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Asymptotic completeness for the Klein-Gordon equation on the Schwarzschild metric

by

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ABSTRACT. – We prove the strong asymptotic completeness of the wave operators, classic at the horizon and Dollard-modified at infinity, describing the scattering of a massive Klein-Gordon field by a Schwarzschild black-hole. The scattering operator is unitarily implementable in the Fock space of free fields.

RÉSUMÉ. – Nous démontrons la complétude asymptotique forte des opérateurs d'onde, classiques à l'horizon, modifiés à la Dollard à l'infini, décrivant la diffraction d'un champ de Klein-Gordon massif par un trou noir de Schwarzschild. L'opérateur de diffraction est unitairement implémentable dans l'espace de Fock des champs libres.

1. INTRODUCTION

The present paper is mainly devoted to proving the asymptotic completeness of the wave operators associated with the massive Klein-Gordon equation on the Schwarzschild metric.

This work is the continuation of a program of rigorous mathematical studies on fields theory on a black-hole-type background. The previous investigations concerned the Maxwell system ([2], [3]), the black-hole resonance's [4], the non linear Klein-Gordon equation [5], the Dirac system [18].

This time dependent approach of scattering theory for scalar fields outside a Schwarzschild black-hole was initiated by Dimock and Kay in a very interesting series of papers [9 to 12]. They constructed the wave operators and, assuming the asymptotic completeness, investigated the quantum states ([11], [12]). Asymptotic completeness is known for massless scalar fields (*see* Dimock [9]). In the massive case, the problem is more difficult due to the long range of the gravitational interaction and we must modify the definition of the wave operators at infinity. Our main contribution, exposed in part II, consists in establishing the existence of the scattering operator constructed with the Dollar-modified wave operators. The main tools are the invariance principle, the results by H. Kitada ([16], [17]), for long range potentials, the analyticity of the gravitational interaction which garantees the absence of singular continuous spectrum, and Kato's two Hilbert spaces scattering theory techniques. This result is used in part III to show, using the second quantization, that the scattering operator is unitarily implemented on the Fock space of the free fields moving at the horizon and at infinity.

2. THE CLASSICAL SCATTERING OPERATOR

A massive scalar field Ψ on the Schwarzschild background is described by the covariant Klein-Gordon equation

$$(\Box_g + m^2)\,\Psi = 0,$$

where $m > 0$ is the mass of the field and $\Box_g f = |g|^{-1/2}\,\partial_\mu (|g|^{1/2}\, g^{\mu\nu}\,\partial_\nu f)$ is the D'Alembertian for the Schwarzschild metric g associated with a spherical black-hole of mass $M > 0$:

$$g_{\mu\nu}\,dx^\mu\,dx^\nu = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2\,(d\theta^2 + \sin^2\theta\, d\varphi^2).$$

We introduce the Regge-Wheeler tortoise radial coordinate r_* defined by

$$r_* = r + 2M\,\mathrm{Log}\,(r - 2M), \qquad r > 2M,$$

$\Psi = \Psi(t, r_*, \omega)$ satisfies

$$\left[\frac{\partial^2}{\partial t^2} - \frac{1}{r^2}\frac{\partial}{\partial r_*} r^2 \frac{\partial}{\partial r_*} + \left(1 - \frac{2M}{r}\right)\left(-\frac{1}{r^2}\Delta_{S^2} + m^2\right)\right]\Psi = 0,$$

$$t \in \mathbb{R}, \qquad r_* \in \mathbb{R}, \qquad \omega \in S^2. \tag{1}$$

Here Δ_{S^2} stands for the Laplacian on the Euclidean two sphere S^2 and $r > 2M$ is an implicit function of r_*.

We write (1) in its Hamiltonian form. Putting

$$\left.\begin{aligned} U &= {}^t(\Psi,\, \partial_t \Psi), \\ H &= \frac{1}{i}\begin{bmatrix} 0 & -1 \\ h & 0 \end{bmatrix}, \\ h &= -\frac{1}{r^2}\frac{\partial}{\partial r_*} r^2 \frac{\partial}{\partial r_*} + \left(1-\frac{2M}{r}\right)\left(-\frac{1}{r^2}\Delta_{S^2} + m^2\right), \end{aligned}\right\} \tag{2}$$

(1) becomes the Schrödinger type equation

$$\frac{\partial U}{\partial t} = -i\, HU. \tag{3}$$

We introduce the Hilbert space $\mathcal{H}$ of finite energy data, completion of $C_0^\infty(\mathbb{R}_{r_*} \times S_\omega^2) \times C_0^\infty(\mathbb{R}_{r_*} \times S_\omega^2)$ for the norm

$$\|{}^t(f,\, g)\|_{\mathcal{H}}^2 = \int \left\{ \left| \frac{\partial}{\partial r_*} f \right|^2 + \left(1 - \frac{2M}{r}\right) \times \left[\frac{1}{r^2} |\nabla_{S^2} f|^2 + m^2 |f|^2\right] + |g|^2 \right\} r^2\, dr_*\, d\omega. \tag{4}$$

The elements of $\mathcal{H}$ are distributions on $\mathbb{R}_{r_*} \times S_\omega^2$ and the differential operator H with domain

$$D(H) = \{U \in \mathcal{H};\ HU \in \mathcal{H}\}$$

is selfadjoint on $\mathcal{H}$ (*see* [9] or lemma 1 below).

Thus, for any F in $\mathcal{H}$, equation (3) has a unique solution $U(t)$ satisfying

$$U \in C^0(\mathbb{R}_t,\, \mathcal{H}), \qquad U(0) = F,$$

and U is given by Stone's theorem:

$$U(t) = e^{-itH} F. \tag{5}$$

Near the black-hole horizon $(r = 2M) \times S_\omega^2$ we compare the solutions of (1) with the plane waves solutions of

$$\frac{\partial^2}{\partial t^2}\Psi_0 - \frac{\partial^2}{\partial r_*^2}\Psi_0 = 0. \tag{6}$$

We introduce the Hilbert spaces $\mathcal{H}_0^\pm$, completions of the sets of regular left/right going data $\mathcal{D}_0^\pm$

$$\mathcal{D}_0^\pm = \left\{{}^t\left(f_1(r_*,\, \omega),\ \pm \frac{\partial}{\partial r_*} f_1(r_*,\, \omega)\right),\ f_1 \in C_0^\infty(\mathbb{R}_{r_*} \times S_\omega^2)\right\}, \tag{7}$$

for the free energy norm

$$\| {}^t(f,\, g) \|^2_{\mathcal{H}_0^\pm} = 4\, M^2 \int \left\{ \left| \frac{\partial}{\partial r_*} f \right|^2 + |\, g\,|^2 \right\} dr_*\, d\omega, \tag{8}$$

and the unitary group $U_0(t)$ defined on $\mathcal{H}_0^+ \oplus \mathcal{H}_0^-$ by

$$F_0^\pm \in \mathcal{D}_0^\pm, \quad U_0(t)\, F_0^\pm = (e^{-itH_0}\, F_0^\pm)(r_*,\, \omega) = F_0^\pm(\pm t + r_*,\, \omega), \tag{9}$$

$$H_0 = \frac{1}{i} \begin{bmatrix} 0 & -1 \\ h_0 & 0 \end{bmatrix}, \qquad h_0 = -\frac{\partial^2}{\partial r_*^2}\,. \tag{10}$$

We construct an (unbounded) identification operator $\mathcal{I}_0$ between $\mathcal{D}_0^+ \oplus \mathcal{D}_0^-$ and $\mathcal{H}$ by putting

$$\left.\begin{aligned} F_0^\pm = {}^t(f,\, g) \in \mathcal{D}_0^\pm, \\ (\mathcal{I}_0\, F_0^\pm)(r_*,\, \omega) = {}^t(\mathcal{X}_0(r_*)\, f(r_*,\, \omega), \quad \chi_0(r_*)\, g(r_*,\, \omega)), \end{aligned}\right\} \tag{11}$$

where χ_0 is defined as follows

$$\left.\begin{aligned} \chi_0 \in C^\infty(\mathbb{R}_{r_*}), \quad r_* < a \quad \Rightarrow \quad \chi_0(r_*) = 1, \\ r_* > b \;\Rightarrow\; \chi_0(r_*) = 0, \quad a < b. \end{aligned}\right\} \tag{12}$$

We introduce the *horizon wave operators* $W_0^\pm$

$$F_0^\pm \in \mathcal{D}_0^\pm, \quad W_0^\pm\, F_0 = s - \lim_{t \to \pm\infty} U(-t)\, \mathcal{I}_0\, U_0(t)\, F_0^\pm \quad \text{in } \mathcal{H}. \tag{13}$$

At the spatial infinity, the term $-2\, M\, m^2\, r^{-1}$ in (1) is a long range perturbation of the Klein-Gordon equation on the Minkowski space-time:

$$\frac{\partial^2}{\partial t^2}\, \Psi_\infty - \Delta_{\mathbb{R}^3_x}\, \Psi_\infty + m^2\, \Psi_\infty = 0, \qquad \Delta_{\mathbb{R}^3_x} = \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2}\,. \tag{14}$$

Following [10] we compare the solutions of (1) at infinity with fields in the Minkowski space-time, governed by a Dollard-modified free dynamics. Let $\mathcal{H}_\infty$ be the Hilbert space

$$\mathcal{H}_\infty = \{ {}^t(f,\, g);\ f,\ \nabla_{\mathbb{R}^3_x} f,\ g \in L^2(\mathbb{R}^3_x,\, dx) \}, \tag{15}$$

with norm

$$\| {}^t(f,\, g) \|^2_{\mathcal{H}_\infty} = \int_{\mathbb{R}^3_x} \{ |\, \nabla_{\mathbb{R}^3_x} f(x)\, |^2 + m^2\, |\, f(x)\, |^2 + |\, g(x)\, |^2 \}\, dx, \tag{16}$$

and $U_\infty^D(t)$ the unitary propagator given for $t \neq 0$ by

$$U_\infty^D(t) = \begin{bmatrix} \cos(t B_\infty + D\log t) & B_\infty^{-1}\sin(t B_\infty + D\log t) \\ -B_\infty \sin(t B_\infty + D\log t) & \cos(t B_\infty + D\log t) \end{bmatrix}, \quad (17)$$

where

$$\left.\begin{array}{c} B_\infty = (-\Delta_{\mathbb{R}_x^3} + m^2)^{1/2}, \quad D = -M\, m^2 \,|\nabla_{\mathbb{R}_x^3}|^{-1}, \\ \log t = \operatorname{sign}(t) \log|t|. \end{array}\right\} \quad (18)$$

Here, $|\nabla_{R_x^3}|^{-1}$ means for $f \in C_0^\infty(R_x^3)$:

$$(|\nabla_{\mathbb{R}_x^3}|^{-1} f)(x) = \mathcal{F}^{-1}[|\xi|^{-1}\mathcal{F}(f)], \quad (19)$$

where $\mathcal{F}$ denotes the Fourier transform

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}_x^3} e^{-ix\cdot\xi} f(x)\,dx,$$

$$\mathcal{F}^{-1}(g)(x) = (2\pi)^{-3}\int_{\mathbb{R}_\xi^3} e^{-ix\cdot\xi} g(\xi)\,d\xi.$$

We also consider the non modified unitary group $U_\infty(t)$ associated with (14)

$$U_\infty(t) = \begin{bmatrix} \cos(t B_\infty) & B_\infty^{-1}\sin(t B_\infty) \\ -B_\infty \sin(t B_\infty) & \cos(t B_\infty) \end{bmatrix}. \quad (20)$$

We introduce the dense subspace $\mathcal{D}_\infty$ of regular wave packets in $\mathcal{H}_\infty$

$$\mathcal{D}_\infty = \{{}^t(f_1, f_2) \in \mathcal{H}_\infty;\ \mathcal{F} f_i \in C_0^\infty(\mathbb{R}_\xi^3 \backslash \{0\})\}. \quad (21)$$

Now we choose a cut-off function χ_∞ such that

$$\left.\begin{array}{c} \chi_\infty \in C^\infty(\mathbb{R}_{r_*}), \quad r_* < c \Rightarrow \chi_\infty(r_*) = 0, \\ r_* > d \Rightarrow \chi_\infty(r_*) = 1, \quad 0 < c < d, \end{array}\right\} \quad (22)$$

and we define a (bounded) identification operator $\mathcal{I}_\infty$ between $\mathcal{H}_\infty$ and $\mathcal{H}$ by identifying the Euclidean norm of $x \in \mathbb{R}^3$ and $r_* \geq 0$, (which avoids artificial long range interactions) and by putting

$$\left.\begin{array}{c} F_\infty \in \mathcal{H}_\infty, \\ 0 \leq r_* \Rightarrow (\mathcal{I}_\infty F_\infty)(r_*, \omega) = \chi_\infty(r_*) F_\infty(x = r_*\omega), \\ r_* \leq 0 \Rightarrow (\mathcal{I}_\infty F_\infty)(r_*, \omega) = 0. \end{array}\right\} \quad (23)$$

We consider the time dependent modified wave operators $W_\infty^\pm$

$$F_\infty \in \mathcal{D}_\infty, \quad W_\infty^\pm F_\infty = s\text{-}\lim_{t\to\pm\infty} U(-t)\mathcal{I}_\infty U_\infty^D(t) F_\infty \quad \text{in } \mathcal{H}. \quad (24)$$

The main result is the following

THEOREM 1. – *For all* $F_0^\pm \in \mathcal{D}_0^\pm$, $F_\infty \in \mathcal{D}_\infty$, *the limits* $W_0^\pm F_0^\pm$ *and* $W_\infty^\pm F_\infty$ *exist in* $\mathcal{H}$ *and are independent of* χ_0 *and* χ_∞ *satisfying* (4), (8). *Furthermore we have:*

$$\| W_0^\pm F_0^\pm \|_{\mathcal{H}} = \| F_0^\pm \|_{\mathcal{H}_0^\pm}, \qquad \| W_\infty^\pm F_\infty \|_{\mathcal{H}} = \| F_\infty \|_{\mathcal{H}_\infty}, \tag{25}$$

$$\left.\begin{array}{c} \forall t \in \mathbb{R}, \quad U(t)\, W_0^\pm F_0^\pm = W_0^\pm U_0(t)\, F_0^\pm, \\ U(t)\, W_0^\pm F_\infty = W_0^\pm U_\infty(t)\, F_\infty, \end{array}\right\} \tag{26}$$

and the strong asymptotic completeness holds, i.e. *the range of* $W_0^\pm \oplus W_\infty^\pm$ *is dense in* $\mathcal{H}$.

Therefore these wave operators can be extended by continuity to $\mathcal{H}_0^\pm \oplus \mathcal{H}_\infty$ and we can define the Scattering Operator S:

$$S = (W_0^+ \oplus W_\infty^+)^{-1} \cdot (W_0^- \oplus W_\infty^-). \tag{27}$$

From the previous result we immediately obtain

THEOREM 2. – *The Scattering Operator* S *is an isometry from* $\mathcal{H}_0^- \oplus \mathcal{H}_\infty$ *onto* $\mathcal{H}_0^+ \oplus \mathcal{H}_\infty$ *and satisfies the intertwining property*

$$\forall t \in \mathbb{R}, \quad [U_0(t) \oplus U_\infty(t)] \cdot S = S \cdot [U_0(t) \oplus U_\infty(t)]. \tag{28}$$

We sketch the ideas of the proof. First we consider the square roots of the second order hyperbolic equation (1), (6), (14) and we construct the wave operators associated with $[h]^{1/2}$, $[h_0]^{1/2}$, B_∞. For this purpose we decouple the study near the horizon from the study at infinity by comparing (1) with the Dirichlet problem for the same equation with homogeneous boundary condition $\Psi = 0$ on the sphere $\{r_* = 0\}$; this is a short range perturbation of the initial problem and since $(1 - 2\,M\,r^{-1})$ is exponentially decreasing as $r_* \to -\infty$, (6) is a short range perturbation of (1) on $]-\infty, 0[_{r_*}$, the existence and asymptotic completeness of the wave operators at the horizon $r = 2\,M$ follow from the Kato-Birman theorem and the invariance principle. As for the problem at infinity, it consists in studying the perturbation of the Klein-Gordon equation (15) by the long range potential $-2\,M\,m^2\,|\,x\,|^{-1}$. We prove the existence and completeness of the wave operators at infinity using the results of Kitada ([16] [17]) about Schrödinger operators and the invariance principle for long range potentials. To prove the *strong* asymptotic completeness we decompose (1) on the spherical harmonics basis and thanks to the analyticity of the gravitational interaction (*see* [4]) the spectrum of

$$-\partial_{r_*}^2 + (1 - 2\,M\,r^{-1})\,[l\,(l+1)\,r^{-2} + 2\,M\,r^{-3} + m^2]$$

is absolutely continuous on $L^2(R_{r_*})$. Making use of the usual two Hilbert spaces scattering techniques, the previous results enable us to conclude to the existence and asymptotic completeness of the wave operators associated to the second order equations (1), (6), (15).

We shall often use the following criterion selfadjointness:

LEMMA. – *Let* $]a, b[$ *be an open real interval,* $\rho \in C^2([a, b])$, *so that* $\frac{d^2}{dx^2}\rho \in L^\infty(]a, b[_x, dx)$, $\rho(x) \geq \alpha > 0$, *and* $A, B \in L^\infty(]a, b[_x, dx)$. *Then the operator*

$$T^0_{A,B} f = -\frac{1}{\rho^2(x)}\frac{\partial}{\partial x}\left[\rho^2(x)\frac{\partial}{\partial x} f\right] - A(x)\Delta_{S^2} f + B(x) f$$

is selfadjoint on the domain

$$D(T^0_{A,B}) = \{f \in L^2(]a, b[_x \times S^2_\omega, \rho^2(x)\, dx\, d\omega;$$
$$T^0_{A,B} f \in L^2(]a, b[_x \times S^2_\omega, \rho^2(x)\, dx\, d\omega),\ f(a, \cdot) = f(b, \cdot) = 0$$

for a, b finite$\}$.

Proof. – We put $T' = \rho T^\rho_{A,B}\rho^{-1}$ and prove the selfadjointness of T' on

$$D(T') = \{f \in L^2(]a, b[_x \times S^2_\omega, dx\, d\omega);\ T' f \in L^2(]a, b[_x \times S^2_\omega, dx\, d\omega),$$
$$f(a, \cdot) = f(b, \cdot) = 0 \text{ for } a, b \text{ finite}\}.$$

We decompose T' on the spherical harmonics basis $Y_{l,m}$ of $L^2(S^2_\omega, d\omega)$:

$$L^2(]a, b[_x \times S^2_\omega, dx\, d\omega) = \bigoplus_{l=0}^{\infty}\bigoplus_{m=-l}^{m=l} L^2(]a, b[_x, dx) \otimes Y_{l,m}.$$

$$T' = \bigoplus_{l=0}^{\infty}\bigoplus_{m=-l}^{m=l} T_{l,m} \otimes 1,$$

$$T_{l,m} = -\partial_x^2 + l(l+1)A(x) + B(x) + \rho^{-2}(x)\frac{d^2}{dx^2}\rho(x).$$

Since A and $B + \rho^{-2}\rho''$ are bounded, by the Kato-Rellich theorem ([19], th. X.12) $T_{l,m}$ is selfadjoint on

$$D(T_{l,m}) = \{f_{l,m} \in L^2(]a, b[_x, dx);\ \partial_x^2 f_{l,m} \in L^2(]a, b[_x, dx),$$
$$f_{l,m}(a) = f_{l,m}(b) = 0 \text{ for } a, b \text{ finite}\}.$$

Here the boundary condition is well defined thanks to the regularity of $f_{l,m}$ which justifies as well the definition of $D(T)$. Therefore T' is selfadjoint on

$$\{\sum f_{l,m};\, f_{l,m} \in D(T_{l,m}),\, \sum \| T_{l,m} f_{l,m} \|^2_{L^2(]a,\, b[_x,\, dx)} < \infty\} = D(T').$$

Q.E.D.

If A_i are selfadjoint operators on Hilbert spaces $\mathcal{H}_i$ and $\mathcal{I}$ an operator between $\mathcal{H}_1$ and $\mathcal{H}_2$ we denote $\Omega^{\pm}(A_1,\, A_2;\, \mathcal{I})$ the wave operator defined by the strong limit

$$\left.\begin{array}{c} f_1 \in \mathcal{H}_1, \\ \Omega^{\pm}(A_2,\, A_1;\, \mathcal{J})\, f_1 = s\text{-}\lim_{t\to\pm\infty} e^{itA_2}\, \mathcal{I}\, e^{-itA_1}\, P_{\mathrm{ac}}(A_1)\, f_1 \quad \text{in } \mathcal{H}_2 \end{array}\right\} \tag{29}$$

where $P_{\mathrm{ac}}(A)$ is the projection onto the absolutely continuous subspace of A. When $\mathcal{H}_1 = \mathcal{H}_2$, $\mathcal{I} = \mathrm{Id}$, we simply write $\Omega^{\pm}(A_2,\, A_1)$. It is well known that

$$e^{itA_2}\, \Omega^{\pm}(A_2,\, A_1;\, \mathcal{I}) = \Omega^{\pm}(A_2,\, A_1;\, \mathcal{I})\, e^{itA_1}. \tag{30}$$

PROPOSITION 1. – *Let ρ, ρ', A, B, A', B' be as in lemma 1. We assume that for some $\delta > 1/2$, we have*

$$A - A',\; B - B' + \rho^{-2}\,\frac{d^2}{dx^2}\,\rho - \rho'^{-2}\,\frac{d^2}{dx^2}\,\rho' \in L^2(]\,a,\, b\,[_x,\, (1 + x^2)^{\delta}\, dx),$$

Then $\Omega^{\pm}(T^{\rho'}_{A',\,B'},\, T^{\rho}_{A,\,B};\, \rho'\,\rho^{-1})$ exist and are complete. If A, A', $B + \rho^{-2}\,\frac{d^2}{dx^2}\,\rho$, $B' + \rho'^{-2}\,\frac{d^2}{dx^2}\,\rho'$ are non negative, then $\Omega^{\pm}([T^{\rho'}_{A',\,B'}]^{1/2}, [T^{\rho}_{A,\,B}]^{1/2};\, \rho'\,\rho^{-1})$exist and are complete.

Proof. – We prove that $\Omega^{\pm}(\rho'\,T^{\rho'}_{A',\,B'}\,\rho'^{-1},\, \rho\,T^{\rho}_{A,\,B}\,\rho^{-1})$ exist and are complete on $L^2(]\,a,\, b\,[_x \times S^2_{\omega},\, dx\, d\omega)$. Taking advantage of the spherical invariance, it is sufficient to establish this result on each constant angular momentum subspace. We write

$$\begin{aligned} &(\rho\,T^{\rho}_{A,\,B}\,\rho^{-1} + i)^{-1} - (\rho'\,T^{\rho'}_{A',\,B'}\,\rho'^{-1} + i)^{-1} \\ &\quad = \bigoplus_{l=0}^{\infty}\, \bigoplus_{m=-1}^{m=l} \mathcal{T}_{l,\,m} \otimes 1 \quad \text{on } \bigoplus_{l=0}^{\infty}\, \bigoplus_{m=-l}^{m=l} L^2(]\,a,\, b\,[_x,\, dx) \otimes Y_{l,\,m}. \end{aligned}$$

If we show that $\mathcal{T}_{l,\,m}$ is trace class, the Kuroda-Birman theorem ([19] th. XI.9) yields the required conclusion and the last part of the proposition is a straightforward consequence of the invariance principle ([19], th. XI.11) since

$$[\rho\, T\, \rho^{-1}]^{1/2} = \rho\, [T]^{1/2}\, \rho^{-1}. \tag{31}$$

We calculate

$$\mathcal{T}_{l,\,m} = [-\partial_x^2 + W(x) + i]^{-1}\, \mathcal{S}_{l,\,m}(x)\, [-\partial_x^2 + W'(x) + i]^{-1},$$

with

$$\begin{aligned}\mathcal{S}_{l,\,m} &= l\,(l+1)\,(A - A') + (B - B') \\ &\quad + \rho^{-2}\,\frac{d^2}{dx^2}\,\rho - \rho'^{-2}\,\frac{d^2}{dx^2}\,\rho' \in L^2\,(]\,a,\, b\,[_x,\, (1+x^2)^\delta\, dx),\end{aligned} \tag{32}$$

$$W = l\,(l+1)\,A + B + \rho^{-2}\,\frac{d^2}{dx^2}\,\rho, \quad W' = l\,(l+1)\,A' + B' + \rho'^{-2}\,\frac{d^2}{dx^2}\,\rho'.$$

We introduce the selfadjoint operators Δ_j

$$\Delta_j\, f = -\,\frac{d^2}{dx^2}\, f, \tag{33}$$

$$\left.\begin{aligned} D\,(\Delta_0) = H^2\,(\mathbb{R}), \quad D\,(\Delta_1) = (H_0^1 \cap H^2)\,(]\,a,\, b\,[), \\ D\,(\Delta_2) = (H_0^1 \cap H^2)\,(\mathbb{R}\backslash[\,a,\, b]), \end{aligned}\right\} \tag{34}$$

where H^2 and H_0^1 are the standard Sobolev spaces

$$\left.\begin{aligned} &H^2\,(]\,a,\, b\,[) = \{f \in L^2\,(]\,a,\, b\,[);\ f'' \in L^2\,(]\,a,\, b\,[)\}, \\ &H_0^1\,(]\,a,\, b\,[) \\ &\quad = \{f \in L^2\,(]\,a,\, b\,[);\ f' \in L^2\,(]\,a,\, b\,[),\ f\,(a) = f\,(b) = 0\}. \end{aligned}\right\} \tag{35}$$

On $L^2\,(\mathbb{R}) = L^2\,(]\,a,\, b\,[) \oplus L^2\,(\mathbb{R}\backslash[a,\, b])$ we have:

$$\begin{aligned}\mathcal{S}_{l,\,m}\,(\Delta_1 + i)^{-1} \oplus 0 &= (\mathcal{S}_{l,\,m} \oplus 0)\,(\Delta_0 + i)^{-1} \\ &\quad + (\mathcal{S}_{l,\,m} \oplus 0)\,[\,(\Delta_1 \oplus \Delta_2 + i)^{-1} - (\Delta_0 + i)^{-1}].\end{aligned}$$

On the one hand $(\mathcal{S}_{l,\,m} \oplus 0)\,(\Delta_0 + i)^{-1}$ is trace class according to (32) and theorem XI.21 in [19], on the other hand $(\Delta_1 \oplus \Delta_2 + i)^{-1} - (\Delta_0 + i)^{-1}$ is of finite rank; hence $\mathcal{S}_{l,\,m}\,(\Delta_1 + i)^{-1}$ is trace class. Now we write

$$\mathcal{T}_{l,\,m} = [-\partial_x^2 + W + i]^{-1}\,\{\mathcal{S}_{l,\,m}\,(\Delta_1 + i)^{-1}\}\,\{(\Delta_1 + i)\,[-\partial_x^2 + W' + i]^{-1}\}.$$

Therefore $\mathcal{T}_{l,\,m} =$ *bounded* $\times$ *trace class* $\times$ *bounded* $=$ *trace class.*

Q.E.D.

Proof of Theorem 1. – 1. First we decouple the problem near the horizon from the problem at the infinity by a Dirichlet decoupling. We consider the operator

$$\left.\begin{aligned} h_1 &= h_1^0 \oplus h_1^\infty, \\ h_1^{0,\infty} f &= \left[-\frac{1}{r^2}\frac{\partial}{\partial r_*} r^2 \frac{\partial}{\partial r_*} + \left(1 - \frac{2M}{r}\right)\left(-\frac{1}{r^2}\Delta_{S^2} + m^2\right)\right] f, \end{aligned}\right\} \quad (36)$$

which is selfadjoint on $L^2(\mathbb{R}_{r_*} \times S^2_\omega, r^2\, dr_*\, d\omega)$ with domain $D(h_1^0) \oplus D(h_1^\infty)$:

$$\begin{aligned} D(h_1^0) = \{ & f \in L^2(\mathbb{R}^-_{r_*} \times S^2_\omega, r^2\, dr_*\, d\omega); \\ & h_1^0 f \in L^2(\mathbb{R}^-_{r_*} \times S^2_\omega, r^2\, dr_*\, d\omega),\ f(0, \omega) = 0\}, \end{aligned} \quad (37)$$

$$\begin{aligned} D(h_1^\infty) = \{ & f \in L^2(\mathbb{R}^+_{r_*} \times S^2_\omega, r^2\, dr_*\, d\omega); \\ & h_1^\infty f \in L^2(\mathbb{R}^+_{r_*} \times S^2_\omega, r^2\, dr_*\, d\omega),\ f(0, \omega) = 0\}. \end{aligned} \quad (38)$$

h_1 is a short range perturbation of operator h given by (2) with domain

$$\begin{aligned} D(h) = \{ & f \in L^2(\mathbb{R}_{r_*} \times S^2_\omega, r^2\, dr_*\, d\omega); \\ & hf \in L^2(\mathbb{R}_{r_*} \times S^2_\omega, r^2\, dr_*\, d\omega)\} \end{aligned} \quad (39)$$

in the sense that on each space of constant angular momentum the operator

$$(h+i)^{-1} - (h_1 + i)^{-1}$$

is of finite rank and thus, therefore trace class. Hence the Birman-Kuroda theorem and the invariance principle give that the wave operators

$$\Omega^\pm([h]^{1/2}, [h_1^0]^{1/2} \oplus [h_1^\infty]^{1/2}) = \Omega^\pm(h, h_1^0 \oplus h_1^\infty) \quad (40)$$

exist and are complete.

2. The study near the horizon is easy thanks to the short range of the gravitational interaction. We compare h_1^0 to the free hamiltonian h_0^0 defined by

$$\left.\begin{aligned} h_0^0 &= -\frac{\partial^2}{\partial r_*^2}, \\ D(h_0^0) &= \{ f \in L^2(\mathbb{R}^-_{r_*} \times S^2_\omega, dr_*\, d\omega); \\ &\qquad h_0^0 f \in L^2(\mathbb{R}^-_{r_*} \times S^2_\omega, dr_*\, d\omega)\}. \end{aligned}\right\} \quad (41)$$

We denote I_0 the operator

$$I_0 : \quad f(r_*, \omega) \to (I_0 f)(r_*, \omega) = r^{-1} f(r_*, \omega). \tag{42}$$

Since $\left(1 - \frac{2M}{r}\right)$ is exponentially decreasing as $r_* \to -\infty$, we can apply proposition 1 and we conclude that the wave operators

$$\Omega^{\pm}([h_1^0]^{1/2}, [h_0^0]^{1/2}; I_0) \tag{43}$$

exist and are complete.

3. For the problem at infinity we use the spherical invariance again. To get rid of the irrelevant singularity which appears at $r_* = 0$, we use Deift and Simon's Dirichlet decoupling [8]. We introduce the selfadjoint operator h_2^{∞} on $L^2(\mathbb{R}_{r_*}^+ \times S_\omega^2, r^2 \, dr_* \, d\omega)$

$$\left.\begin{array}{l} h_2^{\infty} = k_1 \oplus k_2, \\ k_j f = \left[-\frac{1}{r^2} \frac{\partial}{\partial r_*} r^2 \frac{\partial}{\partial r_*} + \left(1 - \frac{2M}{r}\right) \left(-\frac{1}{r^2} \Delta_{S^2} + m^2 \right) \right] f, \end{array}\right\} \tag{44}$$

$$\begin{aligned} D(k_1) = \{ f \in L^2(]0, 1[_{r_*} \times S_\omega^2, r^2 \, dr_* \, d\omega); \\ k_1 f \in L^2(]0, 1[_{r_*} \times S_\omega^2, r^2 \, dr_* \, d\omega), \; f(0, \omega) = f(1, \omega) = 0 \}, \end{aligned} \tag{45}$$

$$\begin{aligned} D(k_2) = \{ f \in L^2(]1, \infty[_{r_*} \times S_\omega^2, r^2 \, dr_* \, d\omega); \\ k_2 f \in L^2(]1, \infty[_{r_*} \times S_\omega^2, r^2 \, dr_* \, d\omega), \; f(1, \omega) = 0 \}, \end{aligned} \tag{46}$$

Again, on each space of constant angular momentum, the operator

$$(h_1^{\infty} + i)^{-1} - (h_2^{\infty} + i)^{-1}$$

is of finite rank and thus trace class. Hence the Birman-Kuroda theorem and the invariance principle imply that the wave operators

$$\Omega^{\pm}([h_1^{\infty}]^{1/2}, [k_1]^{1/2} \oplus [k_2]^{1/2}) \tag{47}$$

exist and are complete. Moreover we note that k_1 is discrete, which gives:

$$\Omega^{\pm}([k_1]^{1/2} \oplus [k_2]^{1/2}, 0 \oplus [k_2]^{1/2}) = P_{\mathrm{ac}}(0 \oplus [k_2]^{1/2}), \tag{48}$$

and therefore

$$\Omega^{\pm}([h_1^{\infty}]^{1/2}, 0 \oplus [k_2]^{1/2}) = \Omega^{\pm}(h_1^{\infty}, 0 \oplus k_2) \tag{49}$$

exist and are complete.

Now we define

$$\left.\begin{aligned} h_3^\infty &= k_1' \oplus k_2', \\ k_j' f &= \left[-\frac{1}{r_*^2} \frac{\partial}{\partial r_*} r_*^2 \frac{\partial}{\partial r_*} - \frac{1}{r_*^2} \Delta_{S^2} + \left(1 - \frac{2M}{r}\right) m^2 \right] f, \end{aligned}\right\} \quad (50)$$

$$\begin{aligned} D(k_1') = \{ & f \in L^2 (]0, 1[_{r_*} \times S_\omega^2, r_*^2 \, dr_* \, d\omega); \\ & k_1' f \in L^2 (]0, 1[_{r_*} \times S_\omega^2, r_*^2 \, dr_* \, d\omega), \; f(1, \omega) = 0 \}, \end{aligned} \quad (51)$$

$$\begin{aligned} D(k_2') = \{ & f \in L^2 (]1, \infty[_{r_*} \times S_\omega^2, r_*^2 \, dr_* \, d\omega); \\ & k_2' f \in L^2 (]1, \infty[_{r_*} \times S_\omega^2, r_*^2 \, dr_* \, d\omega), \; f(1, \omega) = 0 \}. \end{aligned} \quad (52)$$

$$I_* : \quad f(r_*, \omega) \to (I_* f)(r_*, \omega) = r_* f(r_*, \omega), \quad (53)$$

$$I : \quad f(r_*, \omega) \to (I f)(r_*, \omega) = r f(r_*, \omega), \quad (i.e. \, I = I_0^{-1}). \quad (54)$$

These operators are selfadjoint because k_j' is a smooth perturbation of the Laplacian on $\mathbb{R}_x^3$ (in spherical coordinates, $x = r_* \omega$) with Dirichlet boundary condition on the sphere $|x| = 1$.

On the one hand, k_1' is discrete, on the other hand we apply proposition 1 for k_2 and k_2'. Then we conclude that the wave operators

$$\Omega^\pm (0 \oplus [k_2]^{1/2}, \quad [h_3^\infty]^{1/2}; \, I_* \cdot I^{-1}) = \Omega^\pm (0 \oplus k_2, \, h_3^\infty; \, I_* \cdot I^{-1}) \quad (55)$$

exist and are complete.

Al last we consider the operator h^∞ given by

$$h^\infty f = \left[-\frac{1}{r_*^2} \frac{\partial}{\partial r_*} r_*^2 \frac{\partial}{\partial r_*} - \frac{1}{r_*^2} \Delta_{S^2} + \left(1 - \frac{2M}{r}\right) m^2 \right] f, \quad (56)$$

$$\begin{aligned} D(h^\infty) = \{ & f \in L^2 (\mathbb{R}^+{}_{r_*} \times S_\omega^2, r_*^2 \, dr_* \, d\omega); \\ & h^\infty f \in L^2 (\mathbb{R}_{r_*}^+ \times S_\omega^2, r_*^2 \, dr_* \, d\omega) \}. \end{aligned} \quad (57)$$

Since on each space of constant angular momentum the operator

$$(h^\infty + i)^{-1} - (h_3^\infty + i)^{-1}$$

is of finite rank and therefore trace class the wave operators

$$\Omega^\pm ([h_3^\infty]^{1/2}, [h^\infty]^{1/2}) = \Omega^\pm (h_3^\infty, h^\infty) \quad (58)$$

exist and are complete.

4. We note that h^∞ is a smooth long range perturbation of the Klein-Gordon equation on $\mathbb{R}_x^3$ which we identify with $\mathbb{R}_{r_*}^+ \times S_\omega^2$ by putting

$$\mathcal{I}: \quad f(x) \to (\mathcal{I} f)(r_*, \omega) = f(x), \; r_* = |x|, \quad \omega = |x|^{-1} x, \tag{59}$$

$$h^\infty = \mathcal{I}[B_\infty^2 + 2V(x) + 2V_s(x)], \tag{60}$$

$$\left.\begin{aligned} V(x) &= -M m^2 |x|^{-1} \theta_\infty(x), \\ V_s(x) &= -M m^2 (r^{-1} - |x|^{-1} \theta_\infty(x)], \end{aligned}\right\} \tag{61}$$

$$|x| = r + 2M \operatorname{Log}(r - 2M), \tag{62}$$

where $\theta_\infty \in C^3(\mathbb{R}_x^3)$ and satisfies:

$$|x| \le 1/2 \Rightarrow \theta_\infty(x) = 0, \quad |x| \ge 1 \Rightarrow \theta_\infty(x) = 1.$$

We apply the results by Kitada ([15], [16]) to

$$\left.\begin{aligned} H_1 = -\frac{1}{2}\Delta_{\mathbb{R}_x^3}, \quad H_2 = H_1 + V(x) + V_s(x), \\ \varphi(t) = (t + m^2)^{1/2}. \end{aligned}\right\} \tag{63}$$

Then theorems 1.5 and 1.6 in [17], and lemma 2.6 in [16] assure that the strong limits

$$W_\varphi^{\pm \mathrm{as}} = \lim_{t \to \pm\infty} e^{it\varphi(H_2)} Q_\varphi^{\mathrm{as}}(t) \tag{64}$$

where

$$Q_\varphi^{\mathrm{as}}(t) = e^{-it\varphi(H_1) - iX(t\varphi'(H_1))}, \tag{65}$$

$$X(t\varphi'(H_1)) = \mathcal{F}^{-1}\left[\int_0^{t\varphi'(|\xi|^2/2)} V(s\xi)\, ds\right] \mathcal{F}, \tag{66}$$

exist with initial domain $L^2(\mathbb{R}^3)$ and are complete, *i.e.*

$$\operatorname{Ran} W_\varphi^{\pm \mathrm{as}} = P_{\mathrm{ac}}(H_2) L^2(\mathbb{R}^3). \tag{67}$$

A straightforward calculation gives

$$X(t\varphi'(H_1)) = D \log t + \mathcal{F}^{-1}[P(|\xi|)] \mathcal{F} \tag{68}$$

where D is defined by (18) and P is a real valued function. We deduce from these results that the strong limits

$$\Omega_D^\pm([h^\infty]^{1/2}, B_\infty; \mathcal{I}) f \equiv s\text{-}\lim_{t \to \pm\infty} e^{it[h^\infty]^{1/2}} \mathcal{I} e^{-i(tB_\infty + D\log t)} f \tag{69}$$

exist for all f in $L^2(\mathbb{R}^3_x, dx)$ and

$$\text{Range}\, \Omega_D^{\pm}([h^\infty]^{1/2}, B_\infty; \mathcal{I}) = P_{\text{ac}}(h^\infty) L^2(\mathbb{R}^+_{r_*} \times S^2_\omega, r_*^2\, dr_*\, d\omega), \quad (70)$$

$$e^{it[h^\infty]^{1/2}}\, \Omega_D^{\pm}([h^\infty]^{1/2}, B_\infty; \mathcal{I}) = \Omega_D^{\pm}([h^\infty]^{1/2}, B_\infty; \mathcal{I})\, e^{itB_\infty} \quad (71)$$

Now we introduce

$$I_\infty = I_* \cdot I^{-1} \cdot \mathcal{I}, \quad (72)$$

$$\Omega_D^{\pm}([h_1^\infty]^{1/2}, B_\infty; I_\infty) f \equiv s\text{-}\lim_{t\to\pm\infty} e^{it[h_1^\infty]^{1/2}} I_\infty\, e^{-i(tB_\infty + D\log t)} f. \quad (73)$$

The chain rule applied to (49), (55), (58), (69) assures that $\Omega_D^{\pm}([h_1^\infty]^{1/2}, B_\infty; I_\infty)$ is an isometry from $L^2(\mathbb{R}^3_x, dx)$ onto $P_{\text{ac}}(h_1^\infty) L^2(\mathbb{R}^+_{r_*} \times S^2_\omega, r^2\, dr_*\, d\omega)$.

5. Eventually we introduce the operators

$$\Omega_0^{\pm} f_0 \equiv s\text{-}\lim_{t\to\pm\infty} e^{it[h]^{1/2}}\, Y_0\, I_0\, e^{-it[h_0^0]^{1/2}} f_0, \quad (74)$$

$$\Omega_D^{\pm} f_\infty \equiv s\text{-}\lim_{t\to\pm\infty} e^{it[h]^{1/2}}\, Y_\infty\, I_\infty\, e^{-i(tB_\infty + D\log t)} f_\infty, \quad (75)$$

and

$$\left.\begin{array}{l} Y_0: \quad f \in L^1_{\text{loc}}(\mathbb{R}^-_{r_*} \times S^2_\omega) \to Y_0 f = f \quad \text{on } \mathbb{R}^-_{r_*} \times S^2_\omega \\ \qquad Y_0 f = 0 \quad \text{on } \mathbb{R}^+_{r_*} \times S^2_\omega, \end{array}\right\} \quad (76)$$

and

$$\left.\begin{array}{l} Y_\infty: \quad f \in L^1_{\text{loc}}(\mathbb{R}^+_{r_*} \times S^2_\omega) \to Y_\infty f = f \quad \text{on } \mathbb{R}^+_{r_*} \times S^2_\omega \\ \qquad Y_0 f = 0 \quad \text{on } \mathbb{R}^-_{r_*} \times S^2_\omega. \end{array}\right\} \quad (77)$$

By (40), (43), (73), the previous study shows that $\Omega_0^{\pm} \oplus \Omega_D^{\pm}$ is unitary from $L^2(\mathbb{R}^-_{r_*} \times S^2_\omega, dr_*\, d\omega) \oplus L^2(\mathbb{R}^3_x, dx)$ onto $P_{\text{ac}}(h) L^2(\mathbb{R}_{r_*} \times S^2_\omega, r^2\, dr_*\, d\omega)$. Moreover

$$e^{it[h]^{1/2}}\, \Omega_0^{\pm} = \Omega_0^{\pm}\, e^{it[h_0^0]^{1/2}}, \qquad e^{it[h]^{1/2}}\, \Omega_D^{\pm} = \Omega_D^{\pm}\, e^{itB_\infty}. \quad (78)$$

To prove the strong asymptotic completeness we write

$$I_0^{-1} \cdot h \cdot I_0 = \bigoplus_{l=0}^{\infty} \bigoplus_{m=-l}^{m=l} h_{l,m} \otimes 1,$$

$$h_{l,m} = -\frac{\partial^2}{\partial r_*^2} + \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3} + m^2\right).$$

On the one hand $V_l(r_*) \equiv \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3} + m^2\right)$ being non negative, the eigenvalues of $h_{l,m}$ have to be positive. Let $\lambda^2 > 0$ be such an eigenvalue. Since V_l is integrable at $-\infty$, any solution f of

$$h_{l,m} f = \lambda^2 f \tag{79}$$

is a combination of the functions $f^{\pm}(r_*, \lambda)$ solutions of

$$f^{\pm}(r_*, \lambda) = e^{\pm i\lambda r_*} + \int_{-\infty}^{r_*} \frac{\sin\lambda(r_* - y)}{\lambda} V_l(y) f^{\pm}(y, \lambda)\, dy.$$

Since $f^{\pm}$ are linearly independent there is no f in $L^2(\mathbb{R}, dr_*)$ solution of (79): thus the point spectrum of $h_{l,m}$ is empty. On the other hand, we have proved in [4] that $r_* \to r$ extends as an analytic function on $\mathbb{C}\backslash i\mathbb{R}$; hence the Aguilar-Combes theorem [1] implies $h_{l,m}$ has no singular spectrum (we could as well use corollary II.5 of R. Carmona [7]). Therefore we conclude that $\Omega_0^{\pm} \oplus \Omega_D^{\pm}$ is strongly complete on each constant angular momentum subspace and finally:

$$\operatorname{Ran}\Omega_0^{\pm} \oplus \operatorname{Ran}\Omega_D^{\pm} = L^2(\mathbb{R}_{r_*} \times S_\omega^2, r^2\, dr_*\, d\omega). \tag{80}$$

6. We return to the second order hyperbolic equations using the method of the two Hilbert spaces scattering theory. We define

$$\left.\begin{aligned} T = \frac{1}{\sqrt{2}}\begin{bmatrix} [h]^{1/2} & i \\ [h]^{1/2} & -i \end{bmatrix}, \quad T_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} [h_0^0]^{1/2} & i \\ [h_0^0]^{1/2} & -i \end{bmatrix}, \\ T_\infty = \frac{1}{\sqrt{2}}\begin{bmatrix} B_\infty & i \\ B_\infty & -i \end{bmatrix}. \end{aligned}\right\} \tag{81}$$

T is an isometry from $\mathcal{H}$ onto $L^2(\mathbb{R}_{r_*} \times S_\omega^2, r^2\, dr_*\, d\omega) \times L^2(\mathbb{R}_{r_*} \times S_\omega^2, r^2\, dr_*\, d\omega)$, T_∞ is an isometry from $\mathcal{H}_\infty = H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ onto $L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, and T_0 is an isometry from the Hilbert space $\mathcal{H}_0^0$ completion of

$$\mathcal{D}_0^0 = C_0^\infty(]-\infty, 0[_{r_*} \times S_\omega^2) \times C_0^\infty(]-\infty, 0[_{r_*} \times S_\omega^2), \tag{82}$$

for the norm

$$\| {}^t(f, g) \|_{\mathcal{H}_0^0}^2 = \int_{-\infty}^0 \left\{ \left| \frac{\partial}{\partial r_*} f \right|^2 + |g|^2 \right\} dr_*\, d\omega, \tag{83}$$

onto $L^2(\mathbb{R}_{r_*}^- \times S_\omega^2, dr_*\, d\omega) \times L^2(\mathbb{R}_{r_*}^- \times S_\omega^2, dr_*\, d\omega)$. We denote by $U_0^0(t)$ the unitary group on $\mathcal{H}_0^0$ given by

$$U_0^0(t) = e^{-itH_0^0}, \quad H_0^0 = \frac{1}{i}\begin{bmatrix} 0 & 1 \\ h_0^0 & 0 \end{bmatrix}. \tag{84}$$

Now we introduce the wave operators

$$\hat{W}_0^{\pm} = T^{-1}\,(\Omega_0^{\mp} \oplus \Omega_0^{\pm})\,T_0, \quad \hat{W}_\infty^{\pm} = T^{-1}\,(\Omega_D^{\mp} \oplus \Omega_D^{\pm})\,T_\infty. \tag{85}$$

$\hat{W}_0^{\pm}$ (resp. $\hat{W}_\infty^{\pm}$) are isometries from $\mathcal{H}_0^0$ (resp. $\mathcal{H})_\infty$ to $\mathcal{H}$ and thanks to (80), (78) the completeness and the intertwining property hold:

$$\operatorname{Ran}(\hat{W}_0^- \oplus \hat{W}_0^-) = \operatorname{Ran}(\hat{W}_0^+ \oplus \hat{W}_\infty^+) = \mathcal{H}, \tag{86}$$

$$\hat{W}_0^{\pm}\,U_0^0\,(t) = U\,(t)\,\hat{W}_0^{\pm}, \quad \hat{W}_\infty^{\pm}\,U_\infty\,(t) = U\,(t)\,\hat{W}_\infty^{\pm}. \tag{87}$$

We easily check that for all $F_0 \in \mathcal{H}_0^0$, $F_\infty \in \mathcal{H}_\infty$,

$$\hat{W}_0^{\pm}\,F_0 = s\text{-}\lim_{t\to\pm\infty} U\,(-t)\,\hat{\mathcal{I}}_0\,U_0^0\,(t)\,F_0 \quad \text{in } \mathcal{H}, \tag{88}$$

$$\hat{W}_\infty^{\pm}\,F_\infty = s\text{-}\lim_{t\to\pm\infty} U\,(-t)\,\hat{\mathcal{I}}_\infty\,U_\infty^D\,(t)\,F_\infty \quad \text{in } \mathcal{H}, \tag{89}$$

where

$$\left.\begin{aligned} \hat{\mathcal{I}}_0 &= \begin{bmatrix} [h]^{-1/2}\,Y_0\,I_0\,[h_0^0]^{1/2} & 0 \\ 0 & Y_0\,I_0 \end{bmatrix}, \\ \hat{\mathcal{I}}_\infty &= \begin{bmatrix} [h]^{-1/2}\,Y_\infty\,I_\infty\,B_\infty & 0 \\ 0 & Y_\infty\,I_\infty \end{bmatrix}. \end{aligned}\right\} \tag{90}$$

7. Eventually we investigate the links between $\hat{W}_0^{\pm}$, $\hat{W}_\infty^{\pm}$ and $W_0^{\pm}$, $W_\infty^{\pm}$. We introduce the wave operators

$$F_0^{\pm} \in \mathcal{D}_0^{\pm}, \quad \Omega_{00}^{\pm}\,F_0^{\pm} \equiv s\text{-}\lim_{t\to\pm\infty} U_0^0\,(-t)\,\mathcal{I}_{00}^*\,U_0\,(t)\,F_0^{\pm} \quad \text{in } \mathcal{H}_0^0, \tag{91}$$

where

$$\mathcal{I}_{00} = \begin{bmatrix} Y_0 & 0 \\ 0 & Y_0 \end{bmatrix} : \mathcal{H}_0^0 \to \mathcal{H}_0, \quad \mathcal{I}_{00}^* = \begin{bmatrix} Y_0^* & 0 \\ 0 & Y_0^* \end{bmatrix} : \mathcal{H}_0 \to \mathcal{H}_0^0, \tag{92}$$

$$\begin{aligned} Y_0^* : \quad & f \in C_0^\infty\,(\mathbb{R}_{r_*} \times S_\omega^2) \to (Y_0^*\,f)\,(r_*,\,\omega) \\ & = f\,(r_*,\,\omega) - f\,(0,\,\omega) \quad \text{for } r_* < 0. \end{aligned} \tag{93}$$

For $F_0^{\pm} \in \mathcal{D}_0^{\pm}$, $[U_0\,(t)\,F_0^{\pm}]\,(r_*,\omega) = F_0^{\pm}\,(r_* \pm t,\,\omega)$, $F_0^{\pm}$ being compactly supported, we have for $\pm T$ large enough

$$\mathcal{I}_{00}^*\,U_0\,(t+T)\,F_0^{\pm} = U_0^0\,(t)\,\mathcal{I}_{00}^*\,U_0\,(T)\,F_0^{\pm};$$

Hence $\Omega_{00}^{\pm} F_0^{\pm}$ is well defined and $\Omega_{00}^{\pm}$ extends as an isometry from $\mathcal{H}_0^{\pm}$ to $\mathcal{H}_0^0$. Conversely, for $F_0 \in \mathcal{D}_0^0$ we have for $\pm T$ large enough

$$\mathcal{I}_{00}\, U_0^0\,(t+T)\, F_0 = U_0\,(t)\, \mathcal{I}_{00}\, U_0^0\,(T)\, F_0;$$

Then the following limit

$$(\Omega_{00}^{\pm})^*\, F_0 \equiv s\text{-}\lim_{t\to\pm\infty} U_0\,(-t)\, \mathcal{I}_0\, U_0^0\,(t)\, F_0 \quad \text{in } \mathcal{H}_0 \tag{94}$$

exists and $(\Omega_{00}^{\pm})^* = (\Omega_{00}^{\pm})^{-1}$. We conclude that $\Omega_{00}^{\pm}$ is an isometry from $\mathcal{H}_0^{\pm}$ onto $\mathcal{H}_0^0$ and satisfies

$$U_0^0\,(t)\, \Omega_{00}^{\pm} = \Omega_{00}^{\pm}\, U_0\,(t). \tag{95}$$

Now we claim that

$$W_0^{\pm} = 2\, M\, \hat{W}_0^{\pm}\, \Omega_{00}^{\pm}, \tag{96}$$

$$W_{\infty}^{\pm} = \hat{W}_{\infty}^{\pm}; \tag{97}$$

We note that $W_0^{\pm}$ and $W_{\infty}^{\pm}$ are independent of χ_0, χ_∞ since $\hat{W}_0^{\pm}$, $\Omega_{00}^{\pm}$, $\hat{W}_{\infty}^{\pm}$ do not depend on the choice of the cut-off functions. Then theorem 1 follows from (86), (87), (95), (96), (97).

Since the existence of limits (13), (24) for $F_0^{\pm} \in \mathcal{D}_0^{\pm}$, $F_\infty \in \mathcal{D}_\infty$ has been proved in [9], [10], [11], it is sufficient to prove (96), (97) on dense subspaces $\mathcal{E}_0^{\pm}$, $\mathcal{E}_\infty$ defined by

$$\mathcal{E}_0^{\pm} = \left\{ {}^t\!\left(f,\ \pm\frac{\partial}{\partial r_*} f\right);\ f \in C_0^\infty\,(\mathbb{R}_{r_*} \times S_\omega^2),\right.$$
$$\left. f\,(r_*,\, \omega) = \sum_{\text{finite}} f_{l,\,m}\,(r_*) \otimes Y_{l,\,m}\,(\omega) \right\}, \tag{98}$$

$$\mathcal{E}_\infty = \left\{ {}^t(f_1,\, f_2) \in \mathcal{D}_\infty;\ f_j\,(x) = \sum_{\text{finite}} f_{l,\,m}^j\,(|\,x\,|) \otimes Y_{l,\,m}\left(\frac{x}{|\,x\,|}\right) \right\}. \tag{99}$$

It is obvious that $\mathcal{E}_0^{\pm}$ is invariant under $U_0\,(t)$ and $\mathcal{E}_\infty$ is invariant under $U_\infty\,(t)$ and $U_\infty^D\,(t)$ because the Fourier transform of $f\,(|\,x\,|) \otimes Y_{l,\,m}\left(\frac{x}{|\,x\,|}\right)$ has the form $g\,(|\,\xi\,|) \otimes Y_{l,\,m}\left(\frac{\xi}{|\,\xi\,|}\right)$ (*see* for example [20]). Then, given

$$F_0^{\pm} = {}^t\!\left(f_{l,\,m},\ \pm\frac{d}{dr_*}\, f_{l,\,m}\right) \otimes Y_{l,\,m} \in \mathcal{E}_0^{\pm}, \tag{100}$$

$$F_\infty = {}^t(f^1_{l,m},\, f^2_{l,m}) \otimes Y_{l,m} \in \mathcal{E}_\infty, \tag{101}$$

if we show that as $t \to \pm\infty$,

$$\delta_0(t) \equiv \left\| \begin{bmatrix} \chi_0 - 2M[h]^{-1/2} Y_0 I_0 [h_0^0]^{1/2} Y_0^* & 0 \\ 0 & \chi_0 - 2MY_0 I_0 Y_0^* \end{bmatrix} \times \begin{bmatrix} F_0^\pm(r_* \pm t,\, \omega) \\ \pm \dfrac{\partial}{\partial r_*} F_0^\pm(r_* \pm t,\, \omega) \end{bmatrix} \right\|_{\mathcal{H}} \to 0, \tag{102}$$

$$\delta_\infty(t) \equiv \left\| \begin{bmatrix} \chi_\infty Y_\infty \mathcal{I} - [h]^{-1/2} Y_\infty I_\infty B_\infty & 0 \\ 0 & \chi_\infty Y_\infty \mathcal{I} - Y_\infty I_\infty \end{bmatrix} U_\infty^D(t) F_\infty \right\|_{\mathcal{H}} \to 0, \quad t \to \pm\infty, \tag{103}$$

we obtain (96) and (97) as consequences of (102) and (103) respectively.

The investigations for δ_0 and δ_∞ are similar and we shall make use the following

LEMMA 2. – *Let T, T' be two positive, selfadjoint operators densely defined on $L^2(]a, b[_x, \rho^2(x)\,dx)$, $-\infty < a < b \leq \infty$, $\rho > 0$, with domain $D(T)$, $D(T')$. We assume that for any $\lambda > 0$, the operator $(T+\lambda)^{-1} - (T'+\lambda)^{-1}$ is compact. Then, given $u_n \in D(T) \cap D(T')$ satisfying*

$$T u_n = T' u_n, \tag{104}$$

$$u_n \text{ and } T u_n \to 0, \text{ weakly in } L^2(]a, b[_x, \rho^2(x)\,dx), \quad n \to \infty, \tag{105}$$

we have

$$\| ([T]^{1/2} - [T']^{1/2}) u_n \| \to 0, \quad n \to \infty. \tag{106}$$

LEMMA 3. – *Let T, T' be two positive, selfadjoint operators densely defined on $L^2(]a, b[_x, \rho^2(x)\,dx)$, $-\infty < a < b \leq \infty$, $\rho > 0$, with both the same domain $D(T) = D(T')$. We assume that for any $\lambda > 0$, the operator $(T - T')(T'+\lambda)^{-1}$ is compact on $L^2(]a, b[_x, \rho^2(x)\,dx)$. Then, given $u_n \in D(T) = D(T')$ satisfying*

$$u_n \to 0, \text{ weakly in } L^2(]a, b[_x, \rho^2(x)\,dx), \quad n \to \infty, \tag{107}$$

$$\sup_n (\| T u_n \| + \| T' u_n \|) < \infty, \tag{108}$$

we have

$$\| ([T]^{1/2} - [T']^{1/2}) u_n \| \to 0, \quad n \to \infty. \tag{109}$$

We start by proving (102). First we have

$$\begin{aligned} \delta_0 (t) \leq & \| [h]^{1/2} \{\chi_0 - 2M [h]^{-1/2} Y_0 I_0 [h_0^0]^{1/2} Y_0^*\} \\ & \times F_0^{\pm} (r_* \pm t, \omega) \|_{L^2 (\mathbb{R}_{r_*}, r^2 dr_*) \otimes Y_{l,m}} \\ & + \left\| \{\chi_0 - 2M Y_0 I_0 Y_0^*\} \frac{\partial}{\partial r_*} F_0^{\pm} (r_* \pm t, \omega) \right\|_{L^2 (\mathbb{R}_{r_*}, r^2 dr_*) \otimes Y_{l,m}} \\ \equiv & \delta_{01} (t) + \delta_{02} (t). \end{aligned} \tag{110}$$

On the one hand, since $F_0^{\pm}$ is compactly supported in $[-R, R]_{r_*} \times S_\omega^2$

$$\{\chi_0 - 2M Y_0 I_0 Y_0^*\} \frac{\partial}{\partial r_*} F_0^{\pm} (r_* \pm t, \omega) = (1 - 2M r^{-1}) \frac{\partial}{\partial r_*} F_0^{\pm} (r_* \pm t, \omega)$$

for $|t| > R$, and $\delta_{02} (t)$ tends to zero as t tends to $\pm\infty$. On the other hand, we have

$$\begin{aligned} \delta_{01} (t) \leq & \| ([h]^{1/2} - [h_1]^{1/2}) \chi_0 (r_*) F_0^{\pm} (r_* \pm t, \omega) \|_{L^2 (\mathbb{R}_{r_*}, r^2 dr_*) \otimes Y_{l,m}} \\ & + \| [h_1]^{1/2} \chi_0 (r_*) F_0^{\pm} (r_* \pm t, \omega). \\ & - 2M Y_0 I_0 [h_0^0]^{1/2} Y_0^* F_0^{\pm} (r_* \pm t, \omega) \|_{L^2 (\mathbb{R}_{r_*}, r^2 dr_*) \otimes Y_{l,m}} \end{aligned} \tag{111}$$

Because, firstly, $(h + \lambda)^{-1} - (h_1 + \lambda)^{-1}$ is of finite rank on $L^2 (\mathbb{R}_{r_*}, r^2 dr_*) \otimes Y_{l,m}$, secondly, $\chi_0 (r_*) F_0^{\pm} (r_* \pm t, \omega)$ tends weakly to zero in $L^2 (\mathbb{R}_{r_*}, r^2 dr_*) \otimes Y_{l,m}$, thirdly, $h [\chi_0 (r_*) F_0^{\pm} (r_* \pm t, \omega)] = h_1 [\chi_0 (r_*) F_0^{\pm} (r_* \pm t, \omega)]$ for $|t| > R$ since $F_0^{\pm}$ is compactly supported, we have by lemma 2

$$\begin{aligned} & \| ([h]^{1/2} - [h_1]^{1/2}) [\chi_0 (r_*) F_0^{\pm} (r_* \pm t, \omega)] \|_{L^2 (\mathbb{R}_{r_*}, r^2 dr_*) \otimes Y_{l,m}} \\ & \quad \to 0, \quad t \to \pm\infty. \end{aligned} \tag{112}$$

For $|t| > R$ we write

$$\begin{aligned}
&\| [h]^{1/2} [\chi_0 (r_*) F_0^{\pm} (r_* \pm t, \omega)] - 2 M Y_0 I_0 [h_0^0]^{1/2} \\
&\quad \times Y_0^* F_0^{\pm} (r_* \pm t, \omega) \|_{L^2 (\mathbb{R}_{r_*}, r^2 dr_*) \otimes Y_{l, m}} \\
&\quad = \| ([h_1^0]^{1/2} - 2 M r^{-1} [h_0^0]^{1/2}) F_0^{\pm} (r_* \pm t, \omega) \|_{L^2 (\mathbb{R}_{r_*}, r^2 dr_*) \otimes Y_{l, m}} \\
&\quad \leq \| ([r h_1^0 r^{-1}]^{1/2} - [h_0^0]^{1/2}) F_0^{\pm} (r_* \pm t, \omega) \|_{L^2 (\mathbb{R}_{r_*}^-, r^2 dr_*) \otimes Y_{l, m}} \\
&\quad + \| [h_0^0]^{1/2} [r (1 - 2 M r^{-1}) F_0^{\pm} (r_* \pm t, \omega)] \|_{L^2 (\mathbb{R}_{r_*}^-, r^2 dr_*) \otimes Y_{l, m}}. \qquad (113)
\end{aligned}$$

Obviously $r (1-2 M r^{-1}) F_0^{\pm} (r_* \pm t, \omega)$ and $\dfrac{\partial}{\partial r_*} [r (1-2 M r^{-1}) F_0^{\pm} (r_* \pm t, \omega)]$ tend strongly to zero in $L^2 (\mathbb{R}_{r_*}^- \times S_\omega^2, dr_* d\omega)$ as $t \to \pm\infty$, hence

$$\begin{aligned}
&\| [h_0^0]^{1/2} [r (1 - 2 M r^{-1}) F_0^{\pm} (r_* \pm t, \omega) \|_{L^2 (\mathbb{R}_{r_*}^-, r^2 dr_*) \otimes Y_{l, m}} \\
&\quad \to 0, \quad t \to \pm\infty. \qquad (114)
\end{aligned}$$

Now we have for $\lambda > 0$

$$\begin{aligned}
([r h_1^0 r^{-1}] - h_0^0) (h_0^0 + \lambda)^{-1} &= \{\mathcal{S}_{l, m} (\Delta_1 + \lambda)^{-1}\} \{(\Delta_1 + \lambda) (h_0^0 + \lambda)^{-1}\} \\
&\quad \text{on} \quad L^2 (\mathbb{R}_{r_*}^-, dr_*) \otimes Y_{l, m}
\end{aligned}$$

where $\mathcal{S}_{l, m} \in L^2 (\mathbb{R}_{r_*}^-, (1 + | r_* |^2) dr_*)$. As in proposition 1, this operator is trace-class. Therefore we may apply lemma 3 and obtain

$$\begin{aligned}
&\| ([r h_0^0 r^{-1}]^{1/2} - [h_0^0]^{1/2}) [r F_0^{\pm} (r_* \pm t, \omega)] \|_{L^2 (\mathbb{R}_{r_*}^-, r^2 dr_*) \otimes Y_{l, m}} \\
&\quad \to 0, \quad \pm t \to \infty. \qquad (115)
\end{aligned}$$

Finally thanks to (111) to (115) we conclude that $\delta_{01} (t)$ tends to zero as t tends to $\pm\infty$, and (102) is proved.

Now we come to δ_∞. We write

$$\begin{aligned}
\delta_\infty (t) &\leq \| \{[h]^{1/2} \chi_\infty Y_\infty \mathcal{I} - Y_\infty I_\infty B_\infty\} F_1 (t) \|_{L^2 (\mathbb{R}_{r_*}, r^2 dr_*) \otimes Y_{l, m}} \\
&\quad + \| (r r_*^{-1} \chi_\infty - 1) \mathcal{I} F_2 (t) \|_{L^2 (\mathbb{R}_{r_*}^+, r^2 dr_*) \otimes Y_{l, m}} \\
&\quad \equiv \delta_{\infty 1} (t) + \delta_{\infty 2} (t) \qquad (116)
\end{aligned}$$

with

$$U_{\infty}^{D}(t) F_{\infty} = {}^{t}(F_1(t),\ F_2(t)).$$

We state some straightforward consequences of theorem 1 in [10] concerning the decay of this asymptotic dynamics:

$$\left.\begin{array}{c}\forall \mu = 0, \ldots, 3, \quad \forall k \in \mathbb{N}, \quad \forall V \in L^p(\mathbb{R}^3), \\ 2 \leq p < \infty, \quad \| V \partial_{\mu}^{k} F_1(t) \|_{L^2(\mathbb{R}^3)} \to 0, \quad t \to \pm\infty, \end{array}\right\} \tag{117}$$

$$\left.\begin{array}{c}\forall \mu = 0, \ldots, 3, \quad \forall k \in \mathbb{N}, \quad \partial_{\mu}^{k} F_1(t) \to 0, \\ \text{weakly in } L^2(\mathbb{R}^3), \quad t \to \pm\infty. \end{array}\right\} \tag{118}$$

Therefore (117) implies that $\delta_{\infty 2}(t)$ tends to zero as $t \to \pm\infty$, and we have

$$\begin{aligned}\delta_{\infty 1}(t) \leq & \| ([h]^{1/2} - [h_1]^{1/2}) [\chi_{\infty} Y_{\infty} \mathcal{I} F_1(t)] \|_{L^2(\mathbb{R}_{r_*},\, r^2\, dr_*) \otimes Y_{l,m}} \\ & + \| \{[h]^{1/2} \chi_{\infty} Y_{\infty} \mathcal{I} - Y_{\infty} I_{\infty} B_{\infty}\} \\ & \qquad F_1(t) \|_{L^2(\mathbb{R}_{r_*},\, r^2\, dr_*) \otimes Y_{l,m}}. \end{aligned} \tag{119}$$

By (118), $h[\chi_{\infty} Y_{\infty} \mathcal{I} F_1(t)] = h_1[\chi_{\infty} Y_{\infty} \mathcal{I} F_1(t)]$ tends weakly to zero and as above, since $(h+\lambda)^{-1} - (h_1+\lambda)^{-1}$ is of finite rank on $L^2(\mathbb{R}_{r_*}, r^2\, dr_*) \otimes Y_{l,m}$, lemma 2 yields that

$$\begin{aligned}& \| ([h]^{1/2} - [h_1]^{1/2}) \chi_{\infty} Y_{\infty} \mathcal{I} F_1(t) \|_{L^2(\mathbb{R}_{r_*},\, r^2\, dr_*) \otimes Y_{l,m}} \\ & \quad \to 0, \quad t \to \pm\infty. \end{aligned} \tag{120}$$

Now we choose a cut-off function χ_{∞}^{1} satisfying (22) and

$$\operatorname{supp} \chi_{\infty}^{1} \subset]1, \infty[, \tag{121}$$

and we estimate

$$\begin{aligned}& \| \{[h_1]^{1/2} \chi_{\infty} Y_{\infty} \mathcal{I} - Y_{\infty} I_{\infty} B_{\infty}\} F_1(t) \|_{L^2(\mathbb{R}_{r_*},\, r^2\, dr_*) \otimes Y_{l,m}} \\ & \quad = \| \{[h_1^{\infty}]^{1/2} \chi_{\infty} \mathcal{I} - I_{\infty} B_{\infty}\} F_1(t) \|_{L^2(\mathbb{R}_{r_*}^{+},\, r^2\, dr_*) \otimes Y_{l,m}} \\ & \quad \leq \| [h_1^{\infty}]^{1/2} [(\chi_{\infty} - \chi_{\infty}^{1}) \mathcal{I} F_1(t)] \|_{L^2(\mathbb{R}_{r_*}^{+},\, r^2\, dr_*) \otimes Y_{l,m}} \\ & \qquad + \| \{[h_1^{\infty}]^{1/2} - [h_2^{\infty}]^{1/2}\} [\chi_{\infty}^{1} \mathcal{I} F_1(t)] \|_{L^2(\mathbb{R}_{r_*}^{+},\, r^2\, dr_*) \otimes Y_{l,m}} \\ & \qquad + \| [r h_2^{\infty} r^{-1}]^{1/2} [(r - r_*) \chi_{\infty}^{1} \mathcal{I} F_1(t)] \|_{L^2(\mathbb{R}_{r_*}^{+},\, r^2\, dr_*) \otimes Y_{l,m}} \\ & \qquad + \| \{[r h_2^{\infty} r^{-1}]^{1/2} - [r_* h_3^{\infty} r_*^{-1}]^{1/2}\} \end{aligned}$$

$$\times\ [r_*\,\chi^1_\infty\ \mathcal{I}\,F_1\,(t)]\ \|_{L^2\,(\mathbb{R}^+_{r_*},\,r^2\,dr_*)\otimes Y_{l,\,m}}$$
$$+\ \|\ \{[r_*\,h^\infty_3\,r_*^{-1}]^{1/2} - [r_*\,h^\infty - r_*^{-1}]^{1/2}\}$$
$$\times\ [r_*\,\chi^1_\infty\ \mathcal{I}\,F_1\,(t)]\ \|_{L^2\,(\mathbb{R}^+_{r_*},\,r^2\,dr_*)\otimes Y_{l,\,m}}$$
$$+\ \|\ \{[r_*\,h^\infty\,r_*^{-1}]^{1/2}\,[r_*\ \mathcal{I}\,B^2_\infty\,|\,x\,|^{-1}\,\mathcal{I}^{-1}]^{1/2}\}$$
$$\times\ \ [r_*\,\chi^1_\infty\mathcal{I}\ F_1\,(t)]\ \|_{L^2\,(\mathbb{R}^+_{r_*},\,r^2\,dr_*)\otimes Y_{l,\,m}}$$
$$= A_1 + A_2 + A_3 + A_4 + A_5 + A_6. \tag{122}$$

A_1 and A_3 tend to zero according to (117); moreover we note that

$$\dim\,[(h^\infty_1 + \lambda)^{-1} - (h^\infty_2 + \lambda)^{-1}]\,[L^2\,(\mathbb{R}^+_{r_*},\ r^2\,dr_*)\otimes Y_{l,\,m}] < \infty,$$

$$h^\infty_1\,[r_*\,\chi^1_\infty\mathcal{I}\,F_1\,(t)] = h^\infty_2\,[r_*\,\chi^1_\infty\mathcal{I}\,F_1\,(t)]\ \to\ 0$$
$$\text{weakly in } L^2\,(\mathbb{R}^+_{r_*},\ dr_*)\otimes Y_{l,\,m},$$

$$\sup_t\ \|\ \partial_{r_*}\,[r_*\,\chi^1_\infty\mathcal{I}\,F_1\,(t)]\ \|_{L^2\,(]\,1,\,\infty\,[_{r_*},\ dr_*)\otimes Y_{l,\,m}} < \infty,$$

$$\dim\,[(r_*\,h^\infty_3\,r_*^{-1} + \lambda)^{-1} - (r_*\,h^\infty\,r_*^{-1} + \lambda)^{-1}]\,[L^2\,(\mathbb{R}^+_{r_*},\ dr_*)\otimes Y_{l,\,m}] < \infty,$$

$$(r_*\,h^\infty_3\,r_*^{-1})\ [r_*\,\chi^1_\infty\mathcal{I}\,F_1\,(t)] = (r_*\,h^\infty\,r_*^{-1})\,[r_*\,\chi^1_\infty\mathcal{I}\,F_1\,(t)]\ \to\ 0$$
$$\text{weakly in } L^2\,(\mathbb{R}^+_{r_*},\ dr_*)\otimes Y_{l,\,m},$$

then we can apply lemma 2 to A_2 and A_5. Now we remark that

$$\|\ \{[r h^\infty_2\,r^{-1}]^{1/2} - [r_*\,h^\infty_3\,r_*^{-1}]^{1/2}\}\,[r_*\,\chi^1_\infty\mathcal{I}\,F_1\,(t)]\ \|_{L^2\,(\mathbb{R}^+_{r_*},\ dr_*)\otimes Y_{l,\,m}}$$
$$= \|\ \{[r\,k_2\,r^{-1}]^{1/2} - [r_*\,k'_2\ r_*^{-1}]^{1/2}\}$$
$$\times\ [r_*\,\chi^1_\infty\ \mathcal{I}\,F_1\,(t)]\ \|_{L^2\,(]\,1,\,\infty\,[_{r_*},\ dr_*)\otimes Y_{l,\,m}}, \tag{123}$$

and we have

$$\left.\begin{array}{l} r_*\,\chi^1_\infty\mathcal{I}\,F_1\,(t)\ \to\ 0 \text{ weakly in } L^2\,(]\,1,\ \infty\,[_{r_*},\ dr_*)\otimes Y_{l,\,m}, \\ r_*\,\chi^1_\infty\mathcal{I}\,F_1\,(t)\ \to\ 0\quad \text{weakly in } L^2\,(]\,1,\ \infty\,[_{r_*},\ dr_*)\otimes Y_{l,\,m}, \end{array}\right\} \tag{124}$$

$$\begin{aligned} &D\,(r k_2\, r^{-1}\,|_{L^2\,(]\,1,\,\infty\,[_{r_*},\,dr_*)\otimes Y_{l,}})\\ &= D\,(r_*\,k_2'\,r_*^{-1}\,|_{L^2(]1,\,\infty[_{r_*},\,dr_*)\otimes Y_{l,\,m}}\\ &= \{f\in L^2\,(]\,1,\,\infty\,[_{r_*},\,dr_*);\\ &\qquad f'\in L^2\,(]\,1,\,\infty\,[_{r_*},\,dr_*),\ f\,(1)=0\}, \end{aligned}$$

$$\left.\begin{aligned} &([r k_2\, r^{-1}] - [r_*\,k_2'\,r_*^{-1}])\,([r_*\,k_2'\,r_*^{-1}] + \lambda)^{-1}\\ &= \{\mathcal{V}_{l,\,m}\,(\Delta_1+\lambda)^{-1}\}\,\{(\Delta_1+\lambda)\,(r_*\,k_2'\,r_*^{-1}+\lambda)^{-1}\}\\ &\qquad \text{on } L^2\,(]\,1,\,\infty\,[_{r_*},\,dr_*)\otimes Y_{l,\,m}. \end{aligned}\right\} \qquad (125)$$

Since $\mathcal{V}_{l,\,m}\in L^1\,(]\,1,\,\infty\,[_{r_*},\,dr_*)$ operator (125) is *trace class*$\times$*bounded*= *trace class.* Then lemma 3 implies that A_4 tends to zero. Eventually

$$\begin{aligned} &D\,(r_*\,h^\infty\,r_*^{-1}\,|_{L^2\,(\mathbb{R}^+_{r_*},\,dr_*)\otimes Y_{l,\,m}})\\ &= D\,(r_*\,\mathcal{I}\,B_\infty^2\,|\,x\,|^{-1}\,\mathcal{I}^{-1}\,|_{L^2\,(\mathbb{R}^+_{r_*},\,dr_*)\otimes Y_{l,\,m}})\\ &= \{f\in L^2\,(]\,0,\,\infty\,[_{r_*},\,dr_*);\\ &\qquad f''\in L^2\,(]\,0,\,\infty\,[_{r_*},\,dr_*),\ f\,(0)=0\}, \end{aligned}$$

$$\begin{aligned} &([r_*\,h^\infty\,r_*^{-1}] - [r_*\,\mathcal{I}\,B_\infty^2\,|\,x\,|^{-1}\,\mathcal{I}^{-1}])\,([r_*\,\mathcal{I}\,B_\infty^2\,|\,x\,|^{-1}\,\mathcal{I}^{-1}]+\lambda)^{-1}\\ &= \{\mathcal{W}_{l,\,m}\,(\Delta_1+\lambda)^{-1}\}\,\{(\Delta_1+\lambda)\,(r_*\,\mathcal{I}\,B_\infty^2\,|\,x\,|^{-1}\,\mathcal{I}^{-1}+\lambda)^{-1}\}\\ &\qquad \text{on } L^2\,(]\,0,\,\infty\,[_{r_*},\,dr_*)\otimes Y_{l,\,m}, \end{aligned}$$

with

$$\mathcal{W}_{l,\,m}\,(r_*) = -2\,M\,r^{-1}\in L^2\,(]\,0,\,\infty\,[_{r_*},\,dr_*)$$

According to theorem XI.20 in [19], $|\,(\mathcal{W}_{l,\,m}\oplus 0)\,(\Delta_0+\lambda)^{-1}\,|^2$ in trace class, hence as in proposition 1 $\{\mathcal{W}_{l,\,m}\,(\Delta_1+\lambda)^{-1}\}$ is compact. Again we conclude by lemma 3 that A_6 tends to zero. This concludes the proof of (103) and theorem 1 is established.

Q.E.D.

Proof of Lemma 2. – We know (*see* for example [13], p. 284) the square root of a non negative selfadjoint densely defined operator T can be written as

$$[T]^{1/2} = \pi^{-1}\int_0^\infty \lambda^{-1/2}\,(T+\lambda)^{-1}\,T\,d\lambda. \qquad (126)$$

Thus we estimate

$$\begin{aligned}&\| ([T]^{1/2} - [T]^{1/2})\, u_n \| \\ &\quad \leq \pi^{-1} \int_0^{\infty} \lambda^{-1/2} \| [(T + \lambda)^{-1} - (T' + \lambda)^{-1}]\, T\, u_n \| \, d\lambda. \quad (127)\end{aligned}$$

On the one hand the assumptions assure that

$$\forall \lambda > 0, \quad \| [(T + \lambda)^{-1} - (T' + \lambda)^{-1}]\, T\, u_n \| \to 0, \; n \to \infty,$$

and on the other hand the Hille-Yoshida theorem ([19], th. X.47a) implies

$$\begin{aligned}&\lambda^{-1/2} \| [(T + \lambda)^{-1} - (T' + \lambda)^{-1}]\, T\, u_n \| \\ &\quad \leq 2 \ \mathrm{Inf}\, (\lambda^{-1/2},\ \lambda^{-3/2}) \sup_n (\| u_n \|,\ \| T\, u_n \|) \in L^1 (\lambda > 0).\end{aligned}$$

We conclude, by the dominated convergence theorem, that (127) tends to zero.

Q.E.D.

Proof of Lemma 3. – We use formula (126) to obtain

$$\begin{aligned}&\| ([T]^{1/2} - [T]^{1/2})\, u_n \| \\ &\quad \leq \pi^{-1} \int_0^{\infty} \lambda^{-1/2} \| (T + \lambda)^{-1}\, T\, u_n - (T' + \lambda)^{-1}\, T'\, u_n \| \, d\lambda. \quad (128)\end{aligned}$$

On the one hand the assumptions assure that

$$\begin{aligned}&\| (T + \lambda)^{-1}\, T\, u_n - (T' + \lambda)^{-1}\, T'\, u_n \| \\ &\quad = 2\, \lambda \| (T + \lambda)^{-1} (T - T') (T' + \lambda)^{-1} u_n \| \to 0, \quad n \to \infty,\end{aligned}$$

and on the other hand the Hille-Yoshida theorem implies

$$\begin{aligned}&\lambda^{-1/2} \| (T + \lambda)^{-1}\, T\, u_n - (T' + \lambda)^{-1}\, T'\, u_n \| \\ &\quad \leq 2\, \mathrm{Inf}\, (\lambda^{-1/2}, \lambda^{-3/2}) \sup_n (\| u_n \|,\ \| T\, u_n \|,\ \| T'\, u_n \|) \in L^1 (\lambda > 0).\end{aligned}$$

We conclude as above.

Q.E.D.

3. THE QUANTUM SCATTERING OPERATOR

We recall some classic concepts of quantum fields theory ([6], [19], [21]). Given a *real* linear space L, endowed with a skew-symmetric, non degenerate, bilinear form $\sigma(\cdot, \cdot)$, a *Weyl quantization* $(W, \mathfrak{H})$ of (L, σ) is defined as a map $W : x \in L \to W(x)$, from L to $\mathfrak{U}(\mathfrak{H})$ which is the space of unitary operators on a complex Hilbert space $\mathfrak{H}$ satisfying the Weyl version of the canonical commutation relations (CCR's)

$$W(x+y) = e^{i\sigma(x,y)/2} W(x) W(y), \tag{129}$$

and the condition of weak continuity:

$$\forall x \in L, \quad \forall X \in \mathfrak{H}, \quad (t \to W(tx) X) \in C^0(\mathbb{R}_t; \mathfrak{H}). \tag{130}$$

A linear map T on L is said to be *symplectic* if

$$\forall x, y \in L, \quad \sigma(Tx, Ty) = \sigma(x, y). \tag{131}$$

Then $W_T(x) \equiv W(Tx)$ is another Weyl quantization. T will be said unitarily implementable in the representation $(W, \mathfrak{H})$ of the CCR's, if there exists a unitary operator $\mathbb{T}$ on $\mathfrak{H}$ such that

$$\forall x \in L, \quad W_T(x) = \mathbb{T}\, W(x)\, \mathbb{T}^{-1}. \tag{132}$$

In the case of an infinite-dimensional space L, there is a continuous family of pairwise inequivalent representations of CCR's (*cf.* von Neumann's theorem). In the case of boson fields, we describe the most important representation, which preserves the positivity of the energy and the relativistic invariance: the so called *Fock-Cook* representation; it is associated with the *boson single particle space* $(L, \sigma, V(t))$, where $V(t)$ is a one parameter symplectic group on L.

According to Kay [14], it is convenient to introduce the *single particle structure* $(K, \mathfrak{h}_1, \mathcal{U}(t))$ where $\mathfrak{h}_1$ is a complex Hilbert space, $\mathcal{U}(t)$ is a unitary group on $\mathfrak{h}_1$ with strictly positive infinitesimal generator:

$$\mathcal{U}(t) = e^{-itH}, \qquad H > 0, \tag{133}$$

and K is a *real* linear map from L to $\mathfrak{h}_1$, satisfying:

$$\operatorname{Ran} K \textit{ is dense in } \mathfrak{h}_1, \tag{134}$$

$$\sigma(x, y) = 2 \operatorname{Im} \langle Kx, Ky \rangle_{\mathfrak{h}_1}, \tag{135}$$

$$KV(t) = \mathcal{U}(t) K. \tag{136}$$

Note that K is invertible because σ is non degenerate. The fundamental result in [14] asssures that if there exists a single particle structure, then it is unique up to unitary equivalence.

Now the *second quantization* over some complex Hilbert space $\mathfrak{h}_1$ is the Weyl quantization of $(\mathfrak{h}_1,\ 2\,\mathrm{Im}\,\langle\cdot,\ \cdot\rangle_{\mathfrak{h}_1})$ constructed as follows: we take $\mathfrak{H} = \mathcal{F}_s(\mathfrak{h}_1)$ the boson Fock square over $\mathfrak{h}_1$:

$$\mathfrak{H} = \mathcal{F}_s(\mathfrak{h}_1) \equiv \bigoplus_{n=0}^{\infty} [\mathfrak{h}_1^{\otimes n}]_s \tag{137}$$

where $[\mathfrak{h}_1^{\otimes n}]_s$ stands for the n-fold symmetric tensor product of $\mathfrak{h}_1$, and we put:

$$f \in \mathfrak{h}_1, \quad W_{\mathcal{F}}(f) = \exp[a^*(f) - (a^*(f))^*] \in \mathfrak{U}(\mathfrak{H}) \tag{138}$$

where $a^*(f)$ is the standard creation operator ([6], [19]). Then $W_{\mathcal{F}}$ satisfies (129), (130). Moreover, if Σ is unitary on $\mathfrak{h}_1$, the *quantized operator*

$$\Gamma(\Sigma) \equiv [\Sigma^{\otimes n}]_s \tag{139}$$

is unitary on $\mathfrak{H}$ and satisfies

$$f \in \mathfrak{h}_1, \quad W_{\mathcal{F}}(\Sigma f) = \Gamma(\Sigma)\, W_{\mathcal{F}}(f)\, [\Gamma(\Sigma)]^{-1}. \tag{140}$$

Furthermore, from assumption (133) we get

$$\Gamma(\mathcal{U}(t)) = e^{-it\mathbb{H}}, \tag{141}$$

where $\mathbb{H}$ is a densely defined selfadjoint strictly positive operator on $\mathfrak{H}$.

If we assume that the boson single particle space $(L,\ \sigma,\ V(t))$ has a single particle structure $(K,\ \mathfrak{h}_1,\ \mathcal{U}(t))$, then *Fock-Cook quantization* $(W_\Phi,\ \mathfrak{H})$ over $(L,\ \sigma)$ is given by

$$x \in L, \quad W_\Phi(x) \equiv W_{\mathcal{F}}(K\,x) \in \mathfrak{U}(\mathfrak{H}). \tag{142}$$

Thanks to the uniqueness of the single particle structure and the functorial character of the second quantization (139), (140), the Fock-Cook quantization does not depend, up to unitary equivalence, on the single particle structure.

Now we return to the quantum scattering by a black-hole and construct the asymptotic quantizations. For the sake of simplicity, we consider only *real* classical fields, *i.e.* hermitian quantum fields. We denote $\mathcal{E}_{\mathbb{R}}$ the real part of $\mathcal{E} = \mathcal{E}_{\mathbb{R}} + i\,\mathcal{E}_{\mathbb{R}}$, the complex Hilbert space completion of C_0^∞ for some norm $\|\cdot\|_{\mathcal{E}}$, *i.e.* $\mathcal{E}_{\mathbb{R}}$ is obtained by completion for $\|\cdot\|_{\mathcal{E}}$ of the set $C_{0,\mathbb{R}}^\infty$ of *real* test functions.

Obviously, $U_\infty(t)$ is an orthogonal group and $U_\infty^D(t)$ an orthogonal propagator on $\mathcal{H}_{\infty,\mathbb{R}}$; also $U_0(t)$ is an orthogonal group on $\mathcal{H}_{0,\mathbb{R}}^{\pm}$. Therefore $W_0^\pm \oplus W_\infty^\pm$ is an isometry from $\mathcal{H}_{0,\mathbb{R}}^{\pm} \oplus \mathcal{H}_{\infty,\mathbb{R}}$ onto $\mathcal{H}_{\mathbb{R}}$ and the classical Scattering Operator S is an isometry from $\mathcal{H}_{0,\mathbb{R}}^{-} \oplus \mathcal{H}_{\infty,\mathbb{R}}$ onto $\mathcal{H}_{0,\mathbb{R}}^{+} \oplus \mathcal{H}_{\infty,\mathbb{R}}$.

We quantize very simply the fields at the horizon by putting

$$\left.\begin{aligned} F_0^{\pm} = {}^t(f_1, f_2), \quad G_0^{\pm} = {}^t(g_1, g_2) \in \mathcal{D}_{0,\mathbf{R}}^{\pm}, \\ \sigma_0(F_0^{\pm}, G_0^{\pm}) \equiv 4M^2 \int (f_1 g_2 - f_2 g_1)\, dr_*\, d\omega, \end{aligned}\right\} \tag{143}$$

$$\mathfrak{h}_1^0 \equiv L^2_{\mathbb{C}}(\mathbb{R}^+_\xi \times S^2_\omega;\, d\xi\, d\omega). \tag{144}$$

$$\begin{aligned} F_0^{\pm} = {}^t(f_1, f_2) \in \mathcal{D}_{0,\mathbf{R}}^{\pm}, \\ (K_0^{\pm} F_0^{\pm})(\xi, \omega) \equiv \frac{2M}{\sqrt{\pi}}\, \xi^{1/2} \int e^{\pm i\xi r_*} f_1(r_*, \omega)\, dr_* \in \mathfrak{h}_1^0. \end{aligned} \tag{145}$$

Nevertheless we have to be careful because we are concerned with the one dimensional wave equation (6) for which the Hilbert spaces of finite energy $\mathcal{H}_0^{\pm}$ are not spaces of distributions: the first components are not in L^2 and the symplectic form σ_0 is merely defined on subspaces of $\mathcal{H}_0^{\pm}$. Following [11] it is convenient to introduce the dense subspaces

$$\mathcal{K}_0^{\pm} \equiv \{F_0^{\pm} \in \mathcal{H}_0^{\pm};\; H_0^{-1} F_0^{\pm} \in \mathcal{H}_0^{\pm}\} \tag{146}$$

which satisfy

$$\mathcal{D}_0^{\pm} \subset \mathcal{K}_0^{\pm} \subset L^2(\mathbb{R}_{r_*} \times S^2_\omega,\, dr_*\, d\omega) \times L^2(\mathbb{R}_{r_*} \times S^2_\omega,\, dr_*\, d\omega). \tag{147}$$

We see easily that $(\mathcal{K}_{0,\mathbf{R}}^{\pm}, \sigma_0, U_0(t))$ is a boson single particle space, and $(K_0^{\pm}, \mathfrak{h}_1^0, e^{-it\xi})$ is a single particle structure over it. Then we define the Weyl quantizations $(W_{0\Phi}^{\pm}, \mathfrak{H}_0)$ at the past and future horizons by:

$$\mathfrak{H}_0 \equiv \mathcal{F}_s(\mathfrak{h}_1^0), \quad F_0^{\pm} \in \mathcal{K}_{0,\mathbf{R}}^{\pm}, \quad W_{0\Phi}^{\pm}(F_0^{\pm}) \equiv W_{\mathcal{F}}(K_0^{\pm} F_0^{\pm}). \tag{148}$$

Since we work at the horizon on the domain of H_0^{-1}, we must do the same thing the infinity and we put:

$$\mathcal{K}_\infty \equiv \{F_\infty \in \mathcal{H}_\infty;\; H_\infty^{-1} F_\infty \in \mathcal{H}_\infty\}. \tag{149}$$

Thanks to intertwining property (28), the scattering operator S is an isometry from $\mathcal{K}_{0,\mathbf{R}}^{-} \oplus \mathcal{K}_{\infty,\mathbf{R}}$ onto $\mathcal{K}_{0,\mathbf{R}}^{+} \oplus \mathcal{K}_{\infty,\mathbf{R}}$. At infinity we take the usual Fock-Cook quantization for neutral bonsons by choosing:

$$\left.\begin{aligned} F_\infty = {}^t(f_1, f_2), \quad G_\infty = {}^t(g_1, g_2) \in \mathcal{K}_{\infty,\mathbf{R}}, \\ \sigma_\infty(F_\infty, G_\infty) \equiv \int_{\mathbf{R}^3_x} (f_1 g_2 - f_2 g_1)\, dx, \end{aligned}\right\} \tag{150}$$

$$F_\infty = {}^t(f_1,\, f_2) \in \mathcal{K}_{\infty, \mathbb{R}},\; K_\infty F_\infty \equiv \frac{1}{\sqrt{2}}\,(B_\infty^{1/2}\, f_1 + i\, B_\infty^{-1/2}\, f_2), \quad (151)$$

$$\mathfrak{h}_1^\infty \equiv L^2_{\mathbb{C}}\,(\mathbb{R}^3_x;\, dx), \quad (152)$$

$$\mathfrak{H}_\infty \equiv \mathcal{F}_s\,(\mathfrak{h}_1^\infty), \quad F_\infty \in \mathcal{K}_{\infty, \mathbb{R}}, \quad W_{\infty\Phi}\,(F_\infty) \equiv W_{\mathcal{F}}\,(K_\infty\, F_\infty). \quad (153)$$

Now we put for $F_0^\pm,\, G_0^\pm \in \mathcal{K}^\pm_{0, \mathbb{R}},\, F_\infty,\, G_\infty \in \mathcal{K}_{\infty, \mathbb{R}}$,

$$\begin{aligned}&(\sigma_0 \oplus \sigma_\infty)\,[{}^t(F_0^\pm,\, F_\infty),\; {}^t(G_0^\pm,\, G_\infty)]\\ &\quad \equiv \sigma_0\,(F_0^\pm,\, G_0^\pm) + \sigma_\infty\,(F_\infty,\, G_\infty),\end{aligned} \quad (154)$$

$$W_\Phi^\pm\,(F_0^\pm \oplus F_\infty) \equiv W_{0\Phi}^\pm\,(F_0^\pm) \otimes W_{\infty\Phi}\,(F_\infty). \quad (155)$$

Then $(K_0^\pm \oplus K_\infty,\, \mathfrak{h}_1^0 \oplus \mathfrak{h}_1^\infty,\, e^{-it\xi} \oplus e^{-itB_\infty})$ is a single particle structure over the boson single particle space $(\mathcal{K}^\pm_{0, \mathbb{R}} \oplus \mathcal{K}_{\infty, \mathbb{R}},\, \sigma_0 \oplus \sigma_\infty,\, U_0\,(t) \oplus U_\infty\,(t))$ for which the Weyl quantizations $(W_\Phi^\pm,\, \mathfrak{H}_0 \otimes \mathfrak{H}_\infty)$ are called free Fock-Cook representations of the CCR's.

Finally in the same way we quantize the Klein-Gordon equation on the Schwarzschild space-time by putting:

$$\mathcal{K} = \{F \in \mathcal{H};\; H^{-1}\, F \in \mathcal{H}\}, \quad (156)$$

$$\left.\begin{aligned}&F = {}^t(f_1,\, f_2), \quad G = {}^t(g_1,\, g_2) \in \mathcal{K}_{\mathbb{R}},\\ &\sigma\,(F,\, G) \equiv \int (f_1\, g_2 - f_2\, g_1)\, r^2\, dr_*\, d\omega,\end{aligned}\right\} \quad (157)$$

$$F = {}^t(f_1,\, f_2) \in \mathcal{K}_{\mathbb{R}}, \quad KF \equiv \frac{1}{\sqrt{2}}\,(h^{1/4}\, f_1 + ih^{-1/4}\, f_2), \quad (158)$$

$$\mathfrak{h}_1 \equiv L^2_{\mathbb{C}}\,(\mathbb{R}_{r_*} \times S^2_\omega,\, r^2\, dr_*\, d\omega), \quad (159)$$

$$\mathfrak{H} \equiv \mathcal{F}_s\,(\mathfrak{h}_1), \quad F \in \mathcal{K}_{\mathbb{R}}, \quad W_\Phi\,(F) \equiv W_{\mathcal{F}}\,(KF). \quad (160)$$

$(K,\, \mathfrak{h}_1,\, e^{-ith^{1/2}})$ is a single particle structure and $(W_\Phi,\, \mathfrak{H})$ is the standard Fock-Cook quantization of $(\mathcal{K}_{\mathbb{R}},\, \sigma,\, U\,(t))$.

THEOREM 3. – *The classical Wave Operators $W_0^\pm \oplus W_\infty^\pm$ and the Scattering Operator S are unitarily implementable in the Fock-Cook representations*

$(W_\Phi, \mathfrak{H})$, $(W_\Phi^\pm, \mathfrak{H}_0 \otimes \mathfrak{H}_\infty)$, *More precisely, there exists unitary operators* $\mathbb{S}$ *on* $\mathfrak{H}_0 \otimes \mathfrak{H}_\infty$ *and* $\mathbb{W}^\pm$ *from* $\mathfrak{H}_0 \otimes \mathfrak{H}_\infty$ *onto* $\mathfrak{H}$ *such that, for any* $F_0^\pm \in \mathcal{K}_{0,\mathbb{R}}^\pm$, $F_\infty \in \mathcal{K}_{\infty,\mathbb{R}}$, *we have:*

$$W_\Phi (W_0^\pm F_0^\pm \oplus W_\infty^\pm F_\infty) = \mathbb{W}^\pm W_\Phi^\pm (F_0^\pm \oplus F_\infty) [\mathbb{W}^\pm]^{-1}, \tag{161}$$

$$W_0^\pm [S (F_0^- \oplus F_\infty)] = \mathbb{S} W_\Phi^- [F_0^- \oplus F_\infty] \mathbb{S}^{-1}. \tag{162}$$

Remarks. – 1. Operator $\mathbb{S}$ describes entirely the quantum scattering of a neutral scalar massive field by a Schwarzschild balck-hole. Nevertheless this quantum system is not determined by $\mathbb{S}$: it would be necessary to construct the local algebra of observables, and some preferred vacuum state. In particular, the Fock vacuum in $\mathfrak{H}$ assoociated with the Boulware state on the Schwarzschild space-time, is not physically relevant. We refer to [11], [12], [15] where these problems are analysed in depth.

2. As regards the charged fields we can use the same approach since, thanks to intertwining relation (28), the scattering by the black-hole does not mix the particles and antiparticles.

Proof. – Theorem 1 implies that $W_0^\pm \oplus W_\infty^\pm$ is an isometry from $\mathcal{K}_{0,\mathbb{R}}^\pm \oplus \mathcal{K}_{\infty,\mathbb{R}}$, onto $\mathcal{K}_\mathbb{R}$ and satisfies the intertwining relation

$$U(t) [W_0^\pm \oplus W_\infty^\pm] = [W_0^\pm \oplus W_\infty^\pm] [U_0(t) \oplus U_\infty(t)]. \tag{163}$$

Moreover according to [10], [11], $W_0^\pm \oplus W_\infty^\pm$ is symplectic from $(\mathcal{K}_{0,\mathbb{R}}^\pm \oplus \mathcal{K}_{\infty,\mathbb{R}}, \sigma_0 \oplus \sigma_\infty)$ to $(\mathcal{K}_\mathbb{R}, \sigma)$. Therefore $(K[W_0^\pm \oplus W_\infty^\pm], \mathfrak{h}_1, e^{-ith^{1/2}})$ is a single particle structure over $(\mathcal{K}_{0,\mathbb{R}}^\pm \oplus \mathcal{K}_{\infty,\mathbb{R}}, \sigma_0 \oplus \sigma_\infty, U_0(t) \oplus U_\infty(t))$. By uniqueness, there exists a unitary operator $\Sigma^\pm$ from $\mathfrak{h}_1^0 \oplus \mathfrak{h}_1^\infty$ onto $\mathfrak{h}_1$, such that:

$$K[W_0^\pm \oplus W_\infty^\pm] = \Sigma^\pm [K_0^\pm \oplus K_\infty]. \tag{164}$$

We use (160), (164), (148), (155), (153) to obtain:

$$\begin{aligned} W_\Phi (W_0^\pm F_0^\pm \oplus W_\infty^\pm F_\infty) &= W_\mathcal{F} [K (W_0^\pm F_0^\pm \oplus W_\infty^\pm F_\infty] \\ &= W_\mathcal{F} [\Sigma^\pm (K_0^\pm F_0^\pm \oplus K_\infty F_\infty)] \\ &= \Gamma(\Sigma^\pm) W_\mathcal{F} (K_0^\pm F_0^\pm \oplus K_\infty F_\infty) [\Gamma(\Sigma^\pm)]^{-1} \\ &= \Gamma(\Sigma^\pm) W_\Phi^\pm (F_0^\pm \oplus F_\infty) [\Gamma(\Sigma^\pm)]^{-1}. \end{aligned} \tag{165}$$

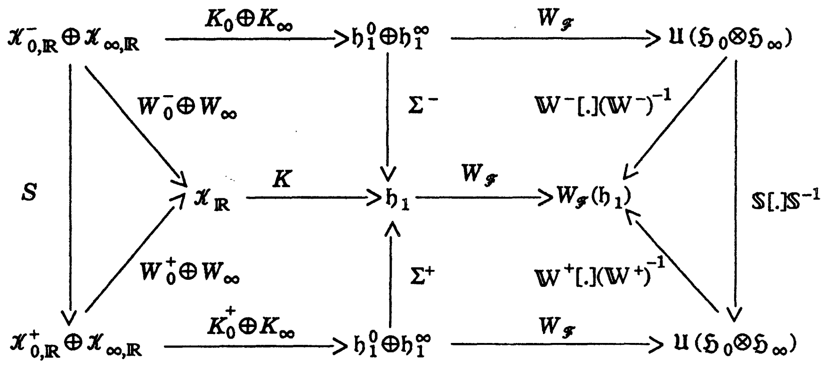
Now it is sufficient to put

$$\mathbb{W}^\pm = \Gamma(\Sigma^\pm), \quad \mathbb{S} = \Gamma(S) = (\mathbb{W}^+)^{-1} \mathbb{W}^-, \tag{166}$$

and (161) (162) are deduced from (165).

Q.E.D.

The classical as well as quantum scattering of a massive scalar field by a Schwarzschild black-hole is summarised by the following commutative diagram:



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