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Quantum Vacuum Polarization at the Black-Hole Horizon

by

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ABSTRACT. – We prove in the case of the Klein-Gordon quantum field, the emergence of the Hawking-Unruh state at the future Black-Hole horizon created by a spherical gravitational collapse.

RÉSUMÉ. – On prouve l'émergence de l'état quantique d'Hawking-Unruh pour un champ de Klein-Gordon, à l'horizon d'un trou noir créé par un effondrement gravitationnel sphérique.

I. INTRODUCTION

The aim of this paper is to give a rigorous mathematical proof of the famous result by S. Hawking [16], on the emergence of a thermal state at the last moment of a gravitational collapse. The only mathematical approach to the quantum states of a Black-Hole-type space-time are due to J. Dimock and B.S. Kay [11], [10], and deal with the eternal Schwarzschild Black-Hole. To get the Hawking effect in the future, these authors assume an *ad hoc* quantum state on the past Black-Hole Horizon. In this paper we consider a spherical star, stationary in the past, and collapsing to a Black-Hole in the future. The quantum state is defined by the standard Fock vacuum in the past. Then we prove that this state is thermal near the future Black-

Hole Horizon with the Hawking temperature. This is a consequence of the infinite Doppler effect caused by the moving star boundary. The effects of this phenomenon on the scattering of classical fields are studied in [3]. The setting is considerably more complicated than for the asymptotically flat space-times [9], or for the eternal Black-Hole [2], and we shall see that the Hawking radiation is associated with a very sharp estimate of the propagator (Remark II.4 below). For the sake of simplicity we only consider scalar fields, but our analysis could be extended to the Dirac field [20].

We recall that the space-time outside a spherical star of mass $M > 0$, and radius $\rho(t) > 2M$, is described in Schwarzschild coordinates by the globally hyperbolic manifold

$$\mathcal{M} = \left\{(t, r, \omega) \in \mathbb{R}\times]\rho(t), \infty[\times S^2\right\} \tag{I.1}$$

with the Schwarzschild metric

$$g_{\mu\nu}dx^\mu dx^\nu = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + sin^2\theta d\varphi^2\right). \tag{I.2}$$

We introduce the Regge-Wheeler tortoise coordinate r_* defined by

$$r_* = r + 2M \ln(r - 2M), \tag{I.3}$$

and we put

$$z(t) = \rho(t) + 2M \ln\left(\rho(t) - 2M\right). \tag{I.4}$$

Then, according to [3], if we assume the star to be stationnary in the past, and collapsing to a black-hole in the future, the natural hypotheses for the function z are

$$\begin{cases} z \in C^2(\mathbb{R}), \\ \forall t \leq 0, \;\; z(t) = z(0) < 0, \\ \forall t \in \mathbb{R}, \;\; -1 < \dot{z}(t) \leq 0, \\ z(t) = -t - Ae^{-2\kappa t} + \zeta(t), \;\; A > 0, \\ \mid \zeta(t) \mid + \mid \dot{\zeta}(t) \mid = O(e^{-4\kappa t}), \;\; t \longrightarrow +\infty, \end{cases} \tag{I.5}$$

where κ is the surface gravity of the future black-hole horizon:

$$\kappa = \frac{1}{4M} \, . \tag{I.6}$$

The *Black-Hole Horizon* is reached as $r_* \rightarrow -\infty$, $t \rightarrow +\infty$, $r_* + t = Cst. > 0$.

We consider the scalar field of mass $m \geq 0$, obeying the Klein-Gordon equation

$$\text{(I.7)} \qquad \left\{\frac{\partial^2}{\partial t^2} - \frac{1}{r^2}\frac{\partial}{\partial r_*}r^2\frac{\partial}{\partial r_*} + \left(1 - \frac{2M}{r}\right)\left(-\frac{\Delta_{S^2}}{r^2} + m^2\right)\right\}\Psi = 0,$$

with the homogeneous Dirichlet boundary condition

$$\text{(I.8)} \qquad \Psi(t, r_* = z(t), \omega) = 0, \quad t \in \mathbb{R}, \quad \omega \in S^2.$$

We have studied the classical solutions in spaces of finite energy, of Sobolev type $H^1 \times L^2$ in [3]. For quantum solutions we need a fine analysis of the propagator in spaces of type $H^{\frac{1}{2}} \times H^{-\frac{1}{2}}$. Taking advantage of the spherical invariance, we reduce the problem to solving an equation in one space dimension, which we do in second part. Then we get the crucial asymptotic behaviour for the three dimensional problem in the third part, and we prove the Hawking effect in part 4.

We end this introduction by giving some bibliographic information. After the historic paper by Hawking [16], a huge litterature has been devoted by physicists to the quantum radiation of black-holes. This work is more particularly connected with the following papers: Candelas [6], Fredenhagen and Haag [12], Gibbons and Hawking [14], Sewell [22], [23], Unruh [24], Wald [25], York [28], and see also the references in the classic monographs on quantum field theory in curved space-time by Birrel and Davies [4], DeWitt [7], Fulling [13], Haag [15], Wald [26], as well as the volume [1].

II. ONE DIMENSIONAL STUDY

Taking advantage of the spherical invariance of the problem, it is convenient to expand solutions Ψ of (I.7) on the basis of spherical harmonics. We note that

$$\text{(II.1)} \qquad \left\{\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + \left(1 - \frac{2M}{r}\right)\left(-\frac{\Delta_{S^2}}{r^2} + \frac{2M}{r^3} + m^2\right)\right\}(r\Psi) = 0,$$

then by puting

$$\text{(II.2)} \qquad \Psi(t, r_*, \omega) = \sum_{l=0}^{l=\infty}\sum_{m=-l}^{m=l}\frac{1}{r}u_{l,m}(t, r_*)Y_{l,m}(\omega),$$

$u_{l,m}(t,r_*)$ is a solution of

$$\frac{\partial^2 u_{l,m}}{\partial t^2} - \frac{\partial^2 u_{l,m}}{\partial r_*^2} + V_l(r_*)u_{l,m} = 0, \ t \in \mathbb{R}, \ r_* > z(t),$$

$$u_{l,m}(t, z(t)) = 0,$$

where the potential V_l is given for $l \in \mathbb{N}$ by

$$V_l(r_*) = \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3} + m^2\right). \tag{II.3}$$

Therefore we consider the general mixed problem

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + V(x)u = 0, \tag{II.4}$$

in

$$\{t \in \mathbb{R}, \ x > z(t)\} \tag{II.5}$$

with the Dirichlet condition

$$u(t, z(t)) = 0, \tag{II.6}$$

where the function z satisfies (I.5) and the potential V is such that there exist $\kappa > 0$, $m \geq 0$, $\mu \in \mathbb{R}$, $\varepsilon > 0$ with

$$\begin{cases} V \in C^1(\mathbb{R}), \ 0 \leq V(x), \\ \lim_{t\to\infty}\left(e^{(\kappa+\varepsilon)t} \sup\{V(x) : x \leq -t\}\right) = 0, \\ \int_0^\infty e^{-2\kappa t} \sup\{\max(-V'(x), 0); \ z(t) \leq x \leq -t\}\, dt < \infty, \\ V(x) = m^2 + \frac{\mu}{x} + O(x^{-1-\varepsilon}), \ x \to +\infty. \end{cases} \tag{II.7}$$

Obviously, the potentials $V_l(x)$ defined by (II.3) satisfy assumptions (II.7) with $\kappa = \frac{1}{4M}$ and r is an implicit function of x given by

$$x = r + 2M \log(r - 2M), \ r > 2M. \tag{II.8}$$

The solution $u(t,x)$ of (II.4), (I.8) at time t is associated with the data at time s by a propagator $U_V(t,s)$:

$$^t(u(t), \partial_t u(t)) = U_V(t,s)(u(s), \partial_t u(s)). \tag{II.9}$$

According to [3], we introduce the Hilbert space of finite energy fields $\mathcal{H}(V,t)$ as the completion of $C_0^\infty(]z(t),\infty[)\times C_0^\infty(]z(t),\infty[)$ for the norm

(II.10)
$$\| {}^t(f,p) \|^2_{\mathcal{H}(V,t)} = \int_{z(t)}^{\infty} | f'(x) |^2 + | p(x) |^2 + V(x) | f(x) |^2 \, dx.$$

On the other hand, because the infinite Doppler effect, we need the Hilbert space $\mathcal{H}_1(V,t)$, completion of $C_0^\infty(]z(t),\infty[)\times C_0^\infty(]z(t),\infty[)$ for the norm

(II.11)
$$\begin{aligned}\| {}^t(f,p) \|^2_{\mathcal{H}_1(V,t)} &= \frac{1}{2}\int_{z(t)}^{-t} | x+t || f'(x)+p(x) |^2 \\ &+ | f'(x)-p(x) |^2 + 2V(x) | f(x) |^2 \, dx \\ &+ \int_{-t}^{\infty} | f'(x) |^2 + | p(x) |^2 + V(x) | f(x) |^2 \, dx.\end{aligned}$$

The main properties of the propagator $U_V(t,s)$ are given by the following

PROPOSITION II.1. – *There exists a constant $C_V > 0$ such that*

(II.12)
$$s \leq t \Rightarrow \| U_V(t,s) \|_{\mathcal{L}(\mathcal{H}(V,s),\mathcal{H}(V,t))} \leq 1,$$

(II.13)
$$C_V^{-1} e^{\kappa t} \leq \| U_V(0,t) \|_{\mathcal{L}(\mathcal{H}(V,t),\mathcal{H}(V,0))},$$

(II.14)
$$s \leq t \Rightarrow \| U_V(s,t) \|_{\mathcal{L}(\mathcal{H}(V,t),\mathcal{H}(V,s))} \leq C_V e^{\kappa(t-s)},$$

(II.15)
$$\sup_{s,t\in\mathbb{R}} \| U_V(t,s) \|_{\mathcal{L}(\mathcal{H}_1(V,s),\mathcal{H}_1(V,t))} \leq C_V < \infty.$$

In fact, the relevant space in Quantum Fields Theory, is a third space, of Sobolev type $H^{1/2}\times H^{-1/2}$. Then we consider the self adjoint operators on $L^2(]z(t),\infty[)$:

(II.16)
$$\mathbb{H}_{V,t} = -\frac{d^2}{dx^2} + V,$$

with dense domains

(II.17)
$$D(\mathbb{H}_{V,t}) = \left\{ f \in L^2(]z(t),\infty[);\ f'' \in L^2(]z(t),\infty[),\ f(z(t)) = 0 \right\},$$

and we define $\mathcal{H}^{\frac{1}{2}}(V,t)$ as the completion of $D(\mathbb{H}_{V,t}^{\frac{1}{4}}) \times D(\mathbb{H}_{V,t}^{-\frac{1}{4}})$ for the norm

$$\text{(II.18)} \quad \| {}^t(f,p) \|^2_{\mathcal{H}^{\frac{1}{2}}(V,t)} = \| \mathbb{H}_{V,t}^{\frac{1}{4}} f \|^2_{L^2(]z(t),\infty[)} + \| \mathbb{H}_{V,t}^{-\frac{1}{4}} p \|^2_{L^2(]z(t),\infty[)} .$$

Our fundamental problem will be to estimate

$$\text{(II.19)} \qquad \lim_{t\to\infty} \| U_V(0,t)F^t \|_{\mathcal{H}^{\frac{1}{2}}(V,0)},$$

where

$$F^t(x) = F(x+t)$$

Therefore we have to develop the scattering theory for (II.4) in $\mathbb{R}_t \times \mathbb{R}_x$, $t \to +\infty$, $x \to -\infty$, $x + t = Cst$. Since the potential $V(x)$ tends to 0 as $x \to -\infty$, we simply compare the solutions of (II.4) with the solutions of the one dimensional wave equation

$$\text{(II.20)} \qquad \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}.$$

So we introduce: the operator $\mathbb{H}_{out}$ on $L^2(\mathbb{R})$ given by
(II.21)

$$\mathbb{H}_{out} = -\frac{d^2}{dx^2}, \; D(\mathbb{H}_{out}) = \{f \in L^2(\mathbb{R});\; f'' \in L^2(\mathbb{R})\} \; (= H^2(\mathbb{R})),$$

and the Hilbert spaces $\mathcal{H}_{out}$ and $\mathcal{H}^{\frac{1}{2}}_{out}$ defined as the completions of $C_0^\infty(\mathbb{R})$ for the norms

$$\text{(II.22)} \qquad \| {}^t(f,p) \|^2_{\mathcal{H}_{out}} = \| \mathbb{H}_{out}^{\frac{1}{2}} f \|^2_{L^2(\mathbb{R})} + \| p \|^2_{L^2(\mathbb{R})},$$

$$\text{(II.23)} \qquad \| {}^t(f,p) \|^2_{\mathcal{H}^{\frac{1}{2}}_{out}} = \| \mathbb{H}_{out}^{\frac{1}{4}} f \|^2_{L^2(\mathbb{R})} + \| \mathbb{H}_{out}^{-\frac{1}{4}} p \|^2_{L^2(\mathbb{R})} .$$

We denote by U^{out} the free propagator associated with (II.20)

$$U^{out}(t) = \exp\left[t \begin{pmatrix} 0 & 1 \\ -\mathbb{H}_{out} & 0 \end{pmatrix} \right]$$

which is unitary on both spaces $\mathcal{H}_{out}$ and $\mathcal{H}^{\frac{1}{2}}_{out}$. It will be useful to introduce the following subspaces of $\mathcal{H}_{out} \cap \mathcal{H}^{\frac{1}{2}}_{out}$:

$$\text{(II.24)} \qquad \mathcal{D}^{out}_{\pm} = \{ F^{out}_{\pm} = {}^t(f, \pm f');\; f \in C_0^\infty(]0,\infty[\}.$$

We remark that

$$F_{\pm}^{out} \in \mathcal{D}_{\pm}^{out} \Longrightarrow \left(U^{out} F_{\pm}^{out}\right)(x) = F_{\pm}^{out}(x \pm t).$$

To investigate the asymptotic behaviour of solutions we choose some function $\theta \in C^{\infty}(\mathbb{R})$ such that

$$x \leq 0 \Rightarrow \theta(x) = 1, \quad 1 \leq x \Rightarrow \theta(x) = 0, \tag{II.25}$$

and we define the cut-off operator

$$\left(\Theta^{out} F\right)(x) = {}^{t}(\theta(x) f(x), \theta(x) p(x)), \quad F = {}^{t}(f, p), \tag{II.26}$$

and we introduce the Wave Operator defined for $F_{+}^{out} \in \mathcal{D}_{+}^{out}$ by

$$\Omega_{V}^{out} F_{+}^{out} = \lim_{t \to \infty} U_{V}(0,t) \Theta^{out} U^{out}(t) F_{+}^{out} \ in\ \mathcal{H}(V,0), \tag{II.27}$$

PROPOSITION II.2. – *Given $F_{+}^{out} \in \mathcal{D}_{+}^{out}$, the strong limit* (II.27) *exists and is independent of the function θ satisfying* (II.25). *Moreover*

$$\Omega_{V}^{out} F_{+}^{out} \in \mathcal{H}^{\frac{1}{2}}(V, 0).$$

Now we can state the fundamental estimate of this part:

THEOREM II.3. – *We assume that the function z satisfies* (I.5) *and the potential V satisfies assumptions* (II.7). *Given $F^0 = F_{+}^0 + F_{-}^0$, $F_{\pm}^0 \in \mathcal{D}_{\pm}^{out}$, we put*

$$F^t(x) = F^0(x + t).$$

Then the norm of $U_V(0,t)F^t$ in $\mathcal{H}^{\frac{1}{2}}(V,0)$ has a limit as $t \to +\infty$ and

$$\lim_{t \to \infty} \| U_V(0,t) F^t \|^2_{\mathcal{H}^{\frac{1}{2}}(V,0)} = \| \Omega_V^{out} F_{+}^0 \|^2_{\mathcal{H}^{\frac{1}{2}}(V,0)} + \| \sqrt{\coth\left(\frac{\pi}{\kappa} \mathbb{H}_{out}^{\frac{1}{2}}\right)} F_{-}^0 \|^2_{\mathcal{H}_{out}^{\frac{1}{2}}} \tag{II.28}$$

REMARK II.4. – *The limit* (II.28) *is a very sharp estimate. Indeed we can show that, on the one hand if $F_{-}^0 \neq 0$:*

$$\| U_V(0,t) F^t \|_{\mathcal{H}(V,0)} \sim e^{\kappa t}, \ t \to +\infty, \tag{II.29}$$

and on the other hand:

(II.30) $$\| U_{V}(0,t)F^{t} \|_{\mathcal{H}_1(V,0)} = O(1),\ t \to +\infty.$$

The first estimate is slightly discouraging, and the second one is not sufficient because if $\mathbf{1}_{[0,\infty[}V \not\equiv 0$, we have according to [3] for $\varepsilon > 0$:

(II.31) $$\mathcal{H}_1(V,0) \cap \mathcal{E}' \subset H^{\frac{1}{2}-\varepsilon} \times H^{-\frac{1}{2}-\varepsilon},$$

but:

(II.32) $$\mathcal{H}_1(V,0) \cap \mathcal{E}' \not\subset H^{\frac{1}{2}} \times H^{-\frac{1}{2}},$$

and

(II.33) $$\mathcal{H}^{\frac{1}{2}}(V,0) \cap \mathcal{E}' \subset H^{\frac{1}{2}} \times H^{-\frac{1}{2}}.$$

Moreover, if $V \geq \alpha > 0$, *then*

$$\mathcal{H}(V,t) \subset \mathcal{H}^{\frac{1}{2}}(V,t),$$

and we have

(II.34) $$\sup_{0 \leq t} \| U_V(0,t) \|_{\mathcal{L}\left(\mathcal{H}(V,t),\mathcal{H}^{\frac{1}{2}}(V,0)\right)} = \infty$$

because the result of the asymptotic completeness part in Theorem III-1 *in* [3]. *Hence* (II.28) *is rather surprising. The key is that we deal with a hyperbolic problem, and the previous functional considerations do not describe the fine phenomenon of the propagation of the field. A precise analysis of the structure of the propagator gives*

(II.35) $$\| U_V(0,t)F^{t} \|_{L^2 \times D(\mathrm{H}^{-\frac{1}{2}})} \sim e^{-\kappa t},\ t \to +\infty,$$

thus by interpolating with (II.29) *we get*

(II.36) $$\| U_{V}(0,t)F^{t} \|_{\mathcal{H}^{\frac{1}{2}}(V,0)} = O(1),\ t \to +\infty.$$

An exact calculus for the case $V = 0$, *and a comparison between* U_V *and* U_0 *give the explicit value of the limit.*

Proof of Proposition II.1. – Estimates (II.12), (II.5) and (II.13) are proved in [3]. To establish (II.14), we remark that the solution u of (II.4), (I.8) with data $F = {}^t(f,p) \in \mathcal{H}(V,t)$ given at time t, satisfies for $0 \leq s \leq t$ the standard energy inequality:

$$\int_{z(t)+t-s}^{\infty} \mid \partial_{r_*} u(s,x) \mid^2 + \mid \partial_t(s,x) \mid^2 + V(x) \mid u(s,x) \mid^2 dx \leq \| F \|^2_{\mathcal{H}(V,t)} .$$

Moreover we have

$$\int_{z(s)}^{z(t)+t-s} \mid \partial_{r_*} u(s,x) \mid^2 + \mid \partial_t(s,x) \mid^2 + V(x) \mid u(s,x) \mid^2 dx$$
$$\leq C e^{2\kappa(t-s)} \| U_V(s,t) F \|^2_{\mathcal{H}_1(V,s)} \leq C' e^{2\kappa(t-s)} \| F \|^2_{\mathcal{H}_1(V,t)} .$$

This completes the proof of (II.14).

Q.E.D.

Proof of Proposition II.2. – Given $F_+^{out} = {}^t(f_+, f'_+) \in \mathcal{D}_+^{out}$, $F_+^{out} = 0$ for $x \geq R$, we have for $t \geq R$:

$$\Theta^{out} U^{out}(t) F_+^{out} = F_{+,t}^{out}$$

with

$$F_{+,t}^{out}(x) = F_+^{out}(x+t).$$

Then we have to establish the existence of

$$\lim_{t \to \infty} U_V(0,t) F_{+,t}^{out} \ \ in \ \ \mathcal{H}(V,0).$$

We apply Cook's method using (II.7), (II.14) and we evaluate

$$\left\| \frac{d}{dt}\left(U_V(0,t) F_{+,t}^{out}\right) \right\|_{\mathcal{H}(V,0)} = \left\| \left(U_V(0,t){}^t(0, V(.)f_+(.+t))\right) \right\|_{\mathcal{H}(V,0)}$$
$$\leq C(V,R) e^{-\varepsilon t}.$$

Therefore $\Omega_V^{out} F_+^{out}$ exists in $\mathcal{H}(V,0)$ and moreover satisfies:

$$x \geq R \Rightarrow F(x) = 0.$$

Then we conclude by Lemma II.8 below that $F \in \mathcal{H}^{\frac{1}{2}}(V,0)$.

Q.E.D.

In this paper we denote by $\mathcal{F}(u) = \hat{u}$, the Fourier transform of a tempered distribution $u \in \mathcal{S}'(\mathbb{R})$.

LEMMA II.5. – *For any* $\beta > 0$, $\frac{1}{(\sinh(\beta z) \pm i\varepsilon)^2}$ *has a limit in* $\mathcal{S}'(\mathbb{R})$, *denoted* $\frac{1}{(\sinh(\beta z) \pm i0)^2}$, *as* $\varepsilon \to 0^+$, *and we have:*

$$\mathcal{F}\left(\frac{1}{(\sinh(\beta z) + i0)^2} + \frac{1}{(\sinh(\beta z) - i0)^2}\right)(\xi) = \frac{2\pi}{\beta^2} \mid \xi \mid \coth\left(\frac{\pi}{2\beta} \mid \xi \mid\right)$$

Proof of Lemma II.5. – Given $\varepsilon > 0$, $\varphi \in \mathcal{S}(\mathbb{R})$ we denote $\psi(z) = \frac{1}{\beta}\left(\frac{\varphi(z)}{\cosh(\beta z)}\right)'$. We have:

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{1}{[\sinh(\beta z) \pm i\varepsilon]^2} \varphi(z) dz \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sinh(\beta z) \pm i\varepsilon} \psi(z) dz \\
&= \psi(0) \int_{-\infty}^{+\infty} \frac{1}{\sinh(\beta z) \pm i\varepsilon} dz + \int_{-\infty}^{+\infty} \frac{\psi(z) - \psi(0)}{\sinh(\beta z) \pm i\varepsilon} dz \\
&\to \mp i\pi\psi(0) + \int_{-\infty}^{+\infty} \frac{\psi(z) - \psi(0)}{\sinh(\beta z)} dz, \ \varepsilon \to 0^+.
\end{aligned}$$

Now given $\varepsilon \neq 0$, $\xi < 0$ and $N > 0$, $M > 0$, we calculate:

$$\begin{aligned}
&\int_{-\infty}^{+\infty} e^{-iz\xi} \frac{1}{[\sinh(\beta z) + i\varepsilon]^2} dz \\
&= -\frac{1}{\beta} \int_{-\infty}^{+\infty} e^{-iz\xi} \frac{1}{(\sinh(\beta z) + i\varepsilon)} \left(\frac{i\xi}{\cosh(\beta z)} + \frac{\beta \sinh(\beta z)}{\cosh^2(\beta z)}\right) dz.
\end{aligned}$$

We evaluate

$$-\frac{1}{\beta} \oint e^{-iz\xi} \frac{1}{(\sinh(\beta z) + i\varepsilon)} \left(\frac{i\xi}{\cosh(\beta z)} + \frac{\beta \sinh(\beta z)}{\cosh^2(\beta z)}\right) dz$$

on the path

$$\{-N \leq \Re z \leq N, \ \Im z = 0, M\} \cup \{0 \leq \Im z \leq M, \ \Re z = \pm N\}.$$

Noting that for $y \in \mathbb{R}$

$$\mid \sinh(\pm\beta N + i\beta y) \mid \geq \mid \sinh(\beta N) \mid, \quad \mid \cosh(\pm\beta N + i\beta y) \mid \geq \mid \sinh(\beta N) \mid.$$

we get:

$$\left| \int_{\pm N}^{\pm N + iM} e^{-iz\xi} \frac{1}{[\sinh(\beta z) + i\varepsilon]^2} dz \right| \longrightarrow 0, \ N \to \infty.$$

Now we choose $M_k = \frac{1}{\beta}\left(\frac{\pi}{4} + 2k\pi\right)$. We have for $x \in \mathbb{R}$:

$$| e^{-i\left(x+\frac{i}{\beta}\left(\frac{\pi}{4}+2k\pi\right)\right)\xi} | \sim e^{\frac{2k\pi\xi}{\beta}} \longrightarrow 0,\ k \to \infty,$$

$$\left|\sinh\left(\beta x + i\left(\frac{\pi}{4} + 2k\pi\right)\right)\right|^2,$$

$$\left|\cosh\left(\beta x + i\left(\frac{\pi}{4} + 2k\pi\right)\right)\right|^2 \geq \frac{1}{2} \mid \sinh(\beta x) \mid^2 .$$

We deduce that:

$$\left| \int_{-\infty+iM_k}^{+\infty+iM_k} \frac{1}{[\sinh(\beta z) + i\varepsilon]^2} dz \right| \longrightarrow 0,\ k \to \infty,$$

and

$$\int_{-\infty}^{+\infty} e^{-iz\xi} \frac{1}{[\sinh(\beta z) + i\varepsilon]^2} dz = 2i\pi \sum_n \rho_n(\varepsilon)$$

where $\rho_n(\varepsilon)$ are the residues of $e^{-iz\xi}\frac{1}{[\sinh(\beta z)+i\varepsilon]^2}$ at the poles $z_n(\varepsilon) \in \{z \in \mathbb{C}; \Im z > 0\}$. We easily check that:

$$z_n(\varepsilon) = \frac{1}{\beta}(n\pi - (-1)^n \arcsin\varepsilon)i$$

and

$$\rho_n(\varepsilon) = -\frac{i\xi}{\beta^2 \cosh^2(\beta z_n(\varepsilon))} e^{-i\xi z_n(\varepsilon)}$$

with

$$n \in \mathbb{N}\ for\ \varepsilon > 0,$$

and

$$n \in \mathbb{N} \setminus \{0\}\ for\ \varepsilon < 0.$$

We get

$$\begin{aligned}
&\mathcal{F}\left(\frac{1}{(\sinh(\beta z) + i0)^2} + \frac{1}{(\sinh(\beta z) - i0)^2}\right)(\xi) \\
&= lim_{\varepsilon\to 0^+} \int_{-\infty}^{+\infty} e^{-iz\xi}\left(\frac{1}{[\sinh(\beta z) + i\varepsilon]^2} + \frac{1}{[\sinh(\beta z) - i\varepsilon]^2}\right) dz \\
&= \frac{2\pi\xi}{\beta^2}\left(1 + 2\sum_{n=1}^{\infty} e^{n\frac{\xi\pi}{\beta}}\right) \\
&= \frac{2\pi\xi}{\beta^2} \coth\left(\frac{\pi\xi}{2\beta}\right).
\end{aligned}$$

Finally, this formula holds by parity for $\xi > 0$.

Q.E.D.

LEMMA II.6. – *For $\beta > 0$, $u \in C_0^\infty(\mathbb{R})$ we define*

$$F(\xi) = \int e^{i\xi e^{\beta x}} u'(x)dx.$$

Then we have:

$$\int |\xi|^{-1} | F(\xi) |^2 d\xi = \int |\xi| \coth\left(\frac{\pi}{\beta} |\xi|\right) | \hat{u}(\xi) |^2 d\xi.$$

Proof of Lemma II.6. – We have:

$$\int |\xi|^{-1} | F(\xi) |^2 d\xi = \lim_{\varepsilon \to 0^+} I_\varepsilon$$

with:

$$I_\varepsilon = \beta^2 \int\int \left(\int |\eta| e^{-\varepsilon|\eta|} e^{i\eta\left(e^{\beta z_1} - e^{\beta z_2}\right)} d\eta\right) e^{\beta(z_1+z_2)} u(z_1)\overline{u}(z_2) dz_1 dz_2.$$

By calculating the Fourier transform of $|\eta| e^{-\varepsilon|\eta|}$ we get:

$$\begin{aligned} I_\varepsilon = 2\beta^2 \iint & \frac{\varepsilon^2 - 4e^{\beta z_1} e^{\beta z_2} \sinh^2\left(\frac{\beta}{2}(z_1 - z_2)\right)}{\left[\varepsilon^2 + 4e^{\beta z_1} e^{\beta z_2} \sinh^2\left(\frac{\beta}{2}(z_1 - z_2)\right)\right]^2} \\ & \times e^{\beta z_1} e^{\beta z_2} u(z_1)\overline{u}(z_2) dz_1 dz_2. \end{aligned}$$

Together with Lemma II.5, this gives:

$$\begin{aligned} &\int |\xi|^{-1} | F(\xi) |^2 d\xi \\ &= \lim_{\alpha \to 0^+} \beta^2 \iint \left(\frac{1}{\left[2\sinh\left(\frac{\beta}{2}(z_1 - z_2)\right) + i\alpha\right]^2} + \frac{1}{\left[2\sinh\left(\frac{\beta}{2}(z_1 - z_2)\right) - i\alpha\right]^2}\right) \\ &\quad \times u(z_1)\overline{u}(z_2) dz_1 dz_2 \\ &= \frac{\beta^2}{8\pi} \left\langle \mathcal{F}\left(\frac{1}{\left(\sinh\left(\frac{\beta}{2}z\right) + i0\right)^2} + \frac{1}{\left(\sinh\left(\frac{\beta}{2}z\right) - i0\right)^2}\right), | \hat{u}(\xi) |^2 \right\rangle_{\mathcal{S}'(\mathbb{R}).\mathcal{S}(\mathbb{R})} \\ &= \int |\xi| \coth\left(\frac{\pi}{\beta} |\xi|\right) | \hat{u}(\xi) |^2 d\xi. \end{aligned}$$

Q.E.D.

LEMMA II.7. – *For any* $R > z(0)$, *there exists* $C_R > 0$ *such that for any* $u \in C_0^\infty(]z(0), R[)$, *and for any* $\alpha > 0$, *we have:*

$$\int_{-\infty}^{\infty} | \xi || \hat{u}(\xi) |^2 \, d\xi$$
$$\leq C_R \left\{ (1+ | \ln(\alpha) |)^3 \int_{z(0)}^{0} (| x | +\alpha) | u'(x) |^2 \, dx + \int_0^R | u'(x) |^2 \, dx \right\}.$$

Proof of Lemma II.7. – We introduce the cut off function

$$\chi(x) = 1 \ \ for \ \ x \leq \frac{1}{3},$$
$$\chi(x) = -6x + 3 \ \ for \ \ \frac{1}{3} \leq x \leq \frac{1}{2},$$
$$\chi(x) = 0 \ \ for \ \ \frac{1}{2} \leq x,$$

and we put:

$$\chi_\alpha(x) = \chi\left(\frac{x}{\alpha}\right),$$

$$v_\alpha(x) = \chi_\alpha(x) u(x) \ \ for \ \ x \leq 0,$$
$$v_\alpha(x) = \chi_\alpha(x) u(0) \ \ for \ \ 0 \leq x.$$

We note that

$$| u(0) |^2 \leq \left(\int_{z(0)}^{0} (\alpha + | x |)^{-1} dx \right) \left(\int_{z(0)}^{0} (\alpha + | x |) \, | u'(x) |^2 \, dx \right),$$

hence

$$\text{(II.37)} \qquad | u(0) |^2 \leq C(1+ | \ln(\alpha) |) \left(\int_{z(0)}^{0} (\alpha + | x |) \, | u'(x) |^2 \, dx \right).$$

We introduce an auxiliary function w_α:

$$w_\alpha(y) = v_\alpha(\alpha - e^y).$$

We have:

$$\hat{v}_\alpha(\xi) = i e^{-i\alpha\xi} \xi^{-1} F(\xi), \quad F(\xi) = \int e^{i\xi e^y} w'_\alpha(y) dy.$$

We deduce from Lemma II.6 that

$$\text{(II.38)}\qquad \begin{aligned}&\int \mid \xi \mid\mid \hat{v}_\alpha(\xi) \mid^2 d\xi\\ &= \int \mid \xi \mid \coth\left(\frac{\pi}{\beta} \mid \xi \mid\right) \mid \hat{w}_\alpha(\xi) \mid^2 d\xi\\ &\leq C\left\{\int \mid w'_\alpha(y) \mid^2 dy + \left(\int \mid w_\alpha(y) \mid dy\right)^2\right\}.\end{aligned}$$

On the one hand, we evaluate

$$\begin{aligned}&\int \mid w'_\alpha(y) \mid^2 dy\\ &= \int_{-\infty}^{\alpha} (\alpha - x) \mid v'_\alpha(x) \mid^2 dx\\ &= \int_{z(0)}^{0} (\alpha + \mid x \mid) \mid u'(x) \mid^2 dx + C' \mid u(0) \mid^2 .\end{aligned}$$

Hence by (II.37) we get:
(II.39)

$$\int \mid w'_\alpha(y) \mid^2 dy \leq C(1 + \mid \ln(\alpha) \mid)^2 \int_{z(0)}^{0} (\alpha + \mid x \mid) \mid u'(x) \mid^2 dx.$$

On the other hand, we see

$$\begin{aligned}&\int \mid w_\alpha(y) \mid dy\\ &= \int_{-\infty}^{\alpha} (\alpha - x)^{-1} \mid v_\alpha(x) \mid dx\\ &= \int_{z(0)}^{0} (\alpha + \mid x \mid)^{-1} \mid u(x) \mid dx + C" \mid u(0) \mid .\end{aligned}$$

We have:

$$\begin{aligned}&\int_{z(0)}^{0} (\alpha + \mid x \mid)^{-1} \mid u(x) \mid dx\\ &\leq \int_{z(0)}^{0} (\alpha + \mid x \mid)^{-1} \left(\int_{z(0)}^{0} \mid u'(t) \mid dt\right) dx\\ &\leq C(1 + \mid \ln(\alpha) \mid)^{\frac{3}{2}} \left(\int_{z(0)}^{0} (\alpha + \mid x \mid) \mid u'(x) \mid^2 dx\right)^{\frac{1}{2}}.\end{aligned}$$

Hence by (II.37) we get:
(II.40)
$$\left(\int \mid w_\alpha(y) \mid dy\right)^2 \leq C(1+\mid \ln(\alpha) \mid)^3 \int_{z(0)}^0 (\alpha+\mid x \mid) \mid u'(x) \mid^2 dx$$

We conclude from (II.38), (II.39), (II.40) that
(II.41)
$$\int \mid \xi \mid\mid \hat{v}_\alpha(\xi) \mid^2 d\xi \leq C(1+\mid \ln(\alpha) \mid)^3 \int_{z(0)}^0 (\alpha+\mid x \mid) \mid u'(x) \mid^2 dx.$$

Now we estimate $\Delta_\alpha = u - v_\alpha$; We denote by $Y(x)$ the Heaviside function and we have:

$$\Delta_\alpha(x) = f_\alpha(x)u(0) + g_\alpha(x)$$

with

$$f_\alpha(x) = \chi(3Rx)(1-\chi_\alpha(x))Y(x),$$
$$g_\alpha(x) = \chi(3Rx)(u(x)-u(0))Y(x).$$

On the one hand, we calculate

$$\hat{f}_\alpha(\xi) = \frac{6}{\alpha}\left(\frac{e^{-i\frac{\alpha}{2}\xi} - e^{-i\frac{\alpha}{3}\xi}}{\xi^2}\right) + \frac{2}{R}\left(\frac{e^{-iR\xi} - e^{-i\frac{3R}{2}\xi}}{\xi^2}\right),$$

then we get:

$$\text{(II.42)} \qquad \int \mid \xi \mid\mid \hat{f}_\alpha(\xi) \mid^2 d\xi \leq C(1+\mid \ln(\alpha) \mid).$$

On the other hand, we easily check that

$$\text{(II.43)} \qquad \int \mid g'_\alpha(x) \mid^2 + \mid g_\alpha(x) \mid^2 dx \leq C \int_0^R \mid u'(x) \mid^2 dx.$$

We deduce from (II.37), (II.42) and (II.43) that

$$\text{(II.44)} \qquad \int \mid \xi \mid\mid \hat{\Delta}_\alpha(\xi) \mid^2 d\xi$$
$$\leq C_R\left\{(1+\mid \ln(\alpha) \mid)^2 \int_{z(0)}^0 (\alpha+\mid x \mid) \mid u'(x) \mid^2 dx + \int_0^R \mid u'(x) \mid^2 dx\right\}.$$

Finally, Lemma II.7 follows from (II.41) and (II.44).

Q.E.D.

LEMMA II.8. – *Let V be satisfy* (II.7). *Then for any $R > 0$ there exists $C_{R,V} > 0$ such that for all $F = {}^t(f,p) \in \mathcal{H}(V,0)$ with $F(x) = 0$ for $x > R$, we have:*

$$F \in \mathcal{H}^{\frac{1}{2}}(V,0),$$

and for any $\alpha > 0$

$$\text{(II.45)} \qquad \| {}^t(f,p) \|^2_{\mathcal{H}^{\frac{1}{2}}(V,0)}$$
$$\leq C_{R,V}\Big\{(1+ \mid \ln(\alpha) \mid)^3 \int_{z_0}^{0} (\mid x \mid +\alpha) \mid f'(x) + p(x) \mid^2 dx$$
$$+ \int_{z_0}^{0} \mid f'(x) - p(x) \mid^2 dx + \int_0^R \mid f'(x) \mid^2 + \mid p(x) \mid^2 dx\Big\}$$

Proof of Lemma II.8. – We start by establishing some preliminary estimates. Given $\varphi \in C_0^\infty(]z(0), R[)$ we put $\mathbb{P}\varphi(x) = \varphi(x)$ for $x \geq z(0)$ and $\mathbb{P}\varphi(x) = -\varphi\big(2z(0) - x\big)$ for $x \leq z(0)$, then we have for $s \geq -1$:

$$\| \mid \xi \mid^s \mathcal{F}(\mathbb{P}\varphi) \|^2_{L^2(\mathbb{R})} = 2 \int \mid \xi \mid^{2s} (\mid \widehat{\varphi}(\xi) \mid^2 - \cos(2\xi z(0))\widehat{\varphi}(\xi)\overline{\widehat{\varphi}(-\xi)})d\xi.$$

Since $\frac{1}{\sqrt{2}}\mathbb{P}$ is an isometry from $L^2(]z(0), \infty[)$ to $L^2(\mathbb{R})$,which satisfies:

$$\frac{d^2}{dx^2}(\mathbb{P}\varphi) = \mathbb{P}(\frac{d^2}{dx^2}\varphi),$$

we get:
(II.46)

$$2\pi \| \mathbb{H}_{0,0}^{\pm\frac{1}{4}} \varphi \|^2_{L^2(]z(0),\infty[)} = \int |\xi|^{\pm 1} (|\widehat{\varphi}(\xi) \mid^2 - \cos(2\xi z(0))\widehat{\varphi}(\xi)\overline{\widehat{\varphi}(-\xi)})d\xi.$$

thus:

$$\| \mathbb{H}_{0,0}^{\frac{1}{4}} \varphi \|^2_{L^2(]z(0),\infty[)} \leq 2 \int \mid \xi \mid\mid \widehat{\varphi}(\xi) \mid^2 d\xi.$$

Hence Lemma II.7 implies:

$$\text{(II.47)} \qquad \| \mathbb{H}_{0,0}^{\frac{1}{4}} \varphi \|^2_{L^2(]z(0),\infty[)}$$
$$\leq C_R \Big\{ (1+ \mid \ln(\alpha) \mid)^3 \int_{z_0}^{0} (\mid x \mid +\alpha) \mid \varphi'(x) \mid^2 dx$$
$$+ \int_0^R \mid \varphi'(x) \mid^2 dx \Big\}.$$

Now we have for $z_0 \leq x \leq R$:

$$\mid \varphi(x) \mid \leq \min\left(\int_{z_0}^{x} [(\alpha + \mid y \mid)^{\frac{1}{2}} \mid \varphi'(y) \mid] \right. \\ \left. (\alpha + \mid y \mid)^{-\frac{1}{2}} dy, \int_{x}^{R} \mid \varphi'(y) \mid dy \right).$$

Thus we get

$$\text{(II.48)} \quad \| \varphi \|_{L^\infty}^2 \leq C_R \left\{ (1 + \mid \ln(\alpha) \mid) \int_{z_0}^{0} (\mid x \mid + \alpha) \mid \varphi'(x) \mid^2 dx \right. \\ \left. + \int_0^R \mid \varphi'(x) \mid^2 dx \right\}.$$

Since the potential V is uniformly bounded we have

$$\text{(II.49)} \qquad 0 < \mathbb{H}_{V,0} \leq \mathbb{H}_{0,0} + \| V \|_{L^\infty} .$$

Thus, the Heinz theorem ([18], Theorem 4.12) implies

$$\text{(II.50)} \qquad 0 < \mathbb{H}_{V,0}^{\frac{1}{4}} \leq (\mathbb{H}_{0,0} + \| V \|_{L^\infty})^{\frac{1}{4}},$$

hence
(II.51)

$$\| \mathbb{H}_{V,0}^{\frac{1}{4}} \varphi \|_{L^2(]z(0),\infty[)}^2 \leq C \left(\| \mathbb{H}_{0,0}^{\frac{1}{4}} \varphi \|_{L^2(]z(0),\infty[)}^2 + \| \varphi \|_{L^2(]z(0),\infty[)}^2 \right).$$

We conclude from (II.47), (II.48) and (II.51) that

$$\text{(II.52)} \quad \| \mathbb{H}_{V,0}^{\frac{1}{4}} \varphi \|_{L^2(]z(0),\infty[)}^2 \\ \leq C_R \left\{ (1 + \mid \ln(\alpha) \mid)^3 \int_{z_0}^{0} (\mid x \mid + \alpha) \mid \varphi'(x) \mid^2 dx \right. \\ \left. + \int_0^R \mid \varphi'(x) \mid^2 dx \right\}.$$

Now we consider ${}^t(f, p) \in [C_0^\infty(]z(0), R[)]^2$ and we choose $\chi \in C_0^\infty(\mathbb{R})$, $\chi(x) = 1$ for $x \in [z(0), R]$. We put

$$P(x) = \chi(x) \int_{z(0)}^{x} p(t) dt,$$

and using (II.52) we evaluate

$$\|\mathbf{H}_{V,0}^{\frac{1}{4}}(f+P)\|_{L^2(]z(0),\infty[)}^2 \leq C(R)\left\{(1+\mid \ln(\alpha)\mid)^3\int_{z_0}^{0}(\mid x\mid+\alpha)\mid f'(x)+p(x)\mid^2 dx + \int_0^\infty \mid f'(x)\mid^2+\mid p(x)\mid^2 dx + \left(\int \mid p(x)\mid dx\right)^2\right\}.$$

We have

$$\text{(II.53)}\quad \left(\int \mid p(x)\mid dx\right)^2 \leq C\left\{\left(\int_{z_0}^{0}\mid f'(x)+p(x)\mid dx\right)^2 + \left(\int_{z_0}^{0}\mid f'(x)-p(x)\mid dx\right)^2 + \left(\int_0^R \mid p(x)\mid dx\right)^2\right\}$$
$$\leq C_R\left\{(1+\mid \ln(\alpha)\mid)\int_{z_0}^{0}(\mid x\mid+\alpha)\mid f'(x)+p(x)\mid^2 dx + \int_{z_0}^{0}\mid f'(x)-p(x)\mid^2 dx + \int_0^R \mid p(x)\mid^2 dx\right\}$$

Hence

$$\text{(II.54)}\quad \|\mathbf{H}_{V,0}^{\frac{1}{4}}(f+P)\|_{L^2(]z(0),\infty[)}^2 \leq C(R)\left\{(1+\mid \ln(\alpha)\mid)^3\int_{z_0}^{0}(\mid x\mid+\alpha)\mid f'(x)+p(x)\mid^2 dx + \int_{z(0)}^{0}\mid f'(x)-p(x)\mid^2 dx + \int_0^\infty \mid f'(x)\mid^2+\mid p(x)\mid^2 dx\right\}.$$

Moreover we have

$$\|\mathbf{H}_{V,0}^{\frac{1}{4}}(f-P)\|_{L^2(]z(0),\infty[)} \leq C(R)\left\{\int_{z_0}^{0}\mid f'(x)-p(x)\mid^2 dx + \int_0^\infty \mid f'(x)\mid^2+\mid p(x)\mid^2 dx + \left(\int \mid p(x)\mid dx\right)^2\right\}.$$

Then we conclude with (II.55) and (II.54) that

(II.55) $$\|\mathbb{H}_{V,0}^{\frac{1}{4}} f\|^2_{L^2(]z(0),\infty[)} \leq C(R)\left\{(1+\mid \ln(\alpha)\mid)^3 \int_{z_0}^{0} (\mid x\mid +\alpha)\mid f'(x)+p(x)\mid^2 + \mid f'(x)-p(x)\mid^2 dx + \int_0^{\infty} \mid f'(x)\mid^2 + \mid p(x)\mid^2 dx\right\}.$$

Now we choose $\theta \in C^\infty(\mathbb{R})$ such that $\theta(x) = 0$ for $x \leq 0$, and $\theta(x) = 1$ for $x \geq \frac{R}{2}$, and we put:

$$\psi(x) = \int_{z(0)}^{x} \phi(y)dy - \left(\int \phi(y)dy\right)\theta(x),$$

so we have:

$$\phi = \psi' + \left(\int \phi(y)dy\right)\theta'.$$

Since

$$0 < \mathbb{H}_{0,0} \leq \mathbb{H}_{V,0},$$

the Heinz theorem implies

(II.56) $$\begin{aligned} &\|\mathbb{H}_{V,0}^{-\frac{1}{4}}\phi\|^2_{L^2(]z(0),\infty[)} \\ &\leq \| \mathbb{H}_{0,0}^{-\frac{1}{4}}\phi\|^2_{L^2(]z(0),\infty[)} \\ &\leq 2\| \mathbb{H}_{0,0}^{-\frac{1}{4}}\psi'\|^2_{L^2(]z(0),\infty[)} \\ &\quad + 2\left(\int \mid \phi(y)\mid dy\right)^2 \| \mathbb{H}_{0,0}^{-\frac{1}{4}}\theta'\|^2_{L^2(]z(0),\infty[)} \cdot \end{aligned}$$

We get by (II.46) and Lemma II.7:

(II.57) $$\begin{aligned} &\|\mathbb{H}_{0,0}^{-\frac{1}{4}}\psi'\|^2_{L^2(]z(0),\infty[)} \\ &\leq \|\mid \xi\mid^{\frac{1}{2}}\hat{\psi}\|^2_{L^2(\mathbb{R})} \\ &\leq C(\theta, R)\left\{(1+\mid \ln(\alpha)\mid)^3 \int_{z(0)}^{0} (\mid x\mid +\alpha)\mid \phi(x)\mid^2 dx + \int_0^R \mid \phi(x)\mid^2 dx + \left(\int \mid \phi(x)\mid dx\right)^2\right\}. \end{aligned}$$

We conclude by (II.56), (II.57) and (II.53) with $f = 0$ that:

$$\text{(II.58)} \qquad \|\mathbb{H}_{V,0}^{-\frac{1}{4}}\phi\|^2_{L^2(]z(0),\infty[)} \leq C_R\left\{(1+\mid \ln(\alpha)\mid)^3\int_{z(0)}^0 (\mid x\mid+\alpha)\mid\phi(x)\mid^2 dx + \int_0^R \mid\phi(x)\mid^2 dx\right\}.$$

Using (II.58) we evaluate

$$\|\mathbb{H}_{V,0}^{-\frac{1}{4}}(f'+p)\|^2_{L^2(]z(0),\infty[)} \leq C(R)\left\{(1+\mid\ln(\alpha)\mid)^3\int_{z_0}^0(\mid x\mid+\alpha)\mid f'(x)+p(x)\mid^2 dx + \int_0^\infty \mid f'(x)\mid^2+\mid p(x)\mid^2 dx\right\}$$

Moreover we have

$$\|\mathbb{H}_{V,0}^{-\frac{1}{4}}(f'-p)\|_{L^2(]z(0),\infty[)} \leq C(R)\left\{\int_{z_0}^0 \mid f'(x)-p(x)\mid^2 dx + \int_0^\infty \mid f'(x)\mid^2+\mid p(x)\mid^2 dx\right\}.$$

Then we conclude that

$$\text{(II.59)} \qquad \|\mathbb{H}_{V,0}^{-\frac{1}{4}}(p)\|^2_{L^2(]z(0),\infty[)} \leq C(R)\left\{(1+\mid\ln(\alpha)\mid)^3\int_{z_0}^0(\mid x\mid+\alpha)\mid f'(x)+p(x)\mid^2 + \mid f'(x)-p(x)\mid^2 dx + \int_0^\infty \mid f'(x)\mid^2+\mid p(x)\mid^2 dx\right\}.$$

Lemma II.8 follows from (II.55) and (II.59).

Q.E.D.

LEMMA II.9. – *Let* $F_- = {}^t(f, p = -f')$ *be in* $\mathcal{D}_-^{out}$. *For* $T > 0$ *we put*

$$\text{(II.60)} \qquad F_-^T(x) = F_-(x+T).$$

Then we have
(II.61)
$$\| U_0(0,T)F_-^T \|_{\mathcal{H}^{\frac{1}{2}}(0,0)} \longrightarrow \sqrt{2} \, \| \sqrt{\coth\left(4\pi M \mathbb{H}_{out}^{\frac{1}{2}}\right)} \, \mathbb{H}_{out}^{\frac{1}{4}} f \, \|_{L^2(\mathbf{R})},$$
$$T \longrightarrow \infty.$$

Proof of Lemma II.9. – We denote

$$U_0(0,T)F_-^T = {}^t(f_T, p_T).$$

According to the explicit formula for the propagator $U_0(t,s)$ in [3], we have for $T > 0$ large enough:

(II.62) $$p_T = f_T' \in C_0^1(]z_0, 0[)$$

and for $x \in]z_0, 0[$:

(II.63) $$p_T(x) = \frac{1 - \dot{z}(\tau(x))}{1 + \dot{z}(\tau(x))} f'(x + 2T - 2\tau(x))$$

where the function τ is defined by the relation

(II.64) $$z(0) \leq x < 0, \quad x - z(\tau(x)) = \tau(x)$$

and satisfies

(II.65) $$\tau(x) = -\frac{1}{2\kappa}\ln(-x) + \frac{1}{2\kappa}\ln(A) + O(x), \ x \to 0^-,$$

(II.66) $$1 + \dot{z}(\tau(x)) = -2\kappa x + O(x^2), \ x \to 0^-.$$

We define for $T > 0$ large enough:
(II.67)
$$\phi_T(x) = \frac{1}{\kappa x} f'\left(2T + \frac{1}{\kappa}\log(-x) - \frac{1}{\kappa}\log(A)\right) \in C_0^\infty(]z(0), \infty[).$$

We calculate:

$$2\pi \, \| \mathbb{H}_{0,0}^{-\frac{1}{4}} \phi_T \|^2_{L^2(]z(0),\infty[)}$$
$$= \int |\xi|^{-1} |F(\xi)|^2 d\xi - \int \cos\left(2\frac{z(0)}{A} e^{2\kappa T}\xi\right) |\xi|^{-1} F(\xi)\overline{F}(-\xi) d\xi.$$

with

$$F(\xi)=\int_0^\infty e^{ix\xi}\frac{1}{\kappa x}f'\left(\frac{1}{\kappa}\log(x)\right)dx\in\mathcal{S}(\mathbf{R}),\ F(0)=0. \tag{II.68}$$

Since $z(0)\neq 0$, we get by the non stationnary phasis theorem:

$$2\pi\parallel \mathbf{H}_{0,0}^{-\frac{1}{4}}\phi_T\parallel^2_{L^2(]z(0),\infty[)}\longrightarrow\int|\xi|^{-1}|F(\xi)|^2\,d\xi,\ T\longrightarrow\infty.$$

Then we conclude with Lemma II.6 that
(II.69)

$$\parallel \mathbf{H}_{0,0}^{-\frac{1}{4}}\phi_T\parallel^2_{L^2(]z(0),\infty[)}\longrightarrow\parallel\sqrt{\coth\left(4\pi M\mathbf{H}_{out}^{\frac{1}{2}}\right)}\,\mathbf{H}_{out}^{\frac{1}{4}}f\parallel^2_{L^2(\mathbf{R})},\ T\longrightarrow\infty.$$

Now we compare p_T and ϕ_T by using (II.65), (II.66):

$$\begin{aligned}
&p_T(x)-\phi_T(x)\\
&\quad=\frac{1+O(x)}{\kappa x+O(x^2)}\left[p\left(2T+\frac{1}{\kappa}\log(-x)-\frac{1}{\kappa}\log(A)+O(x)\right)\right.\\
&\qquad\qquad\left.-p\left(2T+\frac{1}{\kappa}\log(-x)-\frac{1}{\kappa}\log(A)\right)\right]\\
&\quad+\left[\frac{1+O(x)}{\kappa x+O(x^2)}-\frac{1}{\kappa x}\right]p\left(2T+\frac{1}{\kappa}\log(-x)-\frac{1}{\kappa}\log(A)\right)\\
&\quad\equiv I_T(x)+J_T(x)
\end{aligned}$$

On the one hand we have

$$\sup_{T\to\infty}\parallel I_T\parallel_{L^\infty(\mathbf{R})}<\infty,$$

and there exists $\alpha_T\in]z(0),0[$ such that f_T, p_T, I_T and J_T are compactly supported in $[\alpha_T,0[$ and

$$\alpha_T=-Ae^{\kappa R}e^{-2\kappa T}+O(e^{-4\kappa T}),\ T\to\infty.$$

Hence we get

$$\parallel I_T\parallel_{L^2(\mathbf{R})}=O(e^{-\kappa T}),\ T\to\infty.$$

On the other hand

$$\parallel J_T\parallel_{L^2(\mathbf{R})}\leq C\parallel p\left(2T+\frac{1}{\kappa}\log(-x)-\frac{1}{\kappa}\log(A)\right)\parallel_{L^2(\mathbf{R}_x)}=O(e^{-\kappa T}),$$
$$T\to\infty.$$

Therefore we obtain:

$$\| p_T - \phi_T \|_{L^2(\mathbb{R})} = O(e^{-\kappa T}),\ \ T \to \infty, \tag{II.70}$$

hence by Lemma II.8 with $\alpha = 1$:

$$\| \mathbb{H}_{0,0}^{-\frac{1}{4}} (p_T - \phi_T) \|_{L^2(]z(0),\infty[)} = O(e^{-\kappa T}),\ \ T \to \infty. \tag{II.71}$$

We follow the same ideas to estimate f_T. We define for $T > 0$ large enough:

$$\psi_T(x) = -f\left(2T + \frac{1}{\kappa}\log(-x) - \frac{1}{\kappa}\log(A)\right) \in C_0^\infty(]z(0),\infty[). \tag{II.72}$$

We calculate:

$$\begin{aligned} &2\pi \| \mathbb{H}_{0,0}^{\frac{1}{4}} \psi_T \|^2_{L^2(]z(0),\infty[)} \\ &\quad = \int |\xi|^{-1}|F(\xi)|^2 d\xi + \int \cos\left(2\frac{z(0)}{A}e^{2\kappa T}\xi\right)|\xi|^{-1}F(\xi)\overline{F}(-\xi)d\xi. \end{aligned}$$

As previously we get by the non stationnary phasis theorem:
(II.73)

$$\| \mathbb{H}_{0,0}^{\frac{1}{4}} \psi_T \|^2_{L^2(]z(0),\infty[)} \longrightarrow \| \sqrt{\coth\left(4\pi M \mathbb{H}_{out}^{\frac{1}{2}}\right)}\ \mathbb{H}_{out}^{\frac{1}{4}} f \|^2_{L^2(\mathbb{R})},\ \ T \longrightarrow \infty.$$

Now we compare f_T and ψ_T using (II.62):

$$f_T(x) - \psi_T(x) = \int_{z(0)}^{x} p_T(y) - \phi_T(y)dy.$$

We deduce from (II.70) that
(II.74)

$$\| f_T - \psi_T \|_{L^2(]z(0),\infty[)} + \| f_T' - \psi_T' \|_{L^2(]z(0),\infty[)} = O(e^{-\kappa T}),\ \ T \to \infty,$$

hence by Lemma II.8 with $\alpha = 1$:

$$\| \mathbb{H}_{0,0}^{\frac{1}{4}} (f_T - \psi_T) \|_{L^2(]z(0),\infty[)} = O(e^{-\kappa T}),\ \ T \to \infty. \tag{II.75}$$

Now (II.9) follows from (II.69), (II.73), (II.75) and (II.71).

Q.E.D.

LEMMA II.10. – *Given F_- and F_-^T as in Lemma* II.9 *we have:*
(II.76)
$$\| U_0(0,T)F_-^T \|_{\mathcal{H}^{\frac{1}{2}}(V,0)} \longrightarrow \sqrt{2} \| \sqrt{\coth\left(4\pi M \mathbb{H}_{out}^{\frac{1}{2}}\right)}\, \mathbb{H}_{out}^{\frac{1}{4}} f \|_{L^2(\mathbf{R})},$$
$$T \longrightarrow \infty.$$

Proof of Lemma II.10. – Since
$$0 < \mathbb{H}_{0,0} \leq \mathbb{H}_{V,0} \leq \mathbb{H}_{0,0} + \| V \|_{L^\infty(\mathbf{R})},$$
the Heinz theorem implies:
$$\begin{aligned}&\| \left(\mathbb{H}_{0,0} + \| V \|_{L^\infty(\mathbf{R})}\right)^{-\frac{1}{4}} \phi_T \|_{L^2(]z(0),\infty[)}\\ &\quad \leq \| \mathbb{H}_{V,0}^{-\frac{1}{4}} \phi_T \|_{L^2(]z(0),\infty[)} \leq \| \mathbb{H}_{0,0}^{-\frac{1}{4}} \phi_T \|_{L^2(]z(0),\infty[)},\end{aligned}$$
where ϕ_T is given by (II.67). Hence we have:
$$\begin{aligned}0 &\leq \| \mathbb{H}_{0,0}^{-\frac{1}{4}} \phi_T \|^2_{L^2(]z(0),\infty[)} - \| \mathbb{H}_{V,0}^{-\frac{1}{4}} \phi_T \|^2_{L^2(]z(0),\infty[)}\\ &\leq \| \mathbb{H}_{0,0}^{-\frac{1}{4}} \phi_T \|^2_{L^2(]z(0),\infty[)} - \| \left(\mathbb{H}_{0,0} + \| V \|_{L^\infty(\mathbf{R})}\right)^{-\frac{1}{4}} \phi_T \|^2_{L^2(]z(0),\infty[)}\\ &\leq 2 \int \left[1 - \left(1 + |\xi|^{-1} \| V \|_{L^\infty(\mathbf{R})}\, e^{-4\kappa T}\right)^{-\frac{1}{2}}\right] |\xi|^{-1} |F(\xi)|^2 \, d\xi,\end{aligned}$$
where F is defined by (II.68). By the dominated convergence theorem and (II.69) we deduce that
(II.77)
$$\| \mathbb{H}_{V,0}^{-\frac{1}{4}} \phi_T \|^2_{L^2(]z(0),\infty[)} \longrightarrow \| \sqrt{\coth\left(4\pi M \mathbb{H}_{out}^{\frac{1}{2}}\right)}\, \mathbb{H}_{out}^{\frac{1}{4}} f \|^2_{L^2(\mathbf{R})}, \quad T \longrightarrow \infty.$$

We apply Lemma II.8 with $\alpha = 1$:

(II.78) $$\| \mathbb{H}_{V,0}^{-\frac{1}{4}} (p_T - \phi_T) \|_{L^2(]z(0),\infty[)} \leq C \| p_T - \phi_T \|_{L^2(\mathbf{R})} \rightarrow 0, \quad T \rightarrow \infty,$$

and we get from (II.77) and (II.78) that
(II.79)
$$\| \mathbb{H}_{V,0}^{-\frac{1}{4}} p_T \|^2_{L^2(]z(0),\infty[)} \longrightarrow \| \sqrt{\coth(4\pi M \mathbb{H}_{out}^{\frac{1}{2}})}\, \mathbb{H}_{out}^{\frac{1}{4}} f \|^2_{L^2(\mathbf{R})}, \quad T \longrightarrow \infty.$$

The Heinz theorem implies also:
$$\begin{aligned}&\| \mathbb{H}_{0,0}^{\frac{1}{4}} f_T \|_{L^2(]z(0),\infty[)}\\ &\quad \leq \| \mathbb{H}_{V,0}^{\frac{1}{4}} f_T \|_{L^2(]z(0),\infty[)} \leq \| \left(\mathbb{H}_{0,0} + \| V \|_{L^\infty(\mathbf{R})}\right)^{\frac{1}{4}} f_T \|_{L^2(]z(0),\infty[)} .\end{aligned}$$

Hence we have:

$$0 \leq \| \mathbb{H}^{\frac{1}{4}}_{V,0} f_T \|^2_{L^2(]z(0),\infty[)} - \| \mathbb{H}^{\frac{1}{4}}_{0,0} f_T \|^2_{L^2(]z(0),\infty[)}$$
$$\leq \| \left(\mathbb{H}_{0,0} + \| V \|_{L^\infty(\mathbf{R})}\right)^{\frac{1}{4}} f_T \|^2_{L^2(]z(0),\infty[)} - \| \mathbb{H}^{\frac{1}{4}}_{0,0} f_T \|^2_{L^2(]z(0),\infty[)}$$
$$\leq 2 \int \left[1 - \left(1 + | \xi |^{-2} \| V \|_{L^\infty(\mathbf{R})}\right)^{\frac{1}{2}}\right] | \xi || \hat{f}_T(\xi) |^2 \, d\xi$$
$$\leq C \int | f_T(x) |^2 \, dx.$$

We note that (II.74) implies

(II.80 $$\| f_T \|_{L^2(]z(0),\infty[)} = O(e^{-\kappa T}), \quad T \longrightarrow \infty,$$

hence we conclude by (II.73) that:

(II.81)
$$\| \mathbb{H}^{\frac{1}{4}}_{V,0} f_T \|^2_{L^2(]z(0),\infty[)} \longrightarrow \| \sqrt{\coth\left(4\pi M \mathbb{H}^{\frac{1}{2}}_{out}\right)} \, \mathbb{H}^{\frac{1}{4}}_{out} f \|^2_{L^2(\mathbf{R})}, \quad T \longrightarrow \infty.$$

Lemma II.10 follows from (II.79) and (II.81). Q.E.D.

LEMMA II.11. – *Given $R > 0$ there exists $C_R > 0$ such that for any $T > 0$ and for any $F_T \in [C_0^\infty(]z(T), -T + R[)]^2$ we have:*

(II.82)
$$\| U_V(0,T) F_T \|_{\mathcal{H}^{\frac{1}{2}}(V,0)} \leq C_R \left\{ (1+T)^{\frac{3}{2}} \| F_T \|_{\mathcal{H}_1(V,T)} + \| F_T \|_{\mathcal{H}(V,T)} \right\}.$$

Proof of Lemma II.11. – Denoting ${}^t(f_T, p_T) = U_V(0,T) F_T$, we get from Lemma II.8 that for $0 < \alpha < -z(0)$:

(II.83)
$$\|U_V(0,T) F_T \|^2_{\mathcal{H}^{\frac{1}{2}}(V,0)}$$
$$\leq C_R \left\{ (1 + | \ln(\alpha) |)^3 \int_{z(0)}^{-\alpha} | x || f'_T(x) + p_T(x) |^2 \, dx \right.$$
$$+ \alpha (1 + | \ln(\alpha) |)^3 \int_{-\alpha}^{0} | f'_T(x) + p_T(x) |^2 \, dx$$
$$+ \int_{z(0)}^{0} | f'_T(x) + p_T(x) |^2 \, dx$$
$$\left. + \int_0^R | f'_T(x) |^2 + | p_T(x) |^2 \, dx \right\}$$
$$\leq C'_R \left\{ (1 + | \ln(\alpha) |)^3 \| U_V(0,T) F_T \|^2_{\mathcal{H}_1(V,0)} \right.$$
$$\left. + \left[1 + \alpha(1 + | \ln(\alpha) |)^3\right] \int_{-\alpha}^{R} | f'_T(x) |^2 + | p_T(x) |^2 \, dx \right\}.$$

We note that for $\alpha \leq -T - z(T)$ the standard energy inequality yields:

$$\int_{-\alpha}^{R} \mid f'_T(x) \mid^2 + \mid p_T(x) \mid^2 dx \leq \| F_T \|^2_{\mathcal{H}(V,T)}, \tag{II.84}$$

hence (II.11) follows from (II.83), (II.84), and (II.15) with $\alpha = -T - z(T)$.

Q.E.D.

LEMMA II.12. – *For any* $R > 0$, *there exists* $C_R > 0$ *such that for any* $t \leq 0$, φ, $\psi \in C_0^\infty(]z(t), -t+R[)$, *we have*

$$\| \varphi \|_{L^2((]z(t),-t+R[)} \leq C_R \|^t(\varphi, \psi) \|_{\mathcal{H}_1(0,t)} .$$

Proof of Lemma II.12. – For $z(t) \leq x < -t$ we have:

$$\mid \varphi(x) \mid^2 \leq (\ln \mid x+t \mid - \ln \mid z(t) + t \mid) \int_{z(t)}^{x} \mid y+t \| \varphi'(y) + \psi(y) \mid^2 dy$$
$$+ (x - z(t)) \int_{z(t)}^{x} \mid \varphi'(y) - \psi(y) \mid^2 dy,$$

and for $-t \leq x \leq R$, we have:

$$\mid \varphi(x) \mid^2 \leq (R-x) \left(\int_x^R \mid \varphi'(y) + \psi(y) \mid^2 dy + \int_x^R \mid \varphi'(y) - \psi(y) \mid^2 dy \right).$$

Hence for all x:

$$\mid \varphi(x) \mid^2 \leq C_R (1 + \ln \mid x+t \mid) \|^t (\varphi, \psi) \|^2_{\mathcal{H}_1(0,t)},$$

and Lemma II.12 follows by integrating.

Q.E.D.

LEMMA II.13. – *Given* F_- *and* F_-^T *as in Lemma* II.9 *we have:*
(II.85)

$$\| U_V(0,T) F_-^T \|_{\mathcal{H}^{\frac{1}{2}}(V,0)} \longrightarrow \sqrt{2} \| \sqrt{\coth\left(4\pi M \mathbb{H}_{out}^{\frac{1}{2}}\right)} \mathbb{H}_{out}^{\frac{1}{4}} f \|_{L^2(\mathbb{R})},$$

$$T \longrightarrow \infty.$$

Proof of Lemma II.13. – We recall the Duhamel formula
(II.86)

$$U_V(t,s)F = U_W(t,s)F + \int_s^t U_W(t,\sigma) \begin{pmatrix} 0 \\ (V - W)[U_V(\sigma,s)F]_1 \end{pmatrix} d\sigma,$$

hence
(II.87)
$$U_V(0,T)F_-^T = U_0(0,T)F_-^T - \int_0^T U_V(0,\sigma)\begin{pmatrix} 0 \\ V\left[U_0(\sigma,T)F_-^T\right]_1 \end{pmatrix} d\sigma.$$

We denote by $\theta_{R,T}$ the solution of

(II.88)
$$-z(\theta_{R,T}) + \theta_{R,T} = 2T - R.$$

The assumptions on $z(t)$ imply

(II.89)
$$\theta_{R,T} = T - \frac{R}{2} + O(e^{-2\kappa T}) < T - \frac{R}{2}, \quad T \to \infty.$$

The support of $(\sigma, x) \mapsto U_0(\sigma,T)F_-^T(x)$ is described by

(II.90)
$$T - \frac{R}{2} \leq \sigma \leq T, \quad U_0(\sigma,T)F_-^T(x) \neq 0 \Rightarrow x \in]z(\sigma), \sigma + R - 2T[,$$

(II.91)
$$\theta_{R,T} \leq \sigma \leq T - \frac{R}{2}, \quad U_0(\sigma,T)F_-^T(x) \neq 0 \Rightarrow x \in]z(\sigma), -\sigma[,$$

(II.92)
$$0 \leq \sigma \leq \theta_{R,T}, \quad U_0(\sigma,T)F_-^T(x) \neq 0 \Rightarrow x \in]z(\theta_{R,T}) + \theta_{R,T} - \sigma, -\sigma[.$$

By Lemma II.11 we have

(II.93)
$$\|U_V(0,T)F_-^T - U_0(0,T)F_-^T\|_{\mathcal{H}^{\frac{1}{2}}(V,0)} \leq C_R \int_0^T (1+\sigma)^{\frac{3}{2}} \| V(x)u_T(\sigma,x) \|_{L^2(]z(\sigma),\infty[)} d\sigma$$

with
$$u_T(\sigma,x) = \left[U_0(\sigma,T)F_-^T(x)\right]_1.$$

On the one hand, for $\sigma \in [0, \theta_{R,T}]$ we have $u_T(\sigma,x) = u_T(0,x+\sigma)$. Thus (II.80), (II.7) and (II.92) imply

(II.94)
$$\| V(x)u_T(\sigma,x) \|_{L^2(]z(\sigma),\infty[)} \leq Ce^{-\kappa(2\sigma+T)}.$$

On the other hand, for $\sigma \in [\theta_{R,T}, T]$ we have by Lemma II.12, (II.90), (II.91), (II.7) and (II.15):

(II.95)
$$\| V(x)u_T(\sigma,x) \|_{L^2(]z(\sigma),\infty[)} \leq Ce^{-(\kappa+\varepsilon)T}.$$

We deduce from (II.89), (II.93), (II.94) and (II.95) that

$$\text{(II.96)} \qquad \| U_V(0,T)F_-^T - U_0(0,T)F_-^T \|_{\mathcal{H}^{\frac{1}{2}}(V,0)} = O(e^{-\kappa T}).$$

Then we conclude thanks to Lemma II.10. Q.E.D.

Proof of Theorem II.3. – We have:

$$\| U_V(0,T)F^T \|^2_{\mathcal{H}^{\frac{1}{2}}(V,0)} = \| U_V(0,T)F_-^T \|^2_{\mathcal{H}^{\frac{1}{2}}(V,0)} + \| U_V(0,T)F_+^T \|^2_{\mathcal{H}^{\frac{1}{2}}(V,0)} + 2\Re\left(< U_V(0,T)F_-^T, U_V(0,T)F_+^T >_{\mathcal{H}^{\frac{1}{2}}(V,0)}\right).$$

On the one hand, Proposition II.2 and Lemma II.8 imply

$$\text{(II.97)} \qquad U_V(0,T)F_+^T \longrightarrow \Omega_V^{out}F_+^{out} \ \ in\ \mathcal{H}^{\frac{1}{2}}(V,0).$$

On the other hand, denoting $F_-^0 = {}^t(f_-, -f_-')$, Lemma II.13 gives

$$\text{(II.98)} \qquad \lim_{t\to\infty} \| U_V(0,T)F_-^T \|^2_{\mathcal{H}^{\frac{1}{2}}(V,0)} = 2 \| \sqrt{\coth\left(4\pi M \mathbb{H}_{out}^{\frac{1}{2}}\right)}\, \mathbb{H}_{out}^{\frac{1}{4}} f_- \|^2_{L^2(\mathbb{R})} = \| \sqrt{\coth\left(4\pi M \mathbb{H}_{out}^{\frac{1}{2}}\right)}\, F_-^0 \|^2_{\mathcal{H}_{out}^{\frac{1}{2}}},$$

moreover (II.80), (II.62) imply $U_0(0,T)F_-^T \rightharpoonup 0$ in the sense of distributions as $T \to \infty$, and since $U_0(0,T)F_-^T$ is bounded in $\mathcal{H}^{\frac{1}{2}}(V,0)$ according to Lemma II.10, then

$$U_0(0,T)F_-^T \rightharpoonup 0 \ \ in\ \ \mathcal{H}^{\frac{1}{2}}(V,0) - weak* \ \ T \longrightarrow \infty,$$

hence by (II.96) we have:

$$\text{(II.99)} \qquad U_V(0,T)F_-^T \rightharpoonup 0 \ \ in\ \ \mathcal{H}^{\frac{1}{2}}(V,0) - weak* \ \ T \longrightarrow \infty,$$

so we get from (II.97) and (II.99):

$$\text{(II.100)} \quad \Re\left(< U_V(0,T)F_-^T, U_V(0,T)F_+^T >_{\mathcal{H}^{\frac{1}{2}}(V,0)}\right) \longrightarrow 0, \ \ T \longrightarrow \infty.$$

The Theorem follows from (II.97), (II.98) and (II.100).

Q.E.D.

III. ESTIMATES FOR CLASSICAL FIELDS

The mixed problem (I.7)(I.8) with data Φ given at time s

$$\Psi(t = s, r_*, \omega) = \Phi, \tag{III.1}$$

is formally solved by a propagator $U(t, s)$

$$\Psi(t, .) = U(t, s)\Phi. \tag{III.2}$$

More precisely, we proved in [3] that $U(t, s)$ is a strongly continuous propagator on the family of Hilbert spaces of finite energy fields $\mathcal{H}(t)$, defined as the completion of $C_0^\infty(]z(t), \infty[_{r_*} \times S^2_\omega) \times C_0^\infty(]z(t), \infty[_{r_*} \times S^2_\omega)$ for the norm

$$\begin{aligned} &\| {}^t(f, p) \|^2_{\mathcal{H}(t)} \\ &= \int_{z(t)}^{\infty} \int_{S^2} | \partial_{r_*} f(r_*, \omega) |^2 + | p(r_*, \omega) |^2 \\ &+ \left(1 - \frac{2M}{r}\right) \left[\frac{1}{r^2} | \nabla_{S^2_\omega} f(r_*, \omega) |^2 + m^2 | f(r_*, \omega) |^2\right] r^2 dr_* \, d\omega. \end{aligned} \tag{III.3}$$

Moreover we have the following energy estimates

$$s \leq t \Longrightarrow \| U(t, s) \|_{\mathcal{L}(\mathcal{H}(s), \mathcal{H}(t))} = 1, \tag{III.4}$$

$$s \geq t \Longrightarrow \| U(t, s) \|_{\mathcal{L}(\mathcal{H}(s), \mathcal{H}(t))} \geq C_t e^{\kappa s}. \tag{III.5}$$

This last estimate means that the backward propagator is not uniformly bounded in the energy norm of $\mathcal{H}(t)$ because of the infinite Doppler effect due to the collapse to a Black-Hole. This fact makes very delicate the development of a scattering theory. Then, to take account of this phenomenon, it is necessary to introduce a new functional framework, $\mathcal{H}_1(t)$ defined as the completion of $C_0^\infty(]z(t), \infty[_{r_*} \times S^2_\omega) \times C_0^\infty(]z(t), \infty[_{r_*} \times S^2_\omega)$ for the norm

$$\begin{aligned} &\| {}^t(f, p) \|^2_{\mathcal{H}_1(t)} \\ &= \frac{1}{2} \int_{z(t)}^{-t} \int_{S^2} |r_* + t| |\partial_{r_*} f(r_*, \omega) + p(r_*, \omega)|^2 \\ &+ |\partial_{r_*} f(r_*, \omega) - p(r_*, \omega)|^2 \\ &+ 2\left(1 - \frac{2M}{r}\right) \left[\frac{1}{r^2} |\nabla_{S^2_\omega} f(r_*, \omega)|^2 + m^2 |f(r_*, \omega)|^2\right] r^2 dr_* \, d\omega \\ &+ \int_{-t}^{\infty} \int_{S^2} |\partial_{r_*} f(r_*, \omega)|^2 + |p(r_*, \omega)|^2 \\ &+ \left(1 - \frac{2M}{r}\right) \left[\frac{1}{r^2} |\nabla_{S^2_\omega} f(r_*, \omega)|^2 + m^2 |f(r_*, \omega)|^2\right] r^2 dr_* \, d\omega. \end{aligned} \tag{III.6}$$

We can interpret this space in terms of *conormal* distributions associated with the vector fields:

$$\mid t+r_* \mid^{\frac{1}{2}} (\partial_{r_*}+\partial_t),\ \partial_{r_*}-\partial_t,\ \nabla_{S^2_\omega}.$$

The main property is that the propagator is uniformly bounded on $\mathcal{H}_1(t)$ for each spherical harmonic: for $l \in \mathbb{N}$, $m \in \mathbb{Z}$, $\mid m \mid \leq l$, we denote by $\Pi_{l,m}$ the projector from $L^1_{loc}(\mathbb{R}_{r_*} \times S^2_\omega)$ onto $L^1_{loc}(\mathbb{R}_{r_*}) \otimes Y_{l,m}$ defined by

$$\varphi \in L^1_{loc}(\mathbb{R}_{r_*} \times S_2^\omega),\quad (\Pi_{l,m}\varphi)(r_*,\omega) =< \varphi(r_*,.), Y_{l,m} >_{L^2(S^2)} \otimes Y_{l,m}(\omega),$$

where $\{Y_{l,m},\ l \in \mathbb{N},\ m \in \mathbb{Z},\ \mid m \mid \leq l\}$ is the spherical harmonics basis of $L^2(S^2)$. The crucial estimate is the following:

$$\text{(III.7)} \qquad \sup_{0 \leq t,s} \| U(t,s)\Pi_{l,m} \|_{\mathcal{L}(\mathcal{H}_1(s),\mathcal{H}_1(t))} \leq C_l < \infty.$$

For the quantum field theory, we need a third space associated with the generator of the propagator ; so we introduce the positive selfadjoint operator $\mathbb{H}_t$ on

$$\text{(III.8)} \qquad L^2_t \equiv L^2(]z(t),\infty[_{r_*} \times S^2_\omega;\ r^2 dr_* d\omega),$$

given by:

$$\text{(III.9)} \qquad \mathbb{H}_t = -\frac{1}{r^2}\frac{\partial}{\partial r_*} r^2 \frac{\partial}{\partial r_*} + \left(1-\frac{2M}{r}\right)\left(-\frac{\Delta_{S^2}}{r^2}+m^2\right)$$

with dense domain

$$\text{(III.10)} \qquad D(\mathbb{H}_t) = \left\{ f \in L^2_t;\ \mathbb{H}_t f \in L^2_t,\ f(r_* = z(t),\omega) = 0 \right\},$$

and the Hilbert space $\mathcal{H}^{\frac{1}{2}}(t)$ completion of $D(\mathbb{H}_t^{\frac{1}{4}}) \times D(\mathbb{H}_t^{-\frac{1}{4}})$ for the norm

$$\text{(III.11)} \qquad \|^t(f,p)\|^2_{\mathcal{H}^{\frac{1}{2}}(t)} = \| \mathbb{H}_t^{\frac{1}{4}} f \|^2_{L^2_t} + \| \mathbb{H}_t^{-\frac{1}{4}} p \|^2_{L^2_t}.$$

The relations between these spaces are given by the following:

PROPOSITION III.1. – *Denoting by $\mathcal{E}'$ the set of compactly supported distributions, we have:*

$$\text{(III.12)} \qquad \mathcal{H}(t) \cap \mathcal{E}' \subset \mathcal{H}^{\frac{1}{2}}(t),\ 0 < m \Rightarrow \mathcal{H}(t) \subset \mathcal{H}^{\frac{1}{2}}(t),$$

(III.13) $$0 < m \Rightarrow \mathcal{H}(t) \subset \mathcal{H}^{\frac{1}{2}}(t),$$

(III.14) $$\mathcal{H}_1(t) \cap \mathcal{E}' \not\subset \mathcal{H}^{\frac{1}{2}}(t).$$

At last we introduce the tools necessary to study the asymptotic behaviour of fields near the future Black-Hole Horizon. We compare the solutions of (I.7) as $t \to +\infty$, $r \to 2M$, with the solutions of

(III.15) $$\frac{\partial^2}{\partial t^2}\Psi_{BH} - \frac{\partial^2}{\partial r_*^2}\Psi_{BH} = 0, \quad r_* \in \mathbb{R}, \quad t \in \mathbb{R}, \quad \omega \in S^2.$$

So we introduce: the operator $\mathbb{H}_{BH}$ on

$$L^2_{BH} \equiv L^2(\mathbb{R}_{r_*} \times S^2_\omega;\, 4M^2 dr_* d\omega),$$

given by

(III.17) $$\mathbb{H}_{BH} = -\frac{\partial^2}{\partial r_*^2}, \quad D(\mathbb{H}_{BH}) = \left\{f \in L^2_{BH};\ \partial^2_{r_*} f \in L^2_{BH}\right\},$$

the Hilbert spaces $\mathcal{H}_{BH}$ and $\mathcal{H}^{\frac{1}{2}}_{BH}$ defined as the completions of $[C_0^\infty(\mathbb{R}_{r_*} \times S^2_\omega)]^2$ for the norms

(III.18) $$\| {}^t(f,p) \|^2_{\mathcal{H}_{BH}} = \| \mathbb{H}^{\frac{1}{2}}_{BH} f \|^2_{L^2_{BH}} + \| p \|^2_{L^2_{BH}},$$

(III.19) $$\| {}^t(f,p) \|^2_{\mathcal{H}^{\frac{1}{2}}_{BH}} = \| \mathbb{H}^{\frac{1}{4}}_{BH} f \|^2_{L^2_{BH}} + \| \mathbb{H}^{-\frac{1}{4}}_{BH} p \|^2_{L^2_{BH}},$$

and the subspaces

(III.20) $$\mathcal{D}^{\pm}_{BH} = \left\{\Phi^{\pm}_{BH}(r_*,\omega) = \sum_{finite} {}^t(f_{l,m}(r_*), \pm f'_{l,m}(r_*)) \otimes Y_{l,m}(\omega);\ f_{l,m} \in C_0^\infty(]0,\infty[)\right\}.$$

We denote by $U_{BH}(t)$ the unitary group on $\mathcal{H}_{BH}$ associated with (III.15) and, given some function θ satisfying (II.25), we introduce the cut off operator Θ_{BH} defined by

$$(\Theta_{BH}\Phi)(r_*,\omega) = {}^t(\theta(r_*)f(r_*,\omega), \theta(r_*)p((r_*,\omega))), \quad \Phi = {}^t(f,p),$$

and we construct the Horizon Wave Operator defined for $\Phi^+_{BH} \in \mathcal{D}^+_{BH}$ by

$$\text{(III.21)} \qquad \Omega^+_{BH}\Phi^+_{BH} = \lim_{t\to\infty} U(0,t)\Theta_{BH}U_{BH}(t)\Phi^+_{BH} \ \mathit{in}\ \mathcal{H}(0).$$

PROPOSITION III.2. – *For any* $\Phi^+_{BH} \in \mathcal{D}^+_{BH}$ *the strong limit* (III.21) *exists and is independent of the choice of the function* θ *satisfying* (II.25). *Moreover*

$$\Omega^+_{BH}\Phi^+_{BH} \in \mathcal{H}^{\frac{1}{2}}(0).$$

Now we can state the fundamental estimate of this part:

THEOREM III.3. – *We assume that the function* z *satisfies* (I.5). *Given* $\Phi^0_{BH} = \Phi^+_{BH} + \Phi^-_{BH}$, $\Phi^\pm_{BH} \in \mathcal{D}^\pm_{BH}$, we put

$$\Phi^t_{BH}(r_*,\omega) = \Phi^0_{BH}(r_* + t,\omega).$$

Then the norm of $U(0,t)\Phi^t_{BH}$ in $\mathcal{H}^{\frac{1}{2}}(0)$ *has a limit as* $t \to +\infty$ *and*

$$\text{(III.22)} \qquad \lim_{t\to\infty} \| U(0,t)\Phi^t_{BH} \|^2_{\mathcal{H}^{\frac{1}{2}}(0)} = \| \Omega^+_{BH}\Phi^+_{BH} \|^2_{\mathcal{H}^{\frac{1}{2}}(0)} + \left\| \sqrt{\coth\left(\frac{\pi}{\kappa}\mathbb{H}^{\frac{1}{2}}_{BH}\right)}\,\Phi^-_{BH} \right\|^2_{\mathcal{H}^{\frac{1}{2}}_{BH}}$$

REMARK III.4. – *As in Remark* II.4, *we note that estimates* (III.5)(III.7) *and Proposition* III.1 *show that limit* (III.22) *is a very sharp estimate.*

Proof of Proposition III.1. – We can easily express $\mathcal{H}(t)$, $\mathcal{H}_1(t)$, $\mathcal{H}^{\frac{1}{2}}(t)$, $\mathbb{H}_t$, in terms of the spaces and operators of Part II. We introduce the map $\mathcal{R}$ defined by:

$$\text{(III.23)} \qquad \Phi \in \left[L^1_{loc}([z(t),\infty[_{r_*}\times S^2_\omega)\right]^2 \Longrightarrow (\mathcal{R}\Phi)(x,\omega) = r\Phi(r_*,\omega),$$

(III.24)
$$F \in \left[L^1_{loc}([z(t),\infty[_x)\right]^2 \Longrightarrow \mathcal{R}^{-1}(F\otimes Y_{l,m})(r_*,\omega) = r^{-1}F(x)\otimes Y_{l,m}(\omega).$$

with

$$x = r_* = r + 2M\ln(r-2M).$$

Then we have:

$$\text{(III.25)} \qquad \mathbb{H}_t = \mathcal{R}^{-1}\left(\bigoplus_{l=0}^{\infty}\bigoplus_{m=-l}^{m=l} \mathbb{H}_{V_l,t}\otimes \Pi_{l,m}\right)\mathcal{R}.$$

(III.26) $$\mathcal{H}(t) = \mathcal{R}^{-1}\left(\bigoplus_{l=0}^{\infty}\bigoplus_{m=-l}^{m=l}\mathcal{H}(V_l, t)\otimes Y_{l,m}\right),$$

(III.27) $$\mathcal{H}_1(t) = \mathcal{R}^{-1}\left(\bigoplus_{l=0}^{\infty}\bigoplus_{m=-l}^{m=l}\mathcal{H}_1(V_l, t)\otimes Y_{l,m}\right),$$

(III.28) $$\mathcal{H}^{\frac{1}{2}}(t) = \mathcal{R}^{-1}\left(\bigoplus_{l=0}^{\infty}\bigoplus_{m=-l}^{m=l}\mathcal{H}^{\frac{1}{2}}(V_l, t)\otimes Y_{l,m}\right),$$

and for

(III.29) $$\Phi = \sum_{l=0}^{\infty}\sum_{m=-l}^{m=l}\mathcal{R}^{-1}\left(\begin{pmatrix} f_{l,m} \\ p_{l,m}\end{pmatrix}\otimes Y_{l,m}\right),$$

we have:

(III.30)
$$\| \Phi \|^2_{\mathcal{H}(t)} = \sum_{l=0}^{\infty}\sum_{m=-l}^{m=l} \| \mathbb{H}^{\frac{1}{2}}_{V_l,t} f_{l,m} \|^2_{L^2([z(t),\infty[)} + \| p_{l,m} \|^2_{L^2([z(t),\infty[)},$$

(III.31)
$$\| \Phi \|^2_{\mathcal{H}^{\frac{1}{2}}(t)} = \sum_{l=0}^{\infty}\sum_{m=-l}^{m=l} \| \mathbb{H}^{\frac{1}{4}}_{V_l,t} f_{l,m} \|^2_{L^2([z(t),\infty[)} + \| \mathbb{H}^{-\frac{1}{4}}_{V_l,t} p_{l,m} \|^2_{L^2([z(t),\infty[)} .$$

On the one hand we have for $f_{l,m} \in C^2_0(]z(t), R[)$

(III.32)
$$\begin{aligned} &\|\mathbb{H}^{\frac{1}{4}}_{V_l,t} f_{l,m} \|^2_{L^2([z(t),\infty[)} \\ &\leq \| \mathbb{H}^{\frac{1}{2}}_{V_l,t} f_{l,m} \|_{L^2([z(t),\infty[)} \| f_{l,m} \|_{L^2([z(t),\infty[)} \\ &\leq C_R \| \mathbb{H}^{\frac{1}{2}}_{V_l,t} f_{l,m} \|^2_{L^2([z(t),\infty[)}, \end{aligned}$$

and on the other hand the Heinz theorem and Lemma II.8 imply

(III.33)
$$\| \mathbb{H}^{-\frac{1}{4}}_{V_l,t} p_{l,m} \|_{L^2([z(t),\infty[)} \leq \| \mathbb{H}^{-\frac{1}{4}}_{0,t} p_{l,m} \|_{L^2([z(t),\infty[)} \leq C_R \| p_{l,m} \|_{L^2([z(t),\infty[)},$$

where C_R does not depend on $V_l \geq 0$. Therefore (III.12) follows from (III.30), (III.31), (III.32) and (III.33). To establish (III.13) we note that since

$$0 < m^2 \leq \mathbb{H}_t,$$

we have

$$\mathbb{H}_t^{-\frac{1}{4}} \leq m^{-\frac{1}{2}}, \quad \mathbb{H}_t^{\frac{1}{4}} \leq m^{-\frac{1}{2}} \mathbb{H}_t^{\frac{1}{2}}.$$

At last, to prove (III.14) we choose $\chi \in C_0^1(\mathbb{R})$, $\chi(0) = 1$, and we put:

$$f_n(r_*, \omega) = \left(1 + \frac{\ln(n)}{\ln \mid r_* + t \mid}\right) \mathbf{1}_{[-t-\frac{1}{n}, -t[}(r_*) + \chi(r_*) \mathbf{1}_{[-t, \infty[}(r_*).$$

We easily check that ${}^t(f_n, \partial_{r_*} f_n)$ is a Cauchy sequence in $\mathcal{H}_1(t)$, but converges as $n \to \infty$ to ${}^t(\chi(r_*)\mathbf{1}_{[-t,\infty[}(r_*), \delta_0(r_*) + \chi'(r_*)\mathbf{1}_{[-t,\infty[}(r_*)) \otimes 1(\omega)$ in $\mathcal{H}_1(t)$ which does not belong to $H^{\frac{1}{2}} \times H^{-\frac{1}{2}}(]z(t), \infty[\times S^2)$.

Q.E.D.

Proof of Proposition III.2. – We have:

$$U(t, s) = \mathcal{R}^{-1} \left(\bigoplus_{l=0}^{\infty} \bigoplus_{m=-l}^{m=l} U_{V_l}(t, s) \otimes \Pi_{l,m} \right) \mathcal{R}, \tag{III.34}$$

$$\mathbb{H}_{BH} = \bigoplus_{l=0}^{\infty} \bigoplus_{m=-l}^{m=l} \mathbb{H}_{out} \otimes \Pi_{l,m}, \tag{III.35}$$

$$\mathcal{H}_{BH} = \bigoplus_{l=0}^{\infty} \bigoplus_{m=-l}^{m=l} \mathcal{H}_{out} \otimes Y_{l,m}, \quad \mathcal{H}_{BH}^{\frac{1}{2}} = \bigoplus_{l=0}^{\infty} \bigoplus_{m=-l}^{m=l} \mathcal{H}_{out}^{\frac{1}{2}} \otimes Y_{l,m}, \tag{III.36}$$

$$\| \sum F_{l,m}^{out} \otimes Y_{l,m} \|_{\mathcal{H}_{BH}^{\frac{1}{2}}}^2 = 4M^2 \sum \| F_{l,m}^{out} \|_{\mathcal{H}_{out}^{\frac{1}{2}}}^2, \tag{III.37}$$

(III.38)

$$U_{BH}(t) = \bigoplus_{l=0}^{\infty} \bigoplus_{m=-l}^{m=l} U^{out}(t) \otimes \Pi_{l,m}, \quad \Theta_{BH} = \bigoplus_{l=0}^{\infty} \bigoplus_{m=-l}^{m=l} \Theta^{out} \otimes \Pi_{l,m},$$

hence for

$$\Phi_{BH}^+ = \sum_{finite} F_{+,l,m}^{out} \otimes Y_{l,m} \in \mathcal{D}_{BH}^+, \quad F_{+,l,m}^{out} \in \mathcal{D}_+^{out}. \tag{III.39}$$

we have:

(III.40)
$$U(0,t)\Theta_{BH}U_{BH}(t)\Phi^{+}_{BH}$$
$$= 2M\mathcal{R}^{-1}\left(\sum_{finite}\left(U_{V_l}(0,t)\Theta^{out}U^{out}(t)F^{out}_{+,l,m}\right)\otimes Y_{l,m}\right)$$
$$+\mathcal{R}^{-1}\left(\sum_{finite}U_{V_l}(0,t)\otimes\Pi_{l,m}\right)(r-2M)\Theta_{BH}U_{BH}(t)\Phi^{+}_{BH}.$$

Since

(III.41)
$$r-2M,\ \partial_{r_*}(r-2M)=O(e^{2\kappa r_*}),$$

and

$$\Theta_{BH}U_{BH}(t)\Phi^{+}_{BH}(r_*,\omega)=\Phi^{+}_{BH}(r_*+t,\omega),$$

we have with (II.14):

(III.42)
$$\|\mathcal{R}^{-1}(U_{V_l}(0,t)\otimes\Pi_{l,m})(r-2M)\Theta_{BH}U_{BH}(t)\Phi^{+}_{BH}\|_{\mathcal{H}_{BH}}=O(e^{-\kappa t})$$

so Proposition II.14 follows from (III.40), (III.42), and Proposition II.2 and we have

(III.43)
$$\Omega^{+}_{BH}\Phi^{+}_{BH}=2M\mathcal{R}^{-1}\sum\Omega^{out}_{V_l}F^{out}_{+,l,m}\otimes Y_{l,m},$$

(III.44)
$$\|\Omega^{+}_{BH}\Phi^{+}_{BH}\|^2_{\mathcal{H}^{\frac{1}{2}}(0)}=4M^2\sum\|\Omega^{out}_{V_l}F^{out}_{+,l,m}\|^2_{\mathcal{H}^{\frac{1}{2}}(V_l,0)}.$$

Q.E.D.

Proof of Theorem III.3. – Given

$$\Phi^0_{BH}=\Phi^+_{BH}+\Phi^-_{BH},\quad \Phi^{\pm}_{BH}\in\mathcal{D}^{\pm}_{BH},\quad \Phi^{\pm}_{BH}=\sum_{finite}F^0_{\pm,l,m}\otimes Y_{l,m},\quad F^0_{\pm,l,m}\in\mathcal{D}^{out}_{\pm},$$

we put

$$F^t_{l,m}(r_*)=F^0_{+,l,m}(r_*+t)+F^0_{-,l,m}(r_*+t),$$

hence

(III.45)
$$U(0,t)\Phi^t_{BH}=2M\mathcal{R}^{-1}\left(\sum U_{V_l}(0,t)F^t_{l,m}\otimes Y_{l,m}\right)$$
$$+\mathcal{R}^{-1}\left(\sum U_{V_l}(0,t)\left[(r-2M)F^t_{l,m}\right]\otimes Y_{l,m}\right).$$

Firstly, by (III.41), (II.14), (III.28), and Lemma II.8, we have:
(III.46)

$$\| \mathcal{R}^{-1}\Big(\sum U_{V_l}(0,t)\big[(r-2M)F^t_{l,m}\big]\otimes Y_{l,m}\Big) \|_{\mathcal{H}^{\frac{1}{2}}(0)} = O(e^{-\kappa t}).$$

and secondly, by Theorem II.3 and (III.28) we have:

$$\text{(III.47)}\quad \|U_{V_l}(0,t)F^t_{l,m}\|^2_{\mathcal{H}^{\frac{1}{2}}(V_l,0)} \underset{t\to\infty}{\longrightarrow} 4M^2 \| \Omega^{out}_{V_l}F^0_{+,l,m}\|^2_{\mathcal{H}^{\frac{1}{2}}(V_l,0)} + 4M^2 \| \sqrt{\coth\left(\frac{\pi}{\kappa}\mathbb{H}^{\frac{1}{2}}_{out}\right)}F^0_{-,l,m}\|^2_{\mathcal{H}^{\frac{1}{2}}_{out}}$$

therefore we conclude with (III.44) and (III.37) that

$$\text{(III.48)}\quad \|U(0,t)\Phi^t_{BH}\|^2_{\mathcal{H}^{\frac{1}{2}}(0)} \underset{t\to\infty}{\longrightarrow} \| \Omega^+_{BH}\Phi^+_{BH}\|^2_{\mathcal{H}^{\frac{1}{2}}(0)} + \| \sqrt{\coth\left(\frac{\pi}{\kappa}\mathbb{H}^{\frac{1}{2}}_{BH}\right)}\Phi^-_{BH}\|^2_{\mathcal{H}^{\frac{1}{2}}_{BH}}$$

Q.E.D.

IV. QUANTIZATION AND HAWKING'S RADIATION

We recall the basic concepts of the quantum machinery (see e.g. [8], [17], [19] , [5], [21], [27]). Algebraic quantum field theory deals with some $\mathbb{C}^* - algebra$, $\mathfrak{A}$, and some states, ω, which are positive, normalized, linear forms on $\mathfrak{U}$. To construct these objects we start with a Weyl quantization $(\mathfrak{W}, \mathfrak{H})$ on a real linear space $\mathcal{D}$ endowed with a skew-symmetric, non degenerate, bilinear form , $\sigma(.,.)$, where $\mathfrak{H}$ is a complex Hilbert space, and $\mathfrak{W}$ is a map: $\Phi \in \mathcal{D} \to \mathfrak{W}(\Phi)$, from $\mathcal{D}$ to the space $\mathfrak{U}(\mathfrak{H})$ of unitary operators on $\mathfrak{H}$, satisfying the Weyl version of the canonical commutation relations (CCR's):

$$\text{(IV.1)}\qquad \mathfrak{W}(\Phi+\Psi) = e^{-\frac{i}{2}\sigma(\Phi,\Psi)}\mathfrak{W}(\Phi)\mathfrak{W}(\Psi),$$

and

$$\text{(IV.2)}\qquad \Phi\in\mathcal{D}_f \to [\mathfrak{W}(\Phi)](X)\in\mathfrak{H}$$

is continuous for any finite dimensional subspace $\mathcal{D}_f$ of $\mathcal{D}$, and any arbitrary vector $X \in \mathfrak{H}$. The fundamental example is the *Fock-Cook* quantization

$(\mathfrak{W}_{\mathcal{F}}, \mathfrak{H}_{\mathcal{F}})$ of a Hilbert space $\mathfrak{h}$ with $\sigma = 2\Im < .,. >_{\mathfrak{h}}$. It is constructed as follows: we take $\mathfrak{H}_{\mathcal{F}} = \mathcal{F}(\mathfrak{h})$ the boson Fock space over $\mathfrak{h}$:

$$\mathfrak{H}_{\mathcal{F}} = \bigoplus_{n=0}^{n=\infty} \left[\mathfrak{h}^{\otimes n}\right]_s \tag{IV.3}$$

where $\left[\mathfrak{h}^{\otimes n}\right]_s$ stands for the n-fold symmetric tensor product of $\mathfrak{h}$, and we put:

$$\Phi \in \mathfrak{h} \longmapsto \mathfrak{W}_{\mathcal{F}}(\Phi) = \exp\left[a^*(\Phi) - (a^*(\Phi))^*\right] \in \mathfrak{U}(\mathfrak{H}_{\mathcal{F}}) \tag{IV.4}$$

where $a^*(\Phi)$ is the standard creation operator.

We define the algebra of observables $\mathfrak{A}(\mathcal{D})$ as the minimal $\mathbb{C}^* - subalgebra$ in the space $\mathcal{L}(\mathfrak{H})$ of bounded linear maps on $\mathfrak{H}$, containing all the operators $\mathfrak{W}(\Phi)$. The algebra of observables is unique in the following sense: if $(\mathfrak{W}, \mathfrak{H})$ and $(\tilde{\mathfrak{W}}, \tilde{\mathfrak{H}})$ are two Weyl quantizations on $\mathcal{D}$, possibly non unitarily equivalent if the dimension of $\mathcal{D}$ is infinite, if $\mathfrak{A}(\mathcal{D})$ and $\tilde{\mathfrak{A}}(\mathcal{D})$ are the associated algebras, Von Neumann's uniqueness theorem assures that the map $\mathfrak{W}(\Phi) \rightarrow \tilde{\mathfrak{W}}(\Phi)$ can be extended in a norm-preserving and involution-preserving isomorphism from $\mathfrak{A}(\mathcal{D})$ onto $\tilde{\mathfrak{A}}(\mathcal{D})$.

Now given ω a state on $\mathfrak{A}(\mathcal{D})$, the map:

$$\Phi \in \mathcal{D} \rightarrow E_\omega(\Phi) = \omega\big(\mathfrak{W}(\Phi)\big) \tag{IV.5}$$

satisfies

$$E(0) = 1, \tag{IV.6}$$

(IV.7)

$$\forall \Phi_j \in \mathcal{D},\ \ \forall \lambda_j \in \mathbb{C},\ \ \forall n \in \mathbb{N},\ \ \sum_{j,k=1}^{n} \lambda_j \bar{\lambda}_k E(\Phi_j - \Phi_k) e^{-i\sigma(\Phi_j, \Phi_k)} \geq 0,$$

(IV.8) $E \in C^0(\mathcal{D}_f, \mathbb{C})$ for all finite dimensional subspace $\mathcal{D}_f$ of $\mathcal{D}$.

Each functional satisfying properties (IV.6), (IV.7), (IV.8) is called a *generating functional* over $\mathcal{D}$. The importance of this notion is that it provides the possibility of reducing a quantum problem, the study of states on a $\mathbb{C}^* - algebra$, to a classical problem, the study of functionals on $\mathcal{D}$: conversely, each generating functional E determines uniquely a state ω_E

on $\mathfrak{A}(\mathcal{D})$ with a suitable Weyl quantization $(\mathfrak{W}_E, \mathfrak{H}_E)$ and cyclic vector $X_E \in \mathfrak{H}_E$ by formula:

$$E(\Phi) \equiv \omega_E\big(\mathfrak{W}_E(\Phi)\big) = < X_E, [\mathfrak{W}_E(\Phi)](X_E) >_{\mathfrak{H}_E} . \tag{IV.9}$$

For instance the *Fock vacuum* state ω_0 on $\mathfrak{A}(\mathfrak{h})$, associated with the *Fock vacuum* vector $\Omega_{\mathcal{F}} = (1, 0, 0, ...) \in \mathfrak{H}_{\mathcal{F}}$, is defined by the functional:

$$E_0(\Phi) \equiv < \Omega_{\mathcal{F}}, [\mathfrak{W}_{\mathcal{F}}(\Phi)](\Omega_{\mathcal{F}}) >_{\mathfrak{H}_{\mathcal{F}}} = e^{-\frac{1}{2}\|\Phi\|_{\mathfrak{h}}^2}. \tag{IV.10}$$

The above constructions can be generalized to allow the quantization of a *boson single particle space* $(\mathcal{D}_t, \sigma_t, U(s,t))_{t,s\in\mathbb{R}}$ where $\mathcal{D}_t$ is a real linear space endowed with a skew-symmetric, non degenerate, bilinear form $\sigma_t(\ldots)$, and $U(s,t)$ is a symplectic propagator from $(\mathcal{D}_t, \sigma_t)$ onto $(\mathcal{D}_s, \sigma_s)$. A *Weyl quantization* $(\mathfrak{W}_t, \mathfrak{H})_{t\in\mathbb{R}}$ of $(\mathcal{D}_t, \sigma_t, U(s,t))_{t,s\in\mathbb{R}}$ is defined as a family of Weyl quantizations $(\mathfrak{W}_t, \mathfrak{H})$, of $(\mathcal{D}_t, \sigma_t)$ satisfying for all $t, s \in \mathbb{R}$:

$$\mathfrak{W}_s(U(s,t)\Phi_t) = \mathfrak{W}_t(\Phi_t). \tag{IV.11}$$

Then $\mathfrak{A}(\mathcal{D}_t) = \mathfrak{A}(\mathcal{D}_s) \equiv \mathfrak{A}$ and a state ω is characterized by one of the generating functionals

$$E_t : \Phi_t \in \mathcal{D}_t \rightarrow \omega\big(\mathfrak{W}_t(\Phi_t)\big) \tag{IV.12}$$

which satisfy

$$E_t(\Phi_t) = E_s(U(s,t)\Phi_t). \tag{IV.13}$$

In particular, the Fock quantization of a boson single particle space is defined by a real linear map $\mathbb{K}$ from $\mathcal{D}_0$ to some complex Hilbert space $\mathfrak{h}$, satisfying:

$$\forall \Phi_0, \Psi_0 \in \mathcal{D}_0,\ \sigma_0(\Phi_0, \Psi_0) = 2\Im < \mathbb{K}\Phi_0, \mathbb{K}\Psi_0 >_{\mathfrak{h}}, \tag{IV.14}$$

and by putting

$$\mathfrak{W}_t(\Phi_t) = \mathfrak{W}_{\mathcal{F}}(\mathbb{K}U(0,t)\Phi_t) \in \mathfrak{L}(\mathfrak{H}_{\mathcal{F}}) \tag{IV.15}$$

where $(\mathfrak{W}_{\mathcal{F}}, \mathfrak{H}_{\mathcal{F}})$ is the Fock quantization of $\mathfrak{h}$. We call *ground quantum state*, the state ω_0 on $\mathfrak{A}$ associated with the functional:

$$\Phi_0 \in \mathcal{D}_0 \rightarrow E_0(\Phi_0) = e^{-\frac{1}{2}\|\mathbb{K}\Phi_0\|_{\mathfrak{h}}^2}. \tag{IV.16}$$

More generally, given a positive, densely defined, selfadjoint operator $\mathbb{H}$ on $\mathfrak{h}$, satisfying:

$$\mathbb{K}(\mathcal{D}_0) \subset D(\mathbb{H}^{-\frac{1}{4}}), \tag{IV.17}$$

a *thermal quantum state* ω_θ of temperature $\theta > 0$ with respect to $\mathbb{H}$, is associated with the functional

$$\Phi_0 \in \mathcal{D}_0 \to E_\theta(\Phi_0) = \exp\left(-\frac{1}{2} \parallel \sqrt{\coth\left(\frac{1}{2\theta}\mathbb{H}^{\frac{1}{2}}\right)}\, \mathbb{K}\Phi_0 \parallel_{\mathfrak{h}}^2\right) \tag{IV.18}$$

In terms of particles, if $\mathbb{H}$ is the Klein-Gordon hamiltonian, (IV.18) describes a gaz of free bosons at temperature θ.

We apply the previous tools to define the Fock quantization of a spin-0 field outside the collapsing star, $(\mathcal{D}_t, \sigma_t, U(s,t))_{s,t\in\mathbb{R}}$, by putting:

$$\mathcal{D}_t = \left\{ \Re\left(\sum_{finite} \begin{pmatrix} f_{l,m}(r_*) \\ p_{l,m}(r_*) \end{pmatrix} \otimes Y_{l,m}(\omega) \right); \right. \\ (f_{l,m}, p_{l,m}) \in C_0^2 \times C_0^1([z(t), \infty[), \\ \left. f_{l,m}(z(t)) = p_{l,m}(z(t)) + \dot{z}(t) f'_{l,m}(z(t)) = 0 \right\}, \tag{IV.19}$$

$$\sigma_t(\Phi_1, \Phi_2) = \int_{z(t)}^{\infty} \int_{S^2} (f_1 p_2 - f_2 p_1) r^2 dr_* d\omega, \quad \Phi_j = {}^t(f_j, p_j), \tag{IV.20}$$

$$\mathfrak{h}_0 = L^2(]z(0), \infty[\times S^2_\omega,\ r^2 dr_* d\omega\,; \mathbb{C}), \tag{IV.21}$$

$$\mathbb{K}_0 \Phi = \frac{1}{\sqrt{2}}\left(\mathbb{H}_0^{\frac{1}{4}} f + i \mathbb{H}_0^{-\frac{1}{4}} p\right), \quad \Phi = {}^t(f, p), \tag{IV.22}$$

where $U(t,s)$ is the propagator (III.2), and $\mathbb{H}_0$ is the selfadjoint operator (III.9), (III.10) at time 0. In the same way we quantize the fields of particles, falling into the Black-Hole Horizon, $(\mathcal{D}^+_{BH,t}, \sigma_{BH}, U_{BH}(t))_{t\in\mathbb{R}}$, or radiating to infinity, $(\mathcal{D}^-_{BH,t}, \sigma_{BH}, U_{BH}(t))_{t\in\mathbb{R}}$, by putting:

$$\mathcal{D}^{\pm}_{BH,t} = \left\{ \Re\left(\sum_{finite} \begin{pmatrix} f_{l,m}(r_*) \\ \pm f'_{l,m}(r_*) \end{pmatrix} \otimes Y_{l,m}(\omega) \right); \right. \\ \left. f_{l,m} \in C_0^2(] \mp t, \infty[), \right\}, \tag{IV.23}$$

(IV.24)
$$\sigma_{BH}(\Phi_1, \Phi_2) = 4M^2 \int_{-\infty}^{\infty} \int_{S^2} (f_1 p_2 - f_2 p_1) dr_* d\omega, \quad \Phi_j = {}^t(f_j, p_j).$$

$$\mathfrak{h}_{BH} = L^2(\mathbb{R}_{r_*} \times S^2_\omega, 4M^2 dr_* d\omega\,; \mathbb{C}), \tag{IV.25}$$

$$\mathbb{K}_{BH}\Phi = \frac{1}{\sqrt{2}}\left(\mathbb{H}_{BH}^{\frac{1}{4}} f + i\mathbb{H}_{BH}^{-\frac{1}{4}} p\right), \quad \Phi = {}^t(f, p). \tag{IV.26}$$

We investigate the quantum state measured by a fiducial observer falling into the future Black-Hole Horizon. The particles detector that reaches the future Black-Hole Horizon as $t \to +\infty$, is modelized by an observable $\mathfrak{W}_t(\Phi^t_{BH})$.

THEOREM IV.1 (Main Result). – *Given* $\Phi^\pm_{BH} \in \mathcal{D}^\pm_{BH,0}$ *we denote for* $t \geq 0$*:*

$$\Phi^t_{BH}(r_*, \omega) = \Phi^+_{BH}(r_* + t, \omega) + \Phi^-_{BH}(r_* + t, \omega). \tag{IV.27}$$

Then

$$\omega_0(\mathfrak{W}_t(\Phi^t_{BH})) \underset{t \to +\infty}{\longrightarrow} \exp\left(-\frac{1}{2} \| \mathbb{K}_0 \Omega^+_{BH} \Phi^+_{BH} \|^2_{\mathfrak{h}_0}\right) \times \exp\left(-\frac{1}{2} \| \sqrt{\coth\left(\frac{\pi}{\kappa}\mathbb{H}^{\frac{1}{2}}_{BH}\right)} \mathbb{K}_{BH} \Phi^-_{BH} \|^2_{\mathfrak{h}_{BH}}\right). \tag{IV.28}$$

REMARK IV.2. – *The limit* (IV.28) *is the main result of this work. It is the famous statement by S. Hawking* [16]: *For an observer going across the Horizon created by a gravitational collapse, the Black-Hole seems to be radiating to infinity at temperature* $\frac{1}{8\pi M}$. *We will study the case of an observer at rest with respect to the Black-Hole in a future paper.*

Proof of Theorem IV.1. – We apply (IV.10) and Theorem III.3 to get:

$$
\begin{aligned}
&\omega_0\left(\mathfrak{W}_t(\Phi^t_{BH})\right)\\
&= \exp\left(-\frac{1}{2} \parallel \mathbb{K}_0 U(0,t)\Phi^t_{BH} \parallel^2_{\mathfrak{h}_0}\right)\\
&= \exp\left(-\frac{1}{2} \parallel U(0,t)\Phi^t_{BH} \parallel^2_{\mathcal{H}^{\frac{1}{2}}(0)}\right)\\
&\underset{t\to\infty}{\longrightarrow} \exp\left(-\frac{1}{2} \parallel \Omega^+_{BH}\Phi^+_{BH} \parallel^2_{\mathcal{H}^{\frac{1}{2}}(0)}\right)\\
&\quad \times \exp\left(-\frac{1}{2} \parallel \sqrt{\coth\left(\frac{\pi}{\kappa}\mathbb{H}^{\frac{1}{2}}_{BH}\right)}\Phi^-_{BH} \parallel^2_{\mathcal{H}^{\frac{1}{2}}_{BH}}\right)\\
&= \exp\left(-\frac{1}{2} \parallel \mathbb{K}_0\Omega^+_{BH}\Phi^+_{BH} \parallel^2_{\mathfrak{h}_0}\right)\\
&\quad \times \exp\left(-\frac{1}{2} \parallel \sqrt{\coth\left(\frac{\pi}{\kappa}\mathbb{H}^{\frac{1}{2}}_{BH}\right)}\,\mathbb{K}_{BH}\Phi^-_{BH} \parallel^2_{\mathfrak{h}_{BH}}\right).
\end{aligned}
$$

Q.E.D.

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