)^{-m}\sum_{(l,n)\in I} f'\left(l\left(\frac{\pi}{2}-x\right)\right) l \begin{pmatrix} u^l_{-,n}(\psi_-)T^l_{-\frac{1}{2},n} \\ u^l_{+,n}(\chi_-)T^l_{+\frac{1}{2},n} \\ -iu^l_{-,n}(\psi_-)T^l_{-\frac{1}{2},n} \\ iu^l_{+,n}(\chi_-)T^l_{+\frac{1}{2},n} \end{pmatrix} \\
& +\left(\frac{\pi}{2}-x\right)^{-m}\frac{1}{\sin x}\sum_{(l,n)\in I} f\left(l\left(\frac{\pi}{2}-x\right)\right)\left(l+\frac{1}{2}\right)\begin{pmatrix} iu^l_{+,n}(\chi_-)T^l_{-\frac{1}{2},n} \\ -iu^l_{-,n}(\psi_-)T^l_{+\frac{1}{2},n} \\ u^l_{+,n}(\chi_-)T^l_{-\frac{1}{2},n} \\ u^l_{-,n}(\psi_-)T^l_{+\frac{1}{2},n} \end{pmatrix} \\
& -m\left(\frac{\pi}{2}-x\right)^{m-1}\left(1-\frac{\frac{\pi}{2}-x}{\cos x}\right)\sum_{(l,n)\in I} f\left(l\left(\frac{\pi}{2}-x\right)\right)\begin{pmatrix} u^l_{-,n}(\psi_+)T^l_{-\frac{1}{2},n} \\ u^l_{+,n}(\chi_+)T^l_{+\frac{1}{2},n} \\ iu^l_{-,n}(\psi_+)T^l_{-\frac{1}{2},n} \\ -iu^l_{+,n}(\chi_+)T^l_{+\frac{1}{2},n} \end{pmatrix} \\
& -\left(\frac{\pi}{2}-x\right)^{m}\sum_{(l,n)\in I} f'\left(l\left(\frac{\pi}{2}-x\right)\right) l \begin{pmatrix} u^l_{-,n}(\psi_+)T^l_{-\frac{1}{2},n} \\ u^l_{+,n}(\chi_+)T^l_{+\frac{1}{2},n} \\ iu^l_{-,n}(\psi_+)T^l_{-\frac{1}{2},n} \\ -iu^l_{+,n}(\chi_+)T^l_{+\frac{1}{2},n} \end{pmatrix}
\end{aligned}
$$

$$+\left(\frac{\pi}{2}-x\right)^m \frac{1}{\sin x}\sum_{(l,n)\in I} f\left(l\left(\frac{\pi}{2}-x\right)\right)\left(l+\frac{1}{2}\right)\begin{pmatrix} -iu^l_{+,n}(\chi_+)T^l_{-\frac{1}{2},n} \\ iu^l_{-,n}(\psi_+)T^l_{+\frac{1}{2},n} \\ u^l_{+,n}(\chi_+)T^l_{-\frac{1}{2},n} \\ u^l_{-,n}(\psi_+)T^l_{+\frac{1}{2},n}\end{pmatrix}.$$

In this sum, the leading terms have the form

$$\Xi^{\pm}(x,\theta,\varphi)=\left(\frac{\pi}{2}-x\right)^{\pm m}\sum_{(l,n)\in I} h\left(l\left(\frac{\pi}{2}-x\right)\right)\left(l+\frac{1}{2}\right) g^{\pm}_{l,n}(\theta,\varphi),$$

where $h\in C_0^\infty([0,1[)$ and

$$\sum_{(l,n)\in I}\left(l+\frac{1}{2}\right)^{1\mp 2m}\parallel g^{\pm}_{l,n}\parallel^2_{L^2(S^2)}<\infty,\quad \int_{S^2} g^{\pm}_{l,n}(\omega)\overline{g^{\pm}_{l',n'}(\omega)}d\omega=\delta_{l,l'}\delta_{n,n'}.$$

Taking account of the support of h, we evaluate

$$\begin{aligned}\parallel \Xi^{\pm}\parallel^2_{L^2(]0,\pi[\times S^2)} &= \sum_{(l,n)\in I}\left(l+\frac{1}{2}\right)^2\parallel g^{\pm}_{l,n}\parallel^2_{L^2(S^2)}\int_{\frac{\pi}{2}-\frac{1}{l}}^{\frac{\pi}{2}}\left(\frac{\pi}{2}-x\right)^{\pm 2m}\\ &\qquad \left|h\left(l\left(\frac{1}{2}-x\right)\right)\right|^2 dx\\ &\le \int_0^1 t^{\pm 2m}\mid h(t)\mid^2 dt\sum_{(l,n)\in I}\left(l+\frac{1}{2}\right)^{1\mp 2m}\parallel g^{\pm}_{l,n}\parallel^2_{L^2(S^2)}<\infty.\end{aligned}$$

□

Proof of Theorem III.2. Since the map **S** given by (III.8) satisfies (III.36), (III.22) and (III.23) follow from Proposition IV.1 and (V.2), and (V.4). Moreover, since $\frac{\pi}{2}-x\sim\frac{1}{2}(1-\varrho)$, (V.5) and (V.6) imply (III.24) and (III.25). Now, if we put

$$\Psi_{\pm}(\omega)=S(\theta,\varphi)\begin{pmatrix}\psi_{\pm}(\theta,\varphi)\\ \chi_{\pm}(\theta,\varphi)\\ \pm i\psi_{\pm}(\theta,\varphi)\\ \mp i\chi_{\pm}(\theta,\varphi)\end{pmatrix},$$

(III.26) and (III.27) are consequences respectively of (V.7), and:

$$\left(\tilde{\gamma}^1\mp iId\right)\Psi_{\pm}(\omega)=S(\theta,\varphi)\left(\gamma^1\mp iId\right)\begin{pmatrix}\psi_{\pm}(\theta,\varphi)\\ \chi_{\pm}(\theta,\varphi)\\ \pm i\psi_{\pm}(\theta,\varphi)\\ \mp i\chi_{\pm}(\theta,\varphi)\end{pmatrix}=0.$$

Finally, for any $\Psi_- \in \left[H^{\frac{1}{2}+m}(S^2)\right]^4$, $\Psi_+ \in \left[H^{\frac{1}{2}-m}(S^2)\right]^4$ we define $\Phi_\pm(\theta,\varphi) := S^*(\theta,\varphi)\Psi(\omega) \in \mathcal{W}^{\frac{1}{2}\mp m}$. When $\Psi_\pm$ satisfy (III.27), then $\Phi_\pm$ have the form

$$\Phi_\pm(\theta,\varphi) = \begin{pmatrix} \psi_\pm(\theta,\varphi) \\ \chi_\pm(\theta,\varphi) \\ \pm i\psi_\pm(\theta,\varphi) \\ \mp i\chi_\pm(\theta,\varphi) \end{pmatrix}, \quad \psi_- \in W_-^{\frac{1}{2}+m}, \quad \chi_- \in W_+^{\frac{1}{2}+m},$$

$$\psi_+ \in W_-^{\frac{1}{2}-m}, \quad \chi_+ \in W_+^{\frac{1}{2}-m},$$

and there exists $\Phi \in D(H_m)$ satisfying (V.7) and (V.8). We conclude that $\Psi := \mathbf{S}\Phi$, belongs to $D(\mathbf{H}_M)$ and satisfies (III.26) and (III.28). At last, Remark III.3 directly follows from (V.9), Proposition IV.1 and Remark V.2. $\square$

We end this part by an important result of compactness:

Proposition V.4. *Let K be the set*

$$K := \left\{\Phi \in D(H_m), \quad \| \Phi \|^2_{\mathcal{L}^2} + \| \Phi \|^2_{\mathcal{L}^2} \leq 1\right\}. \tag{V.30}$$

Then, when $m > 0$, K is a compact of $\mathcal{L}^2$.

Proof of Proposition V.4. We consider a sequence $(\Phi^\nu)_{\nu\in\mathbb{N}}$ in K. We write

$$\Phi^\nu = \sum_{(l,n)\in I} \begin{pmatrix} u^{l,\nu}_{1,n} T^l_{-\frac{1}{2},n} \\ u^{l,\nu}_{2,n} T^l_{+\frac{1}{2},n} \\ u^{l,\nu}_{3,n} T^l_{-\frac{1}{2},n} \\ u^{l,\nu}_{4,n} T^l_{+\frac{1}{2},n} \end{pmatrix}, \quad H_m\Phi^\nu = \sum_{(l,n)\in I} \begin{pmatrix} f^{l,\nu}_{1,n} T^l_{-\frac{1}{2},n} \\ f^{l,\nu}_{2,n} T^l_{+\frac{1}{2},n} \\ f^{l,\nu}_{3,n} T^l_{-\frac{1}{2},n} \\ f^{l,\nu}_{4,n} T^l_{+\frac{1}{2},n} \end{pmatrix},$$

and we have:

$$\sum_{j=1}^{4} \sum_{(l,n)\in I} \| u^{l,\nu}_{j,n} \|^2_{L^2(0,\frac{\pi}{2})} + \| f^{l,\nu}_{j,n} \|^2_{L^2(0,\frac{\pi}{2})} \leq 1.$$

The Banach-Alaoglu theorem assures that there exists $\Phi \in K$ and a sub-sequence denoted $(\Phi^\nu)_{\nu\in\mathbb{N}}$ again, such that

$$\Phi^\nu \rightharpoonup \Phi = \sum_{(l,n)\in I} \begin{pmatrix} u^{l}_{1,n} T^l_{-\frac{1}{2},n} \\ u^{l}_{2,n} T^l_{+\frac{1}{2},n} \\ u^{l}_{3,n} T^l_{-\frac{1}{2},n} \\ u^{l}_{4,n} T^l_{+\frac{1}{2},n} \end{pmatrix},$$

$$H_m\Phi^\nu \rightharpoonup H_m\Phi = \sum_{(l,n)\in I} \begin{pmatrix} f^{l}_{1,n} T^l_{-\frac{1}{2},n} \\ f^{l}_{2,n} T^l_{+\frac{1}{2},n} \\ f^{l}_{3,n} T^l_{-\frac{1}{2},n} \\ f^{l}_{4,n} T^l_{+\frac{1}{2},n} \end{pmatrix} \quad in \ \mathcal{L}^2 - *, \quad \nu \to \infty.$$

Since for any $(l,n) \in I$, $j = 1, \ldots 4$, $u^{l,\nu}_{j,n} \rightharpoonup u^l_{j,n}$, $f^{l,\nu}_{j,n} \rightharpoonup f^l_{j,n}$, in $L^2(0, \frac{\pi}{2}) - *$ as $\nu \to \infty$, we deduce from (V.11), (V.15), (V.19) and (V.24), that

$$\forall x \in [0, \frac{\pi}{2}], \quad u^{l,\nu}_{j,n}(x) \to u^l_{j,n}(x), \quad \sup_{\nu} \sup_{x \in [0,\frac{\pi}{2}]} \mid u^{l,\nu}_{j,n}(x) \mid < \infty.$$

Therefore

$$\| u^{l,\nu}_{j,n} - u^l_{j,n} \|_{L^2(0,\frac{\pi}{2}]} \to 0, \quad \nu \to \infty. \tag{V.31}$$

Moreove, since $m > 0$, (V.4) implies:

$$\sup_{\nu} \sum_{(l,n)\in I} \left(l + \frac{1}{2}\right)^2 \sum_{j=1}^{4} \| u^{l,\nu}_{j,n} - u^l_{j,n} \|^2_{L^2(0,\frac{\pi}{2})} < \infty. \tag{V.32}$$

For $l \in \mathbb{N} + \frac{1}{2}$ we put

$$\varepsilon^{l,\nu} := \sum_{j=1}^{4} \sum_{n=-l}^{l} \| u^{l,\nu}_{j,n} - u^l_{j,n} \|^2_{L^2(0,\frac{\pi}{2})} .$$

Equations (V.31) and (V.32) show that

$$\forall l \in \mathbb{N} + \frac{1}{2}, \quad \varepsilon^{l,\nu} \to 0, \quad \nu \to \infty, \quad A := \sup_{\nu} \sum_{l \in \mathbb{N}+\frac{1}{2}} \left(l + \frac{1}{2}\right)^2 \varepsilon^{l,\nu} < \infty.$$

Since $\varepsilon^{l,\nu} \leq A \left(l + \frac{1}{2}\right)^{-2}$, the dominated convergence theorem implies that $\sum_l \varepsilon^{l,\nu} \to 0$, as $\nu \to \infty$, that is to say, Φ^ν strongly tends to Φ in $\mathcal{L}^2$. $\square$

VI. Self-Adjoint Extensions

When $0 < m < \frac{1}{2}$, we define the linear map

$$\Gamma : \ \Phi \in D(H_m) \longmapsto \Gamma(\Phi) = \begin{pmatrix} \psi_- \\ \chi_- \\ \psi_+ \\ \chi_+ \end{pmatrix} \in \mathcal{W}^{\frac{1}{2}},$$

where $\psi_\pm$ and $\chi_\pm$ are given by (V.7), and we put $\Gamma(\Phi) = 0$ when $\frac{1}{2} \leq m$. We note that Theorem V.1 assures that

$$\forall m \in]0, \frac{1}{2}[, \quad W_-^{\frac{1}{2}+m} \times W_+^{\frac{1}{2}+m} \times W_-^{\frac{1}{2}-m} \times W_+^{\frac{1}{2}-m} \subset \Gamma\left(D(H_m)\right). \tag{VI.1}$$

We introduce the matrix

$$Q := -\gamma^0\gamma^5 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The basic tool is a nice Green formula:

Lemma VI.1. *Given* $0 < m$, *for any* $\Phi, \tilde{\Phi} \in D(H_m)$ *we have*

$$< H_m\Phi, \tilde{\Phi} >_{\mathcal{L}^2} - < \Phi, H_m\tilde{\Phi} >_{\mathcal{L}^2} = 2 < \Gamma(\Phi), Q\Gamma(\tilde{\Phi}) >_{\mathcal{W}^0} . \tag{VI.2}$$

Proof of Lemma VI.1. Equation (V.4) assures that any $\Phi \in D(H_m)$ belongs to $L^2\left([0, \frac{\pi}{2}]_x; \mathcal{W}^1\right)$, hence for any $\varepsilon > 0$, $\Phi \in H^1\left(]\varepsilon, \frac{\pi}{2} - \varepsilon[_x; \mathcal{W}^0\right)$. Since $(\mathbf{D}, \mathcal{W}^1)$ is selfadjoint on $\mathcal{W}^0$, we evaluate

$$\begin{aligned}&< H_m\Phi, \tilde{\Phi} >_{\mathcal{L}^2} - < \Phi, H_m\tilde{\Phi} >_{\mathcal{L}^2} = \lim_{\varepsilon\to 0} < i\gamma^0\gamma^1\Phi(\frac{\pi}{2} - \varepsilon),\\ &\tilde{\Phi}(\frac{\pi}{2} - \varepsilon) >_{\mathcal{W}^0} - < i\gamma^0\gamma^1\Phi(\varepsilon), \tilde{\Phi}(\varepsilon) >_{\mathcal{W}^0},\end{aligned}$$

and taking account of (V.3), (V.5), (V.6), (V.7) and (V.8) we get (VI.2). □

We now investigate the self-adjoint extensions $(\mathcal{H}, D(\mathcal{H}))$ of H_m, with $C_0^\infty(]0, \frac{\pi}{2}[_x \times]0, \pi[_\theta \times]0, 2\pi[_\varphi; \mathbb{C}^4) \subset D(\mathcal{H})$. The adjoint $\mathcal{H}^*$ is just H_m with domain $D(\mathcal{H}^*) \subset D(H_m)$, and we have:

$$\forall\Phi \in D(\mathcal{H}), \quad \forall\tilde{\Phi} \in D(\mathcal{H}^*), \quad < \Gamma(\Phi), Q\Gamma(\tilde{\Phi}) >_{\mathcal{W}^0} = 0.$$

When $m \geq \frac{1}{2}$ we immediately obtain a first result of self-adjointness of H_m on $\mathcal{L}^2$:

Proposition VI.2. *When* $\frac{1}{2} \leq m$, H_m *is essentially self-adjoint on* $\left[C_0^\infty\left(]0, \frac{\pi}{2}[_x \times]0, \pi[_\theta \times]0, 2\pi[_\varphi\right)\right]^4$.

Proof of Proposition VI.2. Let $\mathcal{H}$ be the operator defined by the differential operator H_m endowed with the domain $D(\mathcal{H}) = C_0^\infty(]0, \frac{\pi}{2}[_x \times]0, \pi[_\theta \times]0, 2\pi[_\varphi; \mathbb{C}^4)$. On the one hand, $\mathcal{H}$ is obviously symmetric, and on the other hand, its adjoint $\mathcal{H}^*$ is just H_m with domain $D(H_m)$. Let any $\Phi_\pm$ be in $D(H_m)$ such that $\mathcal{H}^*\Phi_\pm \pm i\Phi_\pm = 0$, satisfies

$$\mp 2i \parallel \Phi \parallel_{\mathcal{L}^2}^2 = < H_m\Phi_\pm, \Phi_\pm >_{\mathcal{L}^2} - < \Phi_\pm, H_m\Phi_\pm >_{\mathcal{L}^2},$$

and we conclude by (VI.2) that $\Phi_\pm = 0$. □

When $0 < m < \frac{1}{2}$, the situation is much more interesting: there exists a lot of self-adjoint realizations of H_m. First, we introduce the operators $\mathcal{H}_{MIT}$ and $\mathcal{H}_{CHI}$ respectively associated with the *MIT-bag* and the *Chiral* boundary conditions. They are defined as H_m endowed with the domains

$$\begin{aligned}&D\left(\mathcal{H}_{MIT}\right)\\ &:= \left\{\Phi \in D(H_m); \ \parallel \gamma^1\Phi(x, .) + i\Phi(x, .) \parallel_{\mathcal{W}^0} = o\left(\sqrt{\frac{\pi}{2} - x}\right), \ x \to \frac{\pi}{2}\right\},\end{aligned} \tag{VI.3}$$

$$\begin{aligned}&D\left(\mathcal{H}_{CHI}\right)\\ &:= \left\{\Phi \in D(H_m); \ \parallel \gamma^1\Phi(x, .) - i\Phi(x, .) \parallel_{\mathcal{W}^0} = o\left(\sqrt{\frac{\pi}{2} - x}\right), \ x \to \frac{\pi}{2}\right\}.\end{aligned} \tag{VI.4}$$

In fact these asymptotic conditions are reduced to linear constraints on the asymptotic profiles $\Phi_\pm$: we check by (V.7) that

$$\gamma^1\Phi(x,\theta,\varphi)\pm i\Phi(x,\theta,\varphi)=\pm 2i\left(\frac{\pi}{2}-x\right)^{\pm m}\begin{pmatrix}\psi_\pm(\theta,\varphi)\\ \chi_\pm(\theta,\varphi)\\ \pm i\psi_\pm(\theta,\varphi)\\ \mp i\chi_\pm(\theta,\varphi)\end{pmatrix}+\left(\gamma^1\pm i\right)\varphi(x,\theta,\varphi).$$

Thus (V.8) implies that

$$D\left(\mathcal{H}_{MIT}\right)=\left\{\Phi\in D(H_m);\quad \psi_+=\chi_+=0\right\},$$

$$D\left(\mathcal{H}_{CHI}\right)=\left\{\Phi\in D(H_m);\quad \psi_-=\chi_-=0\right\}.$$

We now construct a large family of self-adjoint extensions that are non-local generalizations of the *MIT-bag* and *Chiral* conditions. We consider densely defined self-adjoint operators $(A^\pm, D(A^\pm))$ on $L^2(S^2)\times L^2(S^2)$, satisfying

$$W_-^{\frac{1}{2}}\times W_+^{\frac{1}{2}}\subset D(A^-),\quad D(A^+)=L^2(S^2)\times L^2(S^2), \tag{VI.5}$$

$$A^\pm\left(C_0^\infty(]0,\pi[\times]0,2\pi[;\mathbb{C}^2\right)\subset W_-^{\frac{1}{2}\pm m}\times W_+^{\frac{1}{2}\pm m}. \tag{VI.6}$$

We introduce the operators $\mathcal{H}_{A^\pm}$ defined as H_m endowed with the domain

$$D\left(\mathcal{H}_{A^\pm}\right):=\left\{\Phi\in D(H_m);\quad \begin{pmatrix}\psi_\mp\\ \chi_\mp\end{pmatrix}=A^\pm\begin{pmatrix}\psi_\pm\\ \chi_\pm\end{pmatrix}\right\}.$$

In particular, we have $\mathcal{H}_{A^-=0}=\mathcal{H}_{MIT}$ and $\mathcal{H}_{A^+=0}=\mathcal{H}_{CHI}$.

Proposition VI.3. *When* $0<m<\frac{1}{2}$, $\mathcal{H}_{A^+}$ *and* $\mathcal{H}_{A^-}$ *are self-adjoint on* $\mathcal{L}^2$.

Proof of Proposition VI.3. Since $A^\pm$ are self-adjoint, we have for $\Phi,\tilde\Phi\in D(\mathcal{H}_{A^\pm})$:

$$<\Gamma(\Phi),Q\Gamma(\tilde\Phi)>_{\mathcal{W}^0}=\left\langle\begin{pmatrix}\psi_\pm\\ \chi_\pm\end{pmatrix},\begin{pmatrix}\tilde\psi_\mp\\ \tilde\chi_\mp\end{pmatrix}-A^\pm\begin{pmatrix}\tilde\psi_\pm\\ \tilde\chi_\pm\end{pmatrix}\right\rangle_{L^2(S^2;\mathbb{C}^2)}=0. \tag{VI.7}$$

Therefore $\mathcal{H}_{A^\pm}$ are symmetric. Moreover, given $\tilde\Phi\in D(\mathcal{H}^*_{A^\pm})$, we have (VI.7) for any $\Phi\in D(\mathcal{H}_{A^\pm})$ again. For all $\psi_\pm,\chi_\pm\in C_0^\infty(]0,\pi[\times]0,2\pi[)$, (VI.1) and (VI.6) assure there exists $\Phi\in D(\mathcal{H}_{A^\pm})$ such that

$$\Gamma(\Phi)=\left(A^+(\psi_+,\chi_+),\psi_+,\chi_+\right)\quad or\quad \left(\psi_-,\chi_-,A^-(\psi_-,\chi_-)\right).$$

Therefore

$$\left\langle\begin{pmatrix}\psi_\pm\\ \chi_\pm\end{pmatrix},\begin{pmatrix}\tilde\psi_\mp\\ \tilde\chi_\mp\end{pmatrix}-A^\pm\begin{pmatrix}\tilde\psi_\pm\\ \tilde\chi_\pm\end{pmatrix}\right\rangle_{L^2(S^2;\mathbb{C}^2)}=0.$$

We conclude that $\tilde\Phi\in D(\mathcal{H}_{A^\pm})$. $\square$

Finally we consider the operators $\mathcal{H}_{APS}, \mathcal{H}_{mAPS}$ associated with the *APS* and *mAPS* boundary conditions:

$$D\left(\mathcal{H}_{APS}\right) := \left\{\Phi \in D(H_m);\ \ \| \mathbf{K}_{+}\Phi(x,.) \|_{\mathcal{W}^0} = o\left(\sqrt{\frac{\pi}{2}-x}\right),\ \ x \to \frac{\pi}{2}\right\},$$

$$\begin{aligned} &D\left(\mathcal{H}_{mAPS}\right) \\ &:= \left\{\Phi \in D(H_m);\ \ \| \mathbf{K}_{+}\left(Id+\gamma^1\right)\Phi(x,.) \|_{\mathcal{W}^0} = o\left(\sqrt{\frac{\pi}{2}-x}\right),\ \ x \to \frac{\pi}{2}\right\}, \end{aligned}$$

where $\mathbf{K}_{+}$ is defined by (IV.7).

Proposition VI.4. *When* $0 < m < \frac{1}{2}$, *we have*

$$D\left(\mathcal{H}_{APS}\right) = D\left(\mathcal{H}_{mAPS}\right) = \left\{\Phi \in D(H_m);\ \ \mathbf{K}_{+}\Phi_{+} = \mathbf{K}_{+}\Phi_{-} = 0\right\},$$

and $\mathcal{H}_{APS} = \mathcal{H}_{mAPS}$ *is self-adjoint on* $\mathcal{L}^2$.

Proof of Proposition VI.4. By (V.7), we have

$$\begin{aligned} \mathbf{K}_{+}\Phi(x) = &\left(\frac{\pi}{2}-x\right)^{-m}\begin{pmatrix} 1 \\ -i\mathcal{J}^* \\ -i \\ \mathcal{J}^* \end{pmatrix}\left(\psi_{-}+i\mathcal{J}\chi_{-}\right) \\ &+\left(\frac{\pi}{2}-x\right)^{m}\begin{pmatrix} 1 \\ i\mathcal{J}^* \\ i \\ \mathcal{J}^* \end{pmatrix}\left(\psi_{+}-i\mathcal{J}\chi_{+}\right)+\mathbf{K}_{+}\varphi(x), \end{aligned}$$

$$\begin{aligned} \mathbf{K}_{+}\left(Id+\gamma^1\right)\Phi(x) = &(1-i)\left(\frac{\pi}{2}-x\right)^{-m}\begin{pmatrix} 1 \\ -i\mathcal{J}^* \\ -i \\ \mathcal{J}^* \end{pmatrix}\left(\psi_{-}+i\mathcal{J}\chi_{-}\right) \\ &+(1+i)\left(\frac{\pi}{2}-x\right)^{m}\begin{pmatrix} 1 \\ i\mathcal{J}^* \\ i \\ \mathcal{J}^* \end{pmatrix}\left(\psi_{+}-i\mathcal{J}\chi_{+}\right) \\ &+\mathbf{K}_{+}\left(Id+\gamma^1\right)\varphi(x), \end{aligned}$$

thus we deduce from (V.8) that

$$\begin{aligned} \| \mathbf{K}_{+}\Phi(x,.) \|_{\mathcal{W}^0} = o\left(\sqrt{\frac{\pi}{2}-x}\right) &\Leftrightarrow \| \mathbf{K}_{+}\left(Id+\gamma^1\right)\Phi(x,.) \|_{\mathcal{W}^0} = o\left(\sqrt{\frac{\pi}{2}-x}\right) \\ &\Leftrightarrow \psi_{\pm} = \pm i\mathcal{J}\chi_{\pm} \\ &\Leftrightarrow \mathbf{K}_{+}\Phi_{+} = \mathbf{K}_{+}\Phi_{-} = 0. \end{aligned} \tag{VI.8}$$

This equality assures that for $\Phi, \tilde{\Phi} \in D(\mathcal{H}_{APS})$, we have $< \Gamma\Phi, \mathcal{Q}\Gamma\Phi >_{\mathcal{W}^0} = 0$, i.e. $\mathcal{H}_{APS}$ is symmetric. Moreover, for any $\Phi \in D(\mathcal{H}_{APS})$, $\tilde{\Phi} \in D\left(\mathcal{H}^*_{APS}\right)$, we have

$$< \chi_+, \tilde{\chi}_- - i\mathcal{J}^*\tilde{\psi}_- >_{L^2(S^2)} - < \chi_-, \tilde{\chi}_+ + i\mathcal{J}^*\tilde{\psi}_+ >_{L^2(S^2)} = 0. \qquad \text{(VI.9)}$$

Since $C_0^\infty(]0,\pi[\times]0,2\pi[) \subset W_+^1$, for any $\chi_\pm \in C_0^\infty(]0,\pi[\times]0,2\pi[)$, $\mathcal{J}\chi_\pm$ belongs to W_-^1 and by (VI.1) there exists $\Phi \in D(H_m)$ such that

$$\Gamma(\Phi) = \begin{pmatrix} -i\mathcal{J}\chi_- \\ \chi_- \\ i\mathcal{J}\chi_+ \\ \chi_+ \end{pmatrix}.$$

But such a Φ satisfies (VI.8), that means that Φ is in the domain of $\mathcal{H}_{APS}$. Since $\chi_\pm$ are arbitrary, (VI.9) implies

$$\tilde{\chi}_\pm \pm i\mathcal{J}^*\tilde{\psi}_\pm = 0,$$

that is equivalent to (VI.8). We conclude that $\tilde{\Phi} \in D(\mathcal{H}_{APS})$. □

The remainder of the article is devoted to the demonstrations of the theorems of Part 3. As we have explained above, it is sufficient to consider only the case $M > 0$, since the chiral transform changes the sign of the mass.

Proof of Theorem III.4. We denote by $\mathbb{H}$ the operator $\mathbf{H}_M$ endowed with the domain $D(\mathbb{H}) := \left[C_0^\infty(\mathbb{B})\right]^4$. Since $\mathbf{H}_M = \mathbf{S}H_m\mathbf{S}^{-1}$, Proposition VI.2 assures that $\mathbf{H}_M$ is essentially self-adjoint on $\mathbf{S}\left(C_0^\infty\left(]0,\frac{\pi}{2}\left[_x \times\right]0,\pi[_\theta\times]0,2\pi[_\varphi;\mathbb{C}^4\right)\right)$ when $M \geq \sqrt{\frac{\Lambda}{12}}$. Proposition IV.1 and the Sobolev Imbedding Theorem imply that this set is included in $\left[C_0^\infty(\mathbb{B})\right]^4$. Since $\mathbb{H}$ is symmetric, we deduce that it is essentially self-adjoint.

To determine its domain and establish the elliptic estimate, we prove an inequality of Hardy type. Given a real valued function $f \in C_0^1(]0,1[)$, an integration by part gives:

$$\int_0^1 f^2(\varrho)\frac{\varrho^2}{(1-\varrho^2)^2}d\varrho = -\frac{1}{2}\int_0^1 f^2(\varrho)\frac{\varrho}{1-\varrho^2}d\varrho + \int_0^1\left(f(\varrho)\frac{\varrho}{1-\varrho^2}\right)\left(\varrho f'(\varrho)\right)d\varrho$$
$$\leq \frac{1}{2}\int_0^1 f^2(\varrho)\frac{\varrho^2}{(1-\varrho^2)^2}d\varrho + \frac{1}{2}\int_0^1 \varrho^2 f'^2(\varrho)d\varrho,$$

hence by density we get that $\frac{1}{1-\varrho}\Psi \in \mathbf{L}^2$ when $\Psi \in \left[H_0^1(\mathbb{B})\right]^4$, and we have the following Hardy estimate:

$$\forall\Psi \in H_0^1(\mathbb{B}), \quad \int_{\mathbb{B}} \mid \Psi(\mathbf{x}) \mid^2 \frac{1}{(1-\mid \mathbf{x} \mid^2)^2}d\mathbf{x} \leq \int_{\mathbb{B}} \mid \nabla_{\mathbf{x}}\Psi(\mathbf{x}) \mid^2 d\mathbf{x}. \qquad \text{(VI.10)}$$

Thus we see that $\left[H_0^1(\mathbb{B})\right]^4 \subset D(\mathbf{H}_M)$ and the graph norm of $\mathbf{H}_M$ is bounded by the H_0^1 norm. Conversely, for $\Psi \in \left[C_0^\infty(\mathbb{B})\right]^4$, we use the Fourier transform of Ψ, the Parseval formula and the anticommutations relations (III.2) to remark that

$$\int_{\mathbb{B}} \sum_{1\leq i<j\leq 3} \partial_i\Psi^*\gamma^i\gamma^j\partial_j\Psi + \partial_j\Psi^*\gamma^j\gamma^i\partial_i\Psi \, d\mathbf{x} = 0,$$

then we calculate

$$\int_{\mathbb{B}} \left| \gamma^j \partial_j \Psi + \frac{2iM}{1-\varrho^2}\sqrt{\frac{3}{\Lambda}}\Psi \right|^2 d\mathbf{x} = \int_{\mathbb{B}} \mid \nabla_{\mathbf{x}}\Psi \mid^2 + \frac{12M^2}{\Lambda(1-\varrho^2)^2} \mid \Psi \mid^2 + \frac{4iM}{(1-\varrho^2)^2}\sqrt{\frac{3}{\Lambda}} x_j \Psi^* \gamma^j \Psi \, d\mathbf{x}.$$

Therefore the Hardy inequality (VI.10) shows that when $M > \sqrt{\frac{\Lambda}{12}}$, the elliptic estimate (III.32) holds:

$$\parallel \mathbf{H}_M \Psi \parallel^2_{\mathbf{L}^2} \geq \left(1 - M\sqrt{\frac{12}{\Lambda}}\right)^2 \int_{\mathbb{B}} \mid \nabla_{\mathbf{x}} \Psi \mid^2 d\mathbf{x},$$

and the H^1_0-norm on $\left[C^\infty_0(\mathbb{B})\right]^4$ is bounded by the graph norm of $\mathbf{H}_M$. Since $\mathbb{H}$ is essentially self-adjoint, we have $\mathbb{H}^* = \overline{\mathbb{H}}$. On the one hand $D(\mathbb{H}^*) = D(\mathbf{H}_M)$. On the other hand $D(\overline{\mathbb{H}})$ is the closure of $\left[C^\infty_0(\mathbb{B})\right]^4$ for the graph norm. We conclude that $D(\mathbf{H}_M) = \left[H^1_0(\mathbb{B})\right]^4$ when $M > \sqrt{\frac{\Lambda}{12}}$ and the first part of the theorem is proved.

Now when $0 < M < \sqrt{\frac{\Lambda}{12}}$, and $\mathbf{A}^\pm$ satisfy (III.30) and (III.31), then $A^\pm = S^*_{11}\mathbf{A}^\pm S_{11}$ where S_{11} is defined by (IV.9), satisfy (VI.5) and (VI.6). We deduce fom Proposition VI.3 that $\mathbb{H}_{\mathbf{A}^\pm} = \mathbf{S}\mathcal{H}_{A^\pm}\mathbf{S}^{-1}$ is self-adjoint. On the other hand, we have $\mathbb{H}_{\mathcal{B}_{APS}} = \mathbf{S}\mathcal{H}_{APS}\mathbf{S}^{-1} = \mathbf{S}H_{mAPS}\mathbf{S}^{-1} = \mathbb{H}_{\mathcal{B}_{mAPS}}$ that is self-adjoint by Proposition VI.4.

Finally $\left\{\Psi \in D(\mathbf{H}_M), \;\; \parallel \Psi \parallel^2_{\mathbf{L}^2} + \parallel \mathbf{H}_M \Psi \parallel^2_{\mathbf{L}^2} \leq 1\right\}$ is equal to $\mathbf{S}K$ where K defined by (V.30) is compact by Proposition V.4. We conclude that the resolvent of any self-adjoint realization of $\mathbf{H}_M$ is compact. □

Proof of Theorem III.5. Theorem III.4 provides a lot of solutions of the initial value problem: if $\mathbb{H}$ is a self-adjoint realization of $\mathbf{H}_M$, $\Psi(t) = e^{it\sqrt{\frac{\Lambda}{3}}\mathbb{H}}\Psi_0$ is a solution of (III.9), (III.11), (III.12) and (III.13).

Since the maximal globally hyperbolic domain in $\mathcal{E}$ including $\{t=0\}\times[0,\frac{\pi}{2}[_x \times S^2_{\theta,\varphi}$ is given by $0 \leq\mid t \mid< \sqrt{\frac{3}{\Lambda}}\left(\frac{\pi}{2} - x\right)$, the maximal globally hyperbolic domain in $\mathcal{M}$ including $\{t=0\}\times\mathbb{R}^3$ is defined by the same relation, that is in $(t,\mathbf{x})$ coordinates: $0 \leq\mid t \mid< \sqrt{\frac{3}{\Lambda}}\left(\frac{\pi}{2} - 2\arctan\varrho\right)$. We show that all the solutions are equal in this domain. Given Ψ satisfying (III.9), (III.11), we introduce for all $\varepsilon > 0$,

$$\Psi_\varepsilon(t) = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \Psi(s)ds.$$

It is clear that $\Psi_\varepsilon \in C^1(\mathbb{R}_t;\mathbf{L}^2)$, $\Psi_\varepsilon \to \Psi$ in $C^0(\mathbb{R}_t;\mathbf{L}^2)$ as $\varepsilon \to 0$. Moreover we can see that Ψ_ε is a solution of (III.9), thus $\mathbf{H}_M\Psi_\varepsilon \in C^0(\mathbb{R}_t;\mathbf{L}^2)$ and

$$\frac{\partial}{\partial t}\left(\sqrt{\frac{3}{\Lambda}} \mid \Psi_\varepsilon \mid^2\right) + \sum_{j=1}^3 \frac{1+\varrho^2}{2}\frac{\partial}{\partial x^j}\left(\Psi^*_\varepsilon \gamma^0 \gamma^j \Psi_\varepsilon\right) = 0.$$

We integrate this equality on $\left\{(t,\mathbf{x}),\ 0\leq t\leq T,\ \varrho\leq\tan\left(\frac{\pi}{4}-T\sqrt{\frac{\Lambda}{12}}\right)\right\}$ where $0<T<\frac{\pi}{2}\sqrt{\frac{3}{\Lambda}}$, and applying the Green formula we get

$$\int_{\varrho\leq\tan(\frac{\pi}{4}-T\sqrt{\frac{\Lambda}{12}})}\mid\Psi_\varepsilon(T,\mathbf{x})\mid^2 d\mathbf{x}=\int_{\mathbb{B}}\mid\Psi_\varepsilon(0,\mathbf{x})\mid^2 d\mathbf{x}$$
$$-\int_{0\leq t=\sqrt{\frac{3}{\Lambda}}(\frac{\pi}{2}-2\arctan\varrho)\leq T}\mid\Psi(t,\mathbf{x})\mid^2-\frac{x_j}{\varrho}\Psi^*\gamma^0\gamma^j\Psi(t,\mathbf{x})d\sigma.$$

The last integral is non-negative since $\mid x_j\gamma^j\Psi\mid\leq\varrho\mid\Psi\mid$, and taking the limit as $\varepsilon\to 0$, we obtain

$$\int_{\varrho\leq\tan(\frac{\pi}{4}-T\sqrt{\frac{\Lambda}{12}})}\mid\Psi(T,\mathbf{x})\mid^2 d\mathbf{x}\leq\int_{\mathbb{B}}\mid\Psi(0,\mathbf{x})\mid^2 d\mathbf{x}.$$

We conclude that $\Psi=0$ for $0\leq\mid t\mid<\sqrt{\frac{3}{\Lambda}}\left(\frac{\pi}{2}-2\arctan\varrho\right)$ if $\Psi_0=0$. Finally when $M\geq\sqrt{\frac{\Lambda}{12}}$, we use the fact that $\mathbf{H}_M$ is self-adjoint to write

$$\frac{d}{dt}\left(e^{-it\sqrt{\frac{\Lambda}{3}}\mathbf{H}_M}\Psi_\varepsilon(t)\right)=e^{-it\sqrt{\frac{\Lambda}{3}}\mathbf{H}_M}\left(-i\sqrt{\frac{\Lambda}{3}}\mathbf{H}_M\Psi_\varepsilon(t)+\partial_t\Psi_\varepsilon(t)\right)=0,$$

and we deduce that $\Psi_\varepsilon(t)=e^{it\sqrt{\frac{\Lambda}{3}}\mathbf{H}_M}\Psi_\varepsilon(0)$. Taking the limit in ε again, we conclude that $\Psi(t)=e^{it\sqrt{\frac{\Lambda}{3}}\mathbf{H}_M}\Psi_0$. □

Proof of Theorem III.6. Since the spectrum of $\mathbb{H}$ is discrete, and 0 is not an eigenvalue when $M>0$, there exists an orthonormal basis of eigenvectors, $(\Psi_k)_{k\in\mathbb{N}}$, with $\mathbf{H}_M\Psi_k=\lambda_k\sqrt{\frac{\Lambda}{3}}\Psi_k$, $\lambda_k\in\mathbb{R}^*$. Now the crucial point is that

$$\int_{\mathbb{B}}\Psi_k^*\gamma^0\gamma^5\Psi_k(\mathbf{x})d\mathbf{x}=0. \tag{VI.11}$$

To see that, we note that $\mathbf{H}_M\gamma^0\gamma^5=-\gamma^0\gamma^5\mathbf{H}_M$, and we write

$$\begin{aligned}<\Psi_k,\gamma^0\gamma^5\Psi_k>_{\mathbf{L}^2}&=\frac{1}{\lambda_k}<\mathbf{H}_M\Psi_k,\gamma^0\gamma^5\Psi_k>_{\mathbf{L}^2}\\&=-\frac{1}{\lambda_k}<\Psi_k,\gamma^0\gamma^5\mathbf{H}_M\Psi_k>_{\mathbf{L}^2}\\&=-<\Psi_k,\gamma^0\gamma^5\Psi_k>_{\mathbf{L}^2}.\end{aligned}$$

We can expand Ψ on this basis:

$$\Psi(t,\mathbf{x})=\sum_{k\in\mathbb{N}}c_k e^{i\lambda_k t}\Psi_k(\mathbf{x}),\quad c_k\in\mathbb{C},\quad\sum_{k\in\mathbb{N}}\mid c_k\mid^2<\infty,$$

and taking advantage of (VI.11) we evaluate

$$\frac{1}{T}\int_0^T\int_{\mathbb{B}}\Psi^*\gamma^0\gamma^5\Psi(t,\mathbf{x})d\mathbf{x}dt=\sum_{\lambda_p\neq\lambda_q}c_pc_q^*\frac{e^{i(\lambda_p-\lambda_q)T}-1}{i(\lambda_p-\lambda_q)T}\int_{\mathbb{B}}\Psi_q^*\gamma^0\gamma^5\Psi_p(\mathbf{x})d\mathbf{x}.$$

The dominated convergence theorem assures that this sum tends to 0 as $T\to\infty$. □

VII. Appendix. Breitenlohner-Freedman Bounds for the Scalar Waves

We consider the Klein-Gordon equation on the Anti-de Sitter space-time

$$\mid g \mid^{-\frac{1}{2}} \partial_\mu \left(\mid g \mid^{\frac{1}{2}} g^{\mu\nu} \partial_\nu u \right) - \alpha \frac{\Lambda}{3} u = 0,$$

where $\alpha \in \mathbb{R}$ is a coefficient linked to the mass ; the equation with $\alpha = 2$ is conformally invariant and corresponds to the massless case. Using the radial coordinate x given by (II.1), we introduce $f(t, x, \omega) := ru(t\sqrt{\frac{3}{\Lambda}}, r, \omega)$ that is solution of $\partial_t^2 f + \mathbf{h} f = 0$ with

$$\mathbf{h} := -\partial_x^2 + \frac{2-\alpha}{\cos^2 x} - \frac{1}{\sin^2 x} \Delta_{S^2_\omega}. \tag{VII.1}$$

First we investigate the positivity of the potential energy

$$E(f) := \int_0^{\frac{\pi}{2}} \int_{S^2} \mid \partial_x f \mid^2 + \frac{2-\alpha}{\cos^2 x} \mid f \mid^2 + \frac{1}{\sin^2 x} \mid \nabla_{S^2_\omega} f \mid^2 dx d\omega.$$

To estimate the second term, we employ a Hardy inequality. Given $\phi \in C^1_0([0, \frac{1}{2}[; \mathbb{R})$ an integration by part gives

$$\int_0^{\frac{\pi}{2}} \frac{1}{\cos^2 x} \phi^2(x) dx = -\int_0^{\frac{\pi}{2}} 2 \frac{\phi(x)}{\cos x} \phi'(x) \sin x dx \leq \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{1}{\cos^2 x} \phi^2(x) dx + \int_0^{\frac{\pi}{2}} 2\phi'^2(x) \sin^2 x dx,$$

hence

$$\int_0^{\frac{\pi}{2}} \frac{1}{\cos^2 x} \phi^2(x) dx \leq 4 \int_0^{\frac{\pi}{2}} \phi'^2(x) dx.$$

We deduce that for all $f \in C_0^\infty(]0, \frac{\pi}{2}[_x \times S^2_\omega)$,

$$< \mathbf{h} f, f >_{L^2} \geq \min(9 - 4\alpha, 1) \int_0^{\frac{\pi}{2}} \int_{S^2} \mid \partial_x f \mid^2 dx d\omega + \int_0^{\frac{\pi}{2}} \int_{S^2} \frac{1}{\sin^2 x} \mid \nabla_{S^2_\omega} f \mid^2 dx d\omega \geq \min(\frac{9}{4} - \alpha, \frac{1}{4}) \parallel f \parallel^2_{L^2},$$

and we conclude that the operator $\mathbf{h}$ endowed with the domain $D(\mathbf{h}) = C_0^\infty(]0, \frac{\pi}{2}[_x \times S^2_\omega)$, is (strictly) positive when α is (strictly) smaller than the upper bound of Breitenlohner-Freedman:

$$\alpha \leq \frac{9}{4} \quad (respectively \ \ \alpha < \frac{9}{4}). \tag{VII.2}$$

We note that for $\alpha = 9/4$ and $f(x, \omega) = \sqrt{\frac{\cos x}{1+\sin x}}$, we have $E(f) = 0$, hence $f \in Ker(\mathbf{h}^*) \neq \{0\}$.

To study the self-adjointness, we expand $f(x,.)$ on the basis of the spherical harmonics $\left(Y_l^m\right)_{l,m}$ by writing

$$L^2\left(]0,\frac{\pi}{2}[_x\times S^2_\omega\right)=\bigoplus_{l=0}^{\infty}\mathbf{L}^2_l,\quad \mathbf{L}^2_l:=\bigoplus_{m=-l}^{m=l}L^2\left(]0,\frac{\pi}{2}[_x\right)\otimes Y_l^m,$$

therefore $\mathbf{h}$ is unitarily equivalent to $\bigoplus_{l=0}^{\infty}\mathbf{h}_l$ where:

$$\mathbf{h}_l:=-\frac{d^2}{dx^2}+\frac{2-\alpha}{\cos^2 x}+\frac{l(l+1)}{\sin^2 x},\quad D(\mathbf{h}_l)=\bigoplus_{m=-l}^{m=l}C_0^\infty(]0,\frac{\pi}{2}[)\otimes Y_l^m.$$

Since $\frac{2-\alpha}{\cos^2 x}+\frac{l(l+1)}{\sin^2 x}-\frac{2-\alpha}{(\frac{\pi}{2}-x)^2}-\frac{l(l+1)}{x^2}$ is a real valued function, bounded on $]0,\frac{\pi}{2}[$, the symmetric form of the Kato-Rellich theorem (see [31], Theorem X.13) assures that $\mathbf{h}_l$ is essentially self-adjoint iff

$$\mathbf{k}_l:=-\frac{d^2}{dx^2}+\frac{2-\alpha}{(\frac{\pi}{2}-x)^2}+\frac{l(l+1)}{x^2},\quad D(\mathbf{k}_l)=\bigoplus_{m=-l}^{m=l}C_0^\infty(]0,\frac{\pi}{2}[)\otimes Y_l^m,$$

is essentially self-adjoint. By Theorem X.10 of [31], $\mathbf{k}_l$ is in the limit point case at zero when $l\geq 1$, and in the limit point case at $\frac{\pi}{2}$ if $2-\alpha\geq\frac{3}{4}$, i.e. α is smaller than the lower bound of Breitenlhoner-Freedman

$$\alpha\leq\frac{5}{4},\tag{VII.3}$$

and if $\alpha>\frac{5}{4}$, $\mathbf{k}_l$ is in the limit circle case at $\frac{\pi}{2}$. Then the Weyl's limit point-limit circle criterion (see e.g. [30], Theorems 6.3 and 6.5), assures that $\mathbf{k}_l$ is essentially self-adjoint when $l\geq 1,\alpha\leq\frac{5}{4}$, and there exists an infinity of self-adjoint extensions associated with boundary conditions at $\frac{\pi}{2}$ when $l\geq 1,\alpha>\frac{5}{4}$. The case $l=0$ is particular. For $\alpha<\frac{9}{4}$, the solutions of $-u''+(2-\alpha)(\frac{\pi}{2}-x)^{-2}u=0$ are $u=c(\frac{\pi}{2}-x)^{\frac{1}{2}+\sqrt{\frac{9}{4}-\alpha}}+c'(\frac{\pi}{2}-x)^{\frac{1}{2}-\sqrt{\frac{9}{4}-\alpha}}$, therefore $\mathbf{k}_0$ is always in the limit circle case at $\frac{\pi}{2}$ and there exists a lot of self-adjoint extensions. By the Kato-Rellich theorem ([31], Theorem X.12), the same results are true for $\mathbf{h}_l$. Since the spherically symmetric fields play a peculiar role, we introduce their orthogonal space

$$\mathbf{L}^2_*:=\left\{f\in L^2(]0,\frac{\pi}{2}[\times S^2);\quad\forall g\in L^2(]0,\frac{\pi}{2}[),\ \int f(x,\omega)g(x)dxd\omega=0\right\}=\bigoplus_{l=1}^{\infty}\mathbf{L}^2_l,$$

and $\mathbf{h}_*$ denotes $\mathbf{h}$ endowed with the domain $D(\mathbf{h}_*)=C_0^\infty(]0,\frac{\pi}{2}[\times S^2)\cap\mathbf{L}^2_*$, and considered as a densely defined operator on $\mathbf{L}^2_*$. Since this operator is strictly positive when $\alpha<\frac{9}{4}$, it is essentially selfadjoint iff its range is dense ([31], Theorem X.26). We easily prove that $(Ran(\mathbf{h}_*))^{\perp_{\mathbf{L}^2_*}}=\oplus_{l=1}^{\infty}(Ran(\mathbf{h}_l))^{\perp_{\mathbf{L}^2_l}}$, and we conclude that $\mathbf{h}_*$ is essentially self-adjoint when $\alpha\leq\frac{5}{4}$. Finally we have proved the following:

Theorem VII.1. *When $\alpha\leq\frac{9}{4}$ (resp. $\alpha<\frac{9}{4}$), $\mathbf{h}$ is a positive (resp. strictly positive) symmetric operator on $L^2(]0,\frac{\pi}{2}[\times S^2)$. When $\frac{5}{4}<\alpha<\frac{9}{4}$ there exists an infinity of self-adjoint extensions of $\mathbf{h}_*$ on $\mathbf{L}^2_*$, associated with boundary conditions on $\{\frac{\pi}{2}\}\times S^2$. When $\alpha\leq\frac{5}{4}$, $\mathbf{h}_*$ is essentially self-adjoint.*

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Communicated by G.W. Gibbons