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Creation of Fermions at the Charged Black-Hole Horizon

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Abstract. We investigate the quantum state of the Dirac field at the horizon of a charged black-hole formed by a spherical gravitational collapse. We prove this state satisfies a KMS condition with the Hawking temperature and the chemical potential associated with the mass and the charge of the black-hole. Moreover, the fermions with charge of same sign to that of the black-hole are emitted more readily than those of opposite charge. It is a spontaneous loss of charge of the black-hole due to the quantum vacuum polarization.

I Introduction

The purpose of this paper is to investigate the quantum state of charged spinor fields at the horizon of a black-hole created by the collapse of a spherical charged star. Our main result expresses that the ground state, that is given by the Boulware vacuum in the past, is of Unruh type at the future horizon. It is the famous Hawking effect : a static observer at infinity, interprets this state as a thermal radiation of particles and antiparticles outgoing from the black-hole. Moreover the black-hole emits more readily fermions whose the charge is of same sign as its own charge. A similar phenomenon for the scalar fields has been discussed by G. W. Gibbons in [21].

The space time outside the collapsing star is given by a four-dimensional, globally hyperbolic manifold $(\mathcal{M},g)$, of Kruskal-Reissner-Nordstrøm type, endowed with a 1-form $A_\mu dx^\mu$ (the electromagnetic potential due to the star). The boundary $\partial\mathcal{M}$ of $\mathcal{M}$ has two pieces : a time-like part $\mathcal{S}$ (the moving surface of the star), a characteristic part H^+ (the future black-hole horizon). We first solve the mixed hyperbolic problem for the Dirac system for the particles of mass m and charge q,

$$i\gamma^\mu(\nabla_\mu + iqA_\mu)\Psi - m\Psi = 0, \ \ in \ \ \mathcal{M}, \tag{I.1}$$

with initial data on a space-like hypersurface Σ_0. To take the interaction between the matter of the star and the field, into account, we add some boundary condition,

$$n_\mu\gamma^\mu\Psi = \mathcal{B}\Psi, \ \ on \ \ \mathcal{S}, \tag{I.2}$$

where $\mathcal{B}$ belongs to a large class of operators on $\mathcal{S}$, including in particular the MIT condition [13] :

$$\mathcal{B} = ie^{i\alpha\gamma^5}.$$

Then we construct the local algebra of observables $\mathfrak{A}(\mathcal{M})$, and given the Fock vacuum on Σ_0, corresponding to the Boulware vacuum in the past, we prove that the quantum state on H^+ satisfies a KMS condition involving the mass and the charge of the star (Unruh state). Moreover the temperature and the chemical potential are independent of the history of the collapse, and the boundary condition. We also investigate the role of the cosmological constant in the case of the charged black-holes in an expanding universe (De Sitter-Reissner-Nordstrøm metric).

From a mathematical point of view, we adopt the framework of our previous studies on the Hawking effect for the Klein-Gordon fields [2], [3], [4]. It is convenient to choose a frame for which H^+ is pushed away to the null infinity (Schwarzschild-like coordinates). Then the surface of the star is an asymptotically characteristic moving boundary, and an asymptotically infinite Doppler effect appears (blue shift). Hence the problem is reduced to a very sharp analysis of the asymptotic behaviour of the Dirac propagator. The key point consists in proving that the approximation by the geometrical optics is valid.

This paper is organized as follows : we precise the geometric assumptions in part two; we present the key asymptotic estimate for the classical Dirac equation in the third part; we construct the quantum field and we state the main result on the Hawking effect in part 4; this result is discussed in part 5, especially the interpretation in terms of particles, and we study the role of the cosmological constant; part 6 is devoted to the mathematical proofs of the results of parts 3 and 4; taking advantage of the spherical invariance, we reduce the problem to studying a system in one space dimension, which we investigate. After the conclusion, we give in the appendix, by sake of completeness, the main tools of the quantization of the spin fields on the stationary space-times.

We end by giving some bibliographic information. Obviously, the list of references is very incomplete. We cite only the works that we have used. Among a huge literature on the Hawking effect, we can mention [3], [4], [10], [18], [19], [21], [22], [31], [36], [37], [40], [41], [42]. More generally, the quantum field theory on curved spacetime is investigated in [1], [8], [16], [17], [20], [24], [30]. The Dirac equation on a black-hole background has been studied in [5], [12], [27], [32], [33], [34], [35] [37], [40].

II Geometrical Framework

The space-time outside a static spherical black-hole is a four dimensional, globally hyperbolic manifold $(\mathcal{M}_{BH}, g)$

$$\mathcal{M}_{BH} = \mathbb{R}_t \times]r_0, r_+[_r \times S^2_{\theta,\varphi}, \quad 0 < r_0 < r_+ \leq \infty,$$

$$g_{\mu\nu} dx^\mu dx^\nu = F dt^2 - F^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\varphi^2). \tag{II.1}$$

Here F is a C^1 function of $r > 0$ satisfying

$$F(r_0) = 0, \quad F'(r_0) > 0, \quad r_0 < r < r_+ \Rightarrow F(r) > 0. \tag{II.2}$$

r_0 is the radius of the black-hole horizon, and we introduce

$$\kappa_0 := \frac{1}{2} F'(r_0), \tag{II.3}$$

the *surface gravity* at the black-hole horizon. As regards r_+, we assume, either that

$$r_+ < \infty, \quad F(r_+) = 0, \quad F'(r_+) < 0, \tag{II.4}$$

then $\mathcal{M}_{BH}$ is asymptotically of DeSitter type, and r_+ is the radius of the cosmological horizon, or

$$r_+ = \infty, \quad \lim_{r \to \infty} F(r) = F(\infty) > 0, \quad \lim_{r \to \infty} F'(r) = 0, \tag{II.5}$$

in which case $\mathcal{M}_{BH}$ is asymptotically flat in a weak sense. The fundamental example is the *Reissner-Nordstrøm* metric, which is the unique spherically symmetric solution of the Einstein-Maxwell equations in the vacuum :

$$F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \quad 0 \leq \mid Q \mid < M. \tag{II.6}$$

$0 < M$ and Q are respectively the mass and the electric charge of the black-hole, the radius and the surface gravity of which are :

$$r_0 = M + \sqrt{M^2 - Q^2}, \quad \kappa_0 = \frac{\sqrt{M^2 - Q^2}}{\left(M + \sqrt{M^2 - Q^2}\right)^2} \quad (and \;\; r_+ = \infty). \tag{II.7}$$

More generally we could consider spherical charged black-holes in an expanding universe, described by the *DeSitter-Reissner-Nordstrøm* metric

$$F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3} r^2, \tag{II.8}$$

where $\Lambda > 0$ is the cosmological constant.

It is well-know that the black-hole horizon is a fictitious singularity of the metric, that can be removed by a suitable change of variables. It is convenient to introduce a tortoise radial coordinate $x \in \mathbb{R}$ satisfying :

$$\frac{dx}{dr} = F^{-1}, \tag{II.9}$$

by choosing for $r \in]r_0, r_+[$

$$x = \frac{1}{2\kappa_0} \left\{ \ln \mid r - r_0 \mid - \int_{r_0}^{r} \left[\frac{1}{r - r_0} - \frac{2\kappa_0}{F(r)} \right] dr \right\} + x_0. \tag{II.10}$$

We note that the map :

$$x \in \mathbb{R} \mapsto r \in]r_0, r_+[$$

is one-to-one with the asymptotic behaviour :

$$\mid r - r_0 \mid = O(e^{2\kappa_0 x}), \quad x \to -\infty. \tag{II.11}$$

We can extend $x(r)$ for $r \in]r_-, r_0[$ by the formula (II.10) for r_- such that

$$0 < r_- < r_0, \quad r_- < r < r_0 \Rightarrow F(r) < 0.$$

We define the Kruskal-Szekeres coordinates (T, X, θ, φ) :

$$T = \frac{1}{2} e^{\kappa_0 x} \left(e^{\eta \kappa_0 t} - \eta e^{-\eta \kappa_0 t} \right), \quad X = \frac{1}{2} e^{\kappa_0 x} \left(e^{\eta \kappa_0 t} + \eta e^{-\eta \kappa_0 t} \right), \quad \eta = \frac{r - r_0}{\mid r - r_0 \mid}. \tag{II.12}$$

The Schwarzschild type coordinates (t, r, θ, φ) give two local maps with domains $\mathcal{M} = \mathbb{R}_t \times]r_0, r_+[_r \times S^2_{\theta,\varphi}$ and $\mathbb{R}_t \times]r_-, r_0[_r \times S^2_{\theta,\varphi}$, but fail to represent the black-hole horizon $\{r = r_0\}$. Kruskal-Szekeres coordinates define an atlas with a single map with domain a neighborhood of $\overline{\mathcal{M}_{BH}} = \{(T, X, \omega); \ \ X \geq \mid T \mid, \ \ \omega \in S^2\}$, and the black-hole horizon appears as the characteristic submanifold $\{X = \mid T \mid\} \times S^2_\omega$.

In fact we are concerned with the realistic black-holes created by the gravitational collapse of a spherical star. So we consider a star, stationary in the past, contracting to a black-hole in the future. In Kruskal-Szekeres coordinates, its boundary is given by

$$\{(T, X = Z(T), \omega), \ T \in \mathbb{R}, \ \omega \in S^2\}$$

where Z is a C^2 function of $T \in \mathbb{R}$. Since the star boundary is necessarily time like, we have :

$$-1 < Z'(T) \leq 0, \tag{II.13}$$

and the creation of the black-hole is expressed by :

$$\exists T_0 > 0; \ Z(T_0) = T_0. \tag{II.14}$$

Hence we deal with the manifold

$$\mathcal{M} = \left\{ (T, X, \omega) \in \mathbb{R} \times \mathbb{R} \times S^2; \ T \leq T_0 \Rightarrow X \geq Z(T), \ \ T_0 \leq T \Rightarrow X \geq T \right\}, \tag{II.15}$$

and its boundary $\partial\mathcal{M}$ consists of the world lines of the star boundary :

$$\mathcal{S} = \{(T, X = Z(T)); \ T \leq T_0\} \times S^2, \tag{II.16}$$

and the future black-hole horizon :

$$H^+ = \{(T, X = T); \ T \geq T_0\} \times S^2. \tag{II.17}$$

To study the structure of the quantum state on H^+ we should have to solve a Characteristic Cauchy problem with data on H^+, for a hyperbolic system with T-dependent coefficients, on the manifold $\mathcal{M}$ that is singular at $\mathcal{S}\cap H^+ = \{(T_0, X = T_0)\}\times S^2$. Therefore we prefer to adopt the Schwarzschild-like coordinates (t, x, θ, φ) for wich the coefficients of the Dirac system are t-independent and the singularity is pushed away to infinity. Instead of solving a Goursat problem, we develop a time dependent Scattering type theory on :

$$\mathcal{M} = \left\{(t, x, \omega);\ t \in \mathbb{R},\ x \geq z(t),\ \omega \in S^2\right\}, \tag{II.18}$$

where the boundary is described by the function $z(t)$ defined by

$$X = Z(T) \Longleftrightarrow x = z(t). \tag{II.19}$$

We can easily prove (see [2]) that this function satisfies :

$$\left\{\begin{array}{l} z \in C^2(\mathbb{R}),\\ \forall t \in \mathbb{R},\ -1 < \dot{z}(t) \leq 0,\\ z(t) = -t - Ae^{-2\kappa_0 t} + \zeta(t),\ A > 0,\\ \mid \zeta(t) \mid + \mid \dot{\zeta}(t) \mid = O(e^{-4\kappa_0 t}),\ t \longrightarrow +\infty, \end{array}\right. \tag{II.20}$$

and for commodity, we choose x_0 in (II.10) such that

$$\forall t \leq 0,\ z(t) = z(0) < 0. \tag{II.21}$$

Here A depends only on κ_0, and we remark that the physics of the collapse is hidden in the rest $\zeta(t)$, when the leading term $-t - Ae^{-2\kappa_0 t}$ involves only the surface gravity. This fact leads to the *No Hair* property of the Hawking effect.
With this choice of frame, the star boundary $\mathcal{S}$ seems to be asymptotically characteristic :

$$\mathcal{S} = \left\{(t, x = z(t), \omega);\ t \in \mathbb{R},\ \omega \in S^2\right\}, \tag{II.22}$$

and a point of the future horizon H^+ is reached at the infinity of a null ray $(t, x = -t + t_0, \omega)_{t\in\mathbb{R}}$ as $t \to +\infty$, in short :

$$\lim_{t\to+\infty} \left(t, x = -t + \frac{1}{\kappa_0}\ln(2T_1), \omega\right) = (T_1, X_1 = T_1, \omega) \in H^+. \tag{II.23}$$

Finally the geometrical framework of a generic spherical gravitational collapse is given by (II.1), (II.2), (II.4), (II.5), (II.18), (II.20), (II.21).

III The Classical Dirac Equation

We consider the Dirac equation for particles with charge $q \in \mathbb{R}$ and mass $m \geq 0$ outside a collapsing charged spherical star :

$$i\gamma^\mu(\nabla_\mu + iqA_\mu)\Psi - m\Psi = 0.$$

Here the 1-form

$$A_\mu dx^\mu$$

defines the electromagnetic field created by the star. The geometrical framework is given in Part II. Then the Dirac equation has the form in (t, r, θ, φ) coordinates (see e.g. [33], [34], [27]) :

$$\begin{aligned} 0 = & \\ & \Big\{ iF^{-\frac{1}{2}}\boldsymbol{\gamma}^0\left(\frac{\partial}{\partial t}+iqA_t\right)+iF^{\frac{1}{2}}\boldsymbol{\gamma}^1\left(\frac{\partial}{\partial r}+\frac{1}{r}+\frac{F'}{4F}+iqA_r\right) \\ & +\frac{i}{r}\boldsymbol{\gamma}^2\left(\frac{\partial}{\partial\theta}+\frac{1}{2\tan\theta}+iqA_\theta\right)+\frac{i}{r\sin\theta}\boldsymbol{\gamma}^3\left(\frac{\partial}{\partial\varphi}+iqA_\varphi\right)-m\Big\}\Psi, \end{aligned} \tag{III.1}$$

where the Dirac matrices are :

$$\boldsymbol{\gamma}^0 = i\begin{pmatrix} 0 & \sigma^0 \\ -\sigma^0 & 0 \end{pmatrix}, \quad \boldsymbol{\gamma}^a = i\begin{pmatrix} 0 & \sigma^a \\ \sigma^a & 0 \end{pmatrix}, \quad a = 1, 2, 3,$$

with the Pauli matrices :

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

We have to add a boundary condition on the star surface. We write this condition as

$$n_\mu \gamma^\mu \Psi = \mathcal{B}\Psi \tag{III.2}$$

where n_μ the unit outgoing normal and $\mathcal{B}$ is some operator on the surface of the star. It is natural to assume $\mathcal{B}$ is local in time, rotationally invariant and such that the L^2-norm of the spinor is conserved. Such a boundary condition, which is local in space-time, is the generalized MIT boundary condition [13], [26] (see also [5]) :

$$\mathcal{B}_{MIT}\Psi = ie^{i\alpha\gamma^5}\Psi, \tag{III.3}$$

where $\alpha \in \mathbb{R}$ is the Chiral Angle and

$$\boldsymbol{\gamma}^5 := -i\boldsymbol{\gamma}^0\boldsymbol{\gamma}^1\boldsymbol{\gamma}^2\boldsymbol{\gamma}^3 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}. \tag{III.4}$$

When the spinor field is massless ($m = 0$), the system is chiral invariant : we can choose any real α. When m is non zero, (III.3) defines a family of non equivalent boundary conditions. In this case, and if the space-time is asymptotically flat, we must restrict the range of the chiral angle :

$$m \neq 0, \;\; r_+ = \infty \Longrightarrow \alpha \neq (2k+1)\pi, \;\; k \in \mathbb{Z}. \tag{III.5}$$

We are interested by proving that the Hawking effect does not depend of the interaction between the field and the matter of the star. Hence we consider a very large class of boundary conditions given by the family of zero order pseudo-differential operators

$$\mathcal{B} = \sum_{\ell,n} ie^{i\alpha_{\ell,n}\gamma^5}\Pi_{\ell,n}, \tag{III.6}$$

where the sequence $(\alpha_{\ell,n})$ satisfies (III.5), and $\Pi_{\ell,n}$ is the orthogonal projector on the (ℓ,n)-space of the spinoidal spherical harmonics expansion (VI.4).

We assume that the electromagnetic potential satisfies

$$\left\{\begin{array}{l} A_t = A(r) \in C^1([r_0,r_+]), \\ A_r = A_\theta = A_\varphi = 0. \end{array}\right. \tag{III.7}$$

These hypotheses are fullfilled in the important case of the Reissner-Nordstrøm Black-Hole since (see e.g. [11]) :

$$A = \frac{Q}{r}. \tag{III.8}$$

In fact the Dirac equation is obviously well defined on the whole space-time. In Kruskal coordinates (T,X,ω) we introduce the spinor field

$$\Phi_K(T,X,\omega) = rF^{\frac{1}{4}}(r)e^{itqA(r_0)}\boldsymbol{M}\Psi(t,r,\omega), \tag{III.9}$$

with

$$\boldsymbol{M} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & F^{-\frac{1}{2}} & 0 & 0 \\ 0 & 0 & F^{-\frac{1}{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and the Dirac system becomes :

$$\begin{aligned} 0 = & \\ & \Big\{\frac{\partial}{\partial T} + \boldsymbol{\gamma}^0\boldsymbol{\gamma}^1\frac{\partial}{\partial X} + i\frac{q}{\kappa_0}(A - A(r_0))\left(X\mathbf{1} + T\boldsymbol{\gamma}^0\boldsymbol{\gamma}^1\right)^{-1} \\ & + \frac{1}{4\kappa_0}(X+T)^{-1}F'F^{\frac{1}{2}}\left(\mathbf{1} + \boldsymbol{\gamma}^0\boldsymbol{\gamma}^1\right) + \frac{1}{\kappa_0}F^{\frac{1}{2}}\left(X\mathbf{1} + T\boldsymbol{\gamma}^0\boldsymbol{\gamma}^1\right)^{-1} \\ & \boldsymbol{M}\boldsymbol{\gamma}^0\left\{\frac{1}{r}\boldsymbol{\gamma}^2\left(\frac{\partial}{\partial\theta} + \frac{1}{2\tan\theta}\right) + \frac{1}{r\sin\theta}\boldsymbol{\gamma}^3\frac{\partial}{\partial\varphi} + im\right\}\boldsymbol{M}^{-1}\Big\}\Phi_K \end{aligned} \tag{III.10}$$

We can easily check that all the coefficients, which are (T,X,ω) dependent, are regular on the horizon H^+. We could use the theory of the mixed hyperbolic systems to solve it, but since we want to get some precise information on the fields near the horizon, it is more convenient to use the tortoise coordinate $x \in \mathbb{R}$ given by (II.10) instead of $r \in]r_0, r_+[$, and to change the representation of the spinor again, by introducing

$$\Phi(t,x,\theta,\varphi) = rF^{\frac{1}{4}}\Psi(t,x,\theta,\varphi). \tag{III.11}$$

Given a spinor field Φ_s defined on the Cauchy hypersurface

$$\Sigma_s =]z(s), \infty[_x \times S^2_\omega, \tag{III.12}$$

the mixed problem becomes :

$$\frac{\partial}{\partial t}\Phi + \boldsymbol{\gamma}^0\boldsymbol{\gamma}^1\frac{\partial}{\partial x}\Phi + iqA\Phi + F^{\frac{1}{2}}\boldsymbol{\gamma}^0 \left\{\frac{1}{r}\boldsymbol{\gamma}^2\left(\frac{\partial}{\partial\theta} + \frac{1}{2\tan\theta}\right) + \frac{1}{r\sin\theta}\boldsymbol{\gamma}^3\frac{\partial}{\partial\varphi} + im\right\}\Phi = 0, \quad x > z(t), \tag{III.13}$$

$$x = z(t) \Longrightarrow \frac{1}{\sqrt{1-\dot{z}^2}}\left(\dot{z}\boldsymbol{\gamma}^0 - \boldsymbol{\gamma}^1\right)\Phi = i\mathcal{B}\Phi, \tag{III.14}$$

$$\Phi(t=s,.) = \Phi_s(.). \tag{III.15}$$

To construct the functional framework, we introduce the Hilbert spaces :

$$\mathcal{L}^2_t := \left[L^2(\Sigma_t, dxd\omega)\right]^4, \ \ \mathcal{L}^2_\infty := \left[L^2(\mathbb{R}_x \times S^2_\omega, dxd\omega)\right]^4. \tag{III.16}$$

For $s < t \leq \infty$, $\mathcal{L}^2_s$ is naturally embedded in $\mathcal{L}^2_t$; this amounts to extending the function by zero for $x \leq z(s)$. We denote $\| \ . \ \|$ the norm in $\mathcal{L}^2_t$. We define on $\mathcal{L}^2_t$ the operator $\mathbb{H}_t$:

$$\mathbb{H}_t\Phi = i\boldsymbol{\gamma}^0\boldsymbol{\gamma}^1\frac{\partial}{\partial x}\Phi - qA\Phi + iF^{\frac{1}{2}}\boldsymbol{\gamma}^0\left\{\frac{1}{r}\boldsymbol{\gamma}^2\left(\frac{\partial}{\partial\theta} + \frac{1}{2\tan\theta}\right) + \frac{1}{r\sin\theta}\boldsymbol{\gamma}^3\frac{\partial}{\partial\varphi} + im\right\}\Phi, \tag{III.17}$$

$$t \in \mathbb{R}, \ D(\mathbb{H}_t) = \left\{\Phi \in \mathcal{L}^2_t; \ \mathbb{H}_t\Phi \in \mathcal{L}^2_t, x = z(t) \Rightarrow \frac{1}{\sqrt{1-\dot{z}^2}}\left(\dot{z}\boldsymbol{\gamma}^0 - \boldsymbol{\gamma}^1\right)\Phi = i\mathcal{B}\Phi\right\}, \tag{III.18}$$

$$D(\mathbb{H}_\infty) = \left\{\Phi \in \mathcal{L}^2_\infty; \ \mathbb{H}_\infty\Phi \in \mathcal{L}^2_\infty\right\}. \tag{III.19}$$

Lemma III.1. *The operator $i\mathbb{H}_t$ is maximal accretive for any t, and skew-adjoint for $t \leq 0$ and $t = \infty$. Moreover the point spectrum of $\mathbb{H}_\infty$ is empty.*

Since $\mathbb{H}_\infty$ has no eigenvalue, there exists no time-periodic Dirac fields with finite energy on the whole Reissner-Nordstrøm space-time. We let open the problem of the existence of such solutions outside a stationary star, and we solve the Dirac equation outside the collapsing star. The mixed problem has the form :

$$\frac{\partial}{\partial t}\Phi = i\mathbb{H}_t\Phi, \tag{III.20}$$

and it is solved by a propagator

$$\Phi(t) = U(t,s)\Phi_s. \tag{III.21}$$

More precisely we apply a Trotter-Kato method to get the following :

Proposition III.2. *For $\Phi_s \in D(\mathbb{H}_s)$, there exists a unique solution $\Phi \in C^1(\mathbb{R}_t; \mathcal{L}^2_\infty)$ of (III.13), (III.14), (III.15), (III.6), satisfying for any real t :*

$$\Phi(t) \in D(\mathbb{H}_t). \tag{III.22}$$

Moreover we have :

$$\| \Phi(t) \| = \| \Phi_s \|, \tag{III.23}$$

and $U(t,s)$ can be extended in an isometric strongly continuous propagator from $\mathcal{L}^2_s$ onto $\mathcal{L}^2_t$ satisfying

$$\forall \Phi_s \in D(\mathbb{H}_s),\ \ (t \mapsto U(t,s)\Phi_s) \in C^1\left(\mathbb{R}_t, \mathcal{L}^2_\infty\right),\ \ \frac{d}{dt}U(t,s)\Phi_s = i\mathbb{H}_tU(t,s)\Phi_s, \tag{III.24}$$

$$\forall \Phi_{s_0} \in \left[C_0^\infty(\Sigma_{s_0})\right]^4,\ \exists h > 0, \tag{III.25}$$
$$(s \mapsto U(t,s)\Phi_{s_0}) \in C^1\left(]s_0 - h, s_0 + h[, \mathcal{L}^2_t\right),\ \frac{d}{ds}U(t,s)\Phi_{s_0} = -iU(t,s)\mathbb{H}_s\Phi_{s_0},$$

$$\big(x > R \Rightarrow \Phi_s(x,\omega) = 0\big) \Rightarrow \big(x > R + \mid t - s \mid \Rightarrow [U(t,s)\Phi_s](x,\omega) = 0\big). \tag{III.26}$$

Now we state that, given Ψ_{BH} a field falling into the future black-hole, there exists a unique Dirac field Ψ which is equal to Ψ_{BH} at the horizon, and $\Psi = 0$ at the null infinity. Since we have chosen the Schwarzschild coordinates (t,x,ω), this characteristic problem becomes a scattering problem : the characteristic data becomes an asymptotic data. The link between Φ and Φ_K makes clear the suitable set of asymptotic data near the horizon. Since Φ_K is well defined on H^+, the relation $\Phi = e^{-itqA(r_0)}\boldsymbol{M}^{-1}\Phi_K$ implies that $\Phi_2(t, x = -t+s, \omega)$ and $\Phi_3(t, x =$

$-t+s,\omega)$ tends to zero as $t\to\infty$. Therefore we introduce the subspaces of fields Φ, falling into the black-hole as t tends to infinity :

$$\mathcal{L}^2_{BH}=\left\{\Phi\in\mathcal{L}^2_\infty;\ \ \Phi_2=\Phi_3=0,\ \ x<0\Rightarrow\Phi(x,\omega)=0\right\}, \tag{III.27}$$

or going out to infinity :

$$\mathcal{L}^2_{out}=\left\{\Phi\in\mathcal{L}^2_\infty;\ \ \Phi_1=\Phi_4=0,\ \ x<0\Rightarrow\Phi(x,\omega)=0\right\}. \tag{III.28}$$

At the black-hole horizon the electromagnetic potential equals to $A(r_0)$ and F is zero, hence we compare the dynamics (III.20) with :

$$\frac{\partial}{\partial t}\Phi=i\mathbb{H}_{BH}\Phi, \tag{III.29}$$

$$\mathbb{H}_{BH}\Phi=i\boldsymbol{\gamma}^0\boldsymbol{\gamma}^1\frac{\partial}{\partial x}\Phi-qA(r_0)\Phi,\quad D(\mathbb{H}_{BH})=\left\{\Phi\in\mathcal{L}^2_\infty;\ \ \mathbb{H}_{BH}\Phi\in\mathcal{L}^2_\infty\right\}. \tag{III.30}$$

$\mathbb{H}_{BH}$ is selfadjoint and the Cauchy problem for (III.29) is solved by the unitary group on $\mathcal{L}^2_\infty$:

$$U_{BH}(t):=e^{it\mathbb{H}_{BH}}. \tag{III.31}$$

Since $A(r)\to A(r_0)$ and $F(r)\to 0$ exponentially, the Cook method allows to construct the wave operator that gives the solution of the asymptotic problem :

Proposition III.3. *Assume $\Phi\in\mathcal{L}^2_{BH}$. Then the strong limit :*

$$\Omega_{BH}\Phi=\lim_{T\to+\infty}U(0,T)U_{BH}(T)\Phi\ \ in\ \ \mathcal{L}^2_0 \tag{III.32}$$

exists and defines an isometry from $\mathcal{L}^2_{BH}$ to $\mathcal{L}^2_0$.

We can now state the main result of asymptotic behaviour of the propagator near the horizon. To express this estimate we shift the Cauchy data toward the black-hole horizon by the following way. For $\Phi\in\mathcal{L}^2_\infty$ and $T>0$ we put :

$$\Phi^T(x,\omega)=\Phi(x+T,\omega). \tag{III.33}$$

Theorem III.4 (Key Estimate). *Given $\Phi_{out}\in\mathcal{L}^2_{out}$, $\Phi_{BH}\in\mathcal{L}^2_{BH}$, we have for $J=[0,\infty[$ or $]0,\infty[$:*

$$\begin{aligned}\lim_{T\to\infty}\ &\parallel \mathbf{1}_J(\mathbb{H}_0)U(0,T)\left(\Phi^T_{out}+\Phi^T_{BH}\right)\parallel^2\\ &=<\Phi_{out},\zeta e^{\frac{2\pi}{\kappa_0}\mathbb{H}_{BH}}\left(1+\zeta e^{\frac{2\pi}{\kappa_0}\mathbb{H}_{BH}}\right)^{-1}\Phi_{out}>_{\mathcal{L}^2_\infty}\\ &+\parallel \mathbf{1}_J(\mathbb{H}_0)\left(\Omega_{BH}\Phi_{BH}\right)\parallel^2.\end{aligned} \tag{III.34}$$

with

$$\zeta=e^{\frac{2\pi}{\kappa_0}qA(r_0)}. \tag{III.35}$$

We remark that the term involving Φ_{out} in the right member of (III.34) does *not* depend on the boundary condition of type (III.2), (III.6), and on the mass of the field. It is also independent of the history of the collapse defined by function $z(t)$.

We briefly describe the main ideas of the proof. The key phenomenon is the asymptotically infinite Doppler effect due to the collapse to a black-hole : in Schwarzschild coordinates, the contracting surface of the star is asymptotically characteristic. Hence it appears a blue shift and we establish that the approximation of the geometrical optics is valid : for large T, the main part of the energy of $t \mapsto U(t,T)\Phi_{out}^T$, $0 \leq t \leq T$, propagates near the null hypersurface $\{(t, x = -t, \omega),\ 0 \leq t \leq T,\ \omega \in S^2\}$. Furthermore we can evaluate the leading term of $U(0,T)\Phi_{out}^T$:

$$U(0,T)\Phi_{out}^T \sim \Phi_T^* + o(1), \quad T \to \infty,$$

with

$$\Phi_T^*(x,\omega) := e^{iqA(r_0)T} \frac{1}{\mid \kappa_0 x \mid^{\frac{1}{2}}} \left(-\Phi_{out,3}(y,\omega), 0, 0, \Phi_{out,2}(y,\omega)\right),$$

$$y := 2T + \frac{1}{\kappa_0}\ln(-x) - \frac{1}{\kappa_0}\ln(A).$$

We have :

$$\| \mathbf{1}_J(\mathbb{H}_0)\left(U(0,T)\Phi_{out}^T\right) \| \sim \| \mathbf{1}_J(\mathbb{H}_{BH})\left(\Phi_T^*\right) \|,$$

moreover, an explicit calculation by Fourier transform gives the fundamental identity for any $T > 0$:

$$\| \mathbf{1}_J(\mathbb{H}_{BH})\left(\Phi_T^*\right) \|^2 = < \Phi_{out}, \zeta e^{\frac{2\pi}{\kappa_0}\mathbb{H}_{BH}} \left(1 + \zeta e^{\frac{2\pi}{\kappa_0}\mathbb{H}_{BH}}\right)^{-1} \Phi_{out} >_{\mathcal{L}^2_\infty} .$$

Finally, we remark that $U(0,T)\Phi_{out}^T$ and $U(0,T)\Phi_{BH}^T$ are asymptotically orthogonal as $T \to \infty$, since $U(0,T)\Phi_{out}^T$ weakly converges to zero because of the Doppler effect, and $U(0,T)\Phi_{BH}^T$ strongly converges to $\Omega_{BH}\Phi_{BH}$.

IV The Quantum Dirac Fields

We interpret the crucial result (Theorem III.4) in the framework of the Quantum Field Theory. Since we deal with a curved space-time with moving boundary, the concept of particles is not appropriate, hence we adopt the approach of the algebras of local observables in the spirit of [16], [19]. According to J. Dimock [17], we construct the algebra of quantum spin fields on a curved space-time as following. We consider a globally hyperbolic manifold $\mathcal{U}$ with a foliation by a family of Cauchy hypersurfaces Σ_t, i.e.

$$\mathcal{U} = \cup_{t\in\mathbb{R}}\Sigma_t.$$

We choose a representation of the CAR (= canonical anticommutation relations) on Σ_0. It consists of a Hilbert space $\mathcal{H}$ and some antilinear continuous function $\mathbf{\Psi}_0$ between the space of spinors on Σ_0, that we represent as $L^2(\Sigma_0, \mathbb{C}^4)$, and $\mathcal{L}(\mathcal{H})$, satisfying in particular

$$\mathbf{\Psi}_0^*(F_1)\mathbf{\Psi}_0(F_2) + \mathbf{\Psi}_0(F_2)\mathbf{\Psi}_0^*(F_1) =< F_1, F_2 > \mathbf{1}.$$

Now a classical spin field structure is defined by a propagator $B(t, s)$ that is an isometry from $L^2(\Sigma_s, \mathbb{C}^4)$ onto $L^2(\Sigma_t, \mathbb{C}^4)$. We introduce the operator

$$S : \Phi \in C_0^\infty(\mathcal{U}, \mathbb{C}^4) \longmapsto S(F) := \int_{-\infty}^{\infty} B(0, t)\Phi(t)dt \in L^2(\Sigma_0, \mathbb{C}^4). \tag{IV.1}$$

Then the quantum spin field is the operator valued distribution

$$\mathbf{\Psi} : \Phi \in C_0^\infty(\mathcal{U}, \mathbb{C}^4) \longmapsto \mathbf{\Psi}(\Phi) := \mathbf{\Psi}_0(S\Phi) \in \mathcal{L}(\mathcal{H}). \tag{IV.2}$$

For any open set $\mathcal{O} \subset \mathcal{U}$, the algebra of observables is

$$\mathfrak{A}(\mathcal{O}) := \mathbb{C}^* \textit{ algebra generated by } \mathbf{\Psi}^*(\Phi_1)\mathbf{\Psi}(\Phi_2), \ \textit{supp}\, \Phi_j \subset \mathcal{O}.$$

A beautiful result due to Dimock assures that the collection $\mathfrak{A}(\mathcal{O})$ is independent (up to a net isomorphism) of the representation of the CAR, the Cauchy hypersurface, and the choice of spin structure.

Now we consider a state ω_{Σ_0} on the $\mathbb{C}^*$ algebra generated by $\mathbf{\Psi}_0^*(F_1)\mathbf{\Psi}_0(F_2)$, $F_j \in L^2(\Sigma_0, \mathbb{C}^4)$. Then we define a ground state $\omega_{\mathcal{U}}$ on $\mathfrak{A}(\mathcal{U})$ by putting for $\Phi_j \in C_0^\infty(\mathcal{U}, \mathbb{C}^4)$:

$$\omega_{\mathcal{U}}(\mathbf{\Psi}^*(\Phi_1)\mathbf{\Psi}(\Phi_2)) := \omega_{\Sigma_0}(\mathbf{\Psi}_0^*(S\Phi_1)\mathbf{\Psi}_0(S\Phi_2)). \tag{IV.3}$$

In the case of the stationary space-times, we have $\Sigma_t = \{t\} \times \Sigma_0$, the generator of the propagator is a densely defined self adjoint operator $\mathbb{H}$ on $L^2(\Sigma_0, \mathbb{C}^4)$:

$$B(t, s) = e^{i(t-s)\mathbb{H}},$$

and the Fock vacuum state on Σ_0 is defined by

$$\omega_{\Sigma_0}^0(\mathbf{\Psi}_0^*(F_1)\mathbf{\Psi}_0(F_2)) :=< \mathbf{1}_{]0,\infty[}(\mathbb{H})F_1, F_2 > . \tag{IV.4}$$

More generally, a state $\omega_{\Sigma_0}^{\beta,\mu}$ satisfies the (β, μ)-KMS condition, $0 < \beta$, $\mu \in \mathbb{R}$, if

$$\omega_{\Sigma_0}^{\beta,\mu}(\mathbf{\Psi}_0^*(F_1)\mathbf{\Psi}_0(F_2)) :=< ze^{\beta\mathbb{H}} \left(\mathbf{1} + ze^{\beta\mathbb{H}}\right)^{-1} F_1, F_2 >, \ \ z = e^{\beta\mu}. \tag{IV.5}$$

We immediately associate a state $\omega_{\mathcal{U}}^{\beta,\mu}$ on $\mathcal{U}$ by (IV.1), (IV.3), (IV.5). In fact $\omega_{\mathcal{U}}^{\beta,\mu}$ describes a double Gibbs equilibrium state : on the one hand, an ideal Fermi particle gas with temperature $0 < T = \beta^{-1}$ and chemical potential μ, and on the other hand an ideal Fermi antiparticle gas with the same temperature T but an opposite chemical potential $-\mu$. If q is the charge of the particles, the charge density of the gaz is (see Lemma A.2) :

$$\varrho = \frac{1}{\pi} q\mu. \tag{IV.6}$$

For sake of completeness, we present, in the Appendix, the details of the second quantization of the Dirac field with a time-independent Hamiltonian.

We apply these procedures to our problem. First the quantization on $\mathcal{M}$, the space-time outside the collapsing star, is defined by choosing the foliation

$$\Sigma_t :=]z(t), \infty[_x \times S^2,$$

the Fock quantization and the Fock vacuum on Σ_0 given by (IV.4) with $\mathbb{H} = \mathbb{H}_0$. In the past, this state is the so called Boulware vacuum that corresponds to the familiar concept of an empty state for a static observer. Then the quantum ground state on $\mathcal{M}$ is characterized by the two-point function for $\Phi_j \in C_0^\infty(\mathcal{M}, \mathbb{C}^4)$:

$$\omega_{\mathcal{M}}(\mathbf{\Psi}^*(\Phi_1)\mathbf{\Psi}(\Phi_2)) := \left\langle \mathbf{1}_{]0,\infty[}(\mathbb{H}_0) \int_{-\infty}^{\infty} U(0,t)\Phi_1(t)dt, \int_{-\infty}^{\infty} U(0,t)\Phi_2(t)dt \right\rangle_{\mathcal{L}_0^2}. \tag{IV.7}$$

To describe the fields near the future Black-Hole horizon, we have introduced the self-adjoint operator $\mathbb{H}_{BH}$ on the stationary space-time $\mathcal{M}_{BH} = \mathbb{R}_t \times \mathbb{R}_x \times S^2$. The quantum fields $\mathbf{\Psi}_{BH}(\Phi)$ for $\Phi \in C_0^\infty(\mathcal{M}_{BH}, \mathbb{C}^4)$ are constructed as before, by taking the Fock quantization on $\mathbb{R}_x \times S^2$ and S in (IV.2) equals to :

$$S_{BH}\Phi := \int_{-\infty}^{\infty} U_{BH}(-t)\Phi(t)dt. \tag{IV.8}$$

According to the previous definitions, the two point function given for $\Phi_j \in C_0^\infty(\mathcal{M}_{BH}, \mathbb{C}^4)$ by :

$$\omega_{BH}^{\beta,\mu}(\mathbf{\Psi}^*{}_{BH}(\Phi_1)\mathbf{\Psi}_{BH}(\Phi_2)) := \left\langle ze^{\beta\mathbb{H}_{BH}} \left(\mathbf{1} + ze^{\beta\mathbb{H}_{BH}}\right)^{-1} S_{BH}\Phi_1, S_{BH}\Phi_2 \right\rangle_{\mathcal{L}_\infty^2}, \quad z = e^{\beta\mu}, \tag{IV.9}$$

defines a thermal state on $\mathfrak{A}(\mathcal{M}_{BH})$.

In fact it will be useful to split the fields into a part outgoing to infinity, and a part falling into the black-hole, as $t \to +\infty$, by putting for $\Psi \in \mathbb{C}^4$:

$$P^{out}\Psi := (0, \Psi_2, \Psi_3, 0), \quad P^{in}\Psi := (\Psi_1, 0, 0, \Psi_4), \tag{IV.10}$$

and we have for $F \in \mathcal{L}_\infty^2$:

$$\begin{aligned} \left[e^{it\mathbb{H}_{BH}} P^{out} F\right](x,\omega) &= e^{-iqA(r_0)t} \left[P^{out}F\right](x-t,\omega), \\ \left[e^{it\mathbb{H}_{BH}} P^{in} F\right](x,\omega) &= e^{-iqA(r_0)t} \left[P^{in}F\right](x+t,\omega). \end{aligned} \tag{IV.11}$$

We are mainly concerned with the subalgebra of outgoing local observables. Given an open set $\mathcal{O} \subset \mathcal{M}_{BH}$

$$\mathfrak{A}^{out}(\mathcal{O}) = \mathbb{C}^* \textit{ algebra generated by } \mathbf{\Psi}^*_{BH}(P^{out}\Phi_1)\mathbf{\Psi}_{BH}(P^{out}\Phi_2),\quad \Phi_j \in C_0^\infty(\mathcal{O}, \mathbb{C}^4). \tag{IV.12}$$

We are interested in formulating the Hawking effect at the Black-Hole horizon, in terms of KMS state on a local algebra. According to (II.23), the points of the future horizon are reached at the infinity of the incoming radial null geodesics $\{(t, x = -t + x_0, \omega),\ t \in \mathbb{R}\}$. We introduce

$$\mathcal{M}^{in} := \left\{(t, x = -t + x_0, \omega),\ t \in \mathbb{R},\ \ 0 < x_0,\ \ \omega \in S^2\right\} \tag{IV.13}$$

Then, given $\Phi_0^j \in C_0^\infty(\mathcal{M}^{in}, \mathbb{C}^4)$, the two point function of the ground state at the horizon is characterized by

$$\lim_{T\to\infty} \omega_{\mathcal{M}} \left(\mathbf{\Psi}^*(\Phi_T^1)\mathbf{\Psi}(\Phi_T^2)\right)$$

where we put for $T > 0$:

$$\Phi_T^j(t, x, \theta, \varphi) = \Phi_0^j(t - T, x + T, \theta, \varphi). \tag{IV.14}$$

Theorem IV.1 (Main Result). *Given $\Phi_0^j \in C_0^\infty(\mathcal{M}^{in}, \mathbb{C}^4)$ we have*

$$\lim_{T\to\infty} \omega_{\mathcal{M}} \left(\mathbf{\Psi}^*(\Phi_T^1)\mathbf{\Psi}(\Phi_T^2)\right) = \omega_{BH}^{\beta,\mu} \left(\mathbf{\Psi}^*_{BH}(P^{out}\Phi_0^1)\mathbf{\Psi}_{BH}(P^{out}\Phi_0^2)\right)$$

$$+\omega^0_{\Sigma_0} \left(\mathbf{\Psi}^*_0(\Omega_{BH}S_{BH}P^{in}\Phi_0^1)\mathbf{\Psi}_0(\Omega_{BH}S_{BH}P^{in}\Phi_0^2)\right), \tag{IV.15}$$

with

$$\beta = \frac{2\pi}{\kappa_0}, \quad \mu = qA(r_0). \tag{IV.16}$$

We remark that, near the horizon, the ground state $\omega_{\mathcal{M}}$ is asymptotically equals, on $\mathfrak{A}^{out}(\mathcal{M}^{in})$, to a thermal state. Therefore this theorem expresses that the ground state, that is the Boulware vacuum in the past, has exactly the structure of the Unruh state near the future horizon (see e.g. [3], [18], [40]). According to Lemma A.2, $\omega_{BH}^{\beta,\mu} \left(\mathbf{\Psi}^*_{BH}(P^{out}\Phi_0^1)\mathbf{\Psi}_{BH}(P^{out}\Phi_0^2)\right)$ corresponds to a flux of particles leaving the vicinity of the black hole, and streaming outwards. These outgoing modes are thermally distributed with the Hawking temperature :

$$T_{BH} = \frac{\kappa_0}{2\pi}, \tag{IV.17}$$

and the charge density of the flux is equal to :

$$\varrho_{BH} = \frac{1}{\pi} q^2 A(r_0). \tag{IV.18}$$

Moreover, the state of outgoing modes is independent of the nature of the collapse, and of the boundary condition : one need not worry about the exact history of the collapse, or about interactions between the quantum field and the matter of the star subsumed in the large class of boundary conditions (III.2), (III.6) (*No Hair result*).
In the case of the Reissner-Nordstrøm Black-Hole created by a star of mass $0 < M$ and charge Q, $\mid Q \mid < M$, we have :

$$T_{BH} = \frac{\sqrt{M^2 - Q^2}}{2\pi \left(M + \sqrt{M^2 - Q^2} \right)^2}, \tag{IV.19}$$

$$\varrho_{BH} = \frac{q^2 Q}{\pi \left(M + \sqrt{M^2 - Q^2} \right)}. \tag{IV.20}$$

An important fact is that Q and ϱ_{BH} have the same sign, hence the black-hole preferentially emits fermions whose charge is of same sign as its own charge, rather than fermions of opposite charge.

V Discussion

In fact, the previous interpretation of the main result in terms of particles, is relevant only for a static observer at infinity. In the framework of the curved space-times, we have to be very carefull to describe a field with some "particles". Such a description of the field crucially depends on how the "particles" are defined and detected, i.e. the description of some state, as a vacuum state, a thermal state, etc., specifically depends on the choice of the observer.

We first consider a radially freely falling observer released from rest at infinity in the distant past. Its radial velocity is $V^r = -(1-F)^{\frac{1}{2}}$ and he carries a natural orthonormal frame $\left(F^{-1}\partial_t + V^r\partial_r, F^{-1}V^r\partial_t + \partial_r, r^{-1}\partial_\theta, (r\sin\theta)^{-1}\partial_\varphi\right)$. On the hand, the nature of the outgoing particles in the vicinity of the future horizon is ill defined for such an observer, because (IV.16) and (A.57) show that the average wavelength of the emitted quanta is comparable with the size of the hole. In some sense, this observer is inside these particles. On the other hand, a particle detector (e.g. an Unruh box) will react to states which have positive frequency with respect its propertime. Hence a geodesic detector freely falling across the future horizon, will respond to the presence of $\frac{\partial}{\partial U}$-positive frequency with $U = T - X$, i.e. of the in-modes $P^{in}\Phi$. We conclude that the response of this detector is determined by $\omega^0_{\Sigma_0}\left(\mathbf{\Psi}^*_0(\Omega_{BH}S_{BH}P^{in}\Phi^1_0)\mathbf{\Psi}_0(\Omega_{BH}S_{BH}P^{in}\Phi^2_0)\right)$. (In fact, an Unruh box is a "fluctuometer" and contains information both about the fluctuations of the field and its own motion; see e.g. [10] and a lucid contribution by Unruh in [1]). Finally an observer falling through the horizon will see no particles pouring out of the collapsing star.

In opposite, a static observer at infinity defines the particles outgoing from the black hole, as the positive frequency modes for $\frac{\partial}{\partial V}$ with $V = T + X$, which exactly are the out-modes $P^{out}\Phi$. Therefore this observer interprets $\omega_{BH}^{\beta,\mu}\left(\mathbf{\Psi}_{BH}^*(P^{out}\Phi_0^1)\,\mathbf{\Psi}_{BH}(P^{out}\Phi_0^2)\right)$ as a thermal radiation of particle and anti-particles leaving the black hole. We mention an alternative analysis based on the equivalence principle [37] : an observer in free fall defines locally a field theory and a local vacuum state that always look the same, just like flat space time field theory in an empty local neighborhood; this quantum field theory and this vacuum state are viewed by a static external observer as constantly redefined; the set of these local vacua determines the global ground state, which is interpreted by the static observer as an outgoing stream of particles.

In summary, the two terms in the right member of (IV.15), correspond to two different kinds of particles that cannot be detected by the same observer. The in-modes of the state $\omega_{\Sigma_0}^0\left(\mathbf{\Psi}_0^*(\Omega_{BH}S_{BH}P^{in}\Phi_0^1)\mathbf{\Psi}_0(\Omega_{BH}S_{BH}P^{in}\Phi_0^2)\right)$ are detected by the freely falling observer across the horizon, who cannot see the outgoing modes. Nevertheless, the gravitational disturbance produced by the collapsing star, actually induces the creation of an outgoing thermal charged flux of radiation, as seen by a static observer at infinity. Our result (IV.15) completely agrees with the analysis by Hawking [22] (see so [40]). We shall investigate, in a future work, a more precise estimate of the response of the detectors, involving the computation of the renormalized stress energy momentum tensor.

We now investigate the rather subtle role of the cosmological constant in the case of the DeSitter-Reissner-Nordstrøm Black-Hole. The spherical black-hole with mass $M > 0$, electric charge $Q \in \mathbb{R}$, in an asymptotically flat universe (cosmological constant $\Lambda = 0$), or expanding universe ($\Lambda > 0$), is described by the DeSitter-Reissner-Nordstrøm metric

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2\right)dt^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

which, by the uniqueness theorem of Birkhoff, is the unique spherically symmetric solution of the Einstein-Maxwell equations (with cosmological constant $\Lambda \geq 0$) (see e.g. [11]). Given $M > 0$ the mass of the black-hole, we deal with the radius of the black-hole horizon $r_0(Q,\Lambda)$, the Surface Gravity at the black-hole horizon $\kappa_0(Q,\Lambda)$, the Temperature of the quantum state at the horizon $T_{BH}(Q,\Lambda)$ and the Charge Density of the gaz of particles and antiparticles outgoing from the black-hole to infinity $\varrho_{BH}(Q,\Lambda)$. We could give the terrifying expressions of these quantities computed using some program of formal calculus (e.g. MAPLE is convenient). It is more interesting to investigate their behaviours with respect to the charge of the black-hole Q and the cosmological constant Λ. We deduce these results from Theorem IV.1, (IV.17) and (IV.18), by elementary but tedious calculations of expansions with respect to the small parameter.

V.1 Charged black-hole in an asymptotically flat universe (Reissner-Nordstrøm Black-Hole)

It is the case where $\Lambda = 0$ and $0 \leq \mid Q \mid < M$.

$$r_0(Q,0) = M + \sqrt{M^2 - Q^2}, \tag{V.1}$$

$$\kappa_0(Q,0) = \frac{\sqrt{M^2-Q^2}}{\left(M+\sqrt{M^2-Q^2}\right)^2}, \tag{V.2}$$

$$T_{BH}(Q,0) = \frac{\sqrt{M^2-Q^2}}{2\pi\left(M+\sqrt{M^2-Q^2}\right)^2}, \tag{V.3}$$

$$\varrho_{BH}(Q,0) = \frac{q^2 Q}{\pi\left(M+\sqrt{M^2-Q^2}\right)}. \tag{V.4}$$

We note that the radius of the black-hole and the temperature are decreasing functions of $\mid Q \mid \in [0, M[$, and we have :

$$T_{BH}(Q,0) = T_{BH}(0,0) - \frac{1}{128\pi M^5} Q^4 + O(Q^6). \tag{V.5}$$

V.2 Neutral black-hole in an expanding universe (DeSitter-Schwarzschild Black-Hole)

It is the case where $0 < 9\Lambda M^2 < 1$ and $Q = 0$ (introduced by Kottler in 1918).

$$r_0(0,\Lambda) = \frac{2}{\Lambda^{\frac{1}{2}}} \cos\left(\frac{5\pi}{3} - \frac{1}{3}\arccos(3M\Lambda^{\frac{1}{2}})\right), \tag{V.6}$$

$$\kappa_0(0,\Lambda) = \frac{1}{r_0^2(0,\Lambda)}(3M - r_0(0,\Lambda)), \tag{V.7}$$

$$T_{BH}(0,\Lambda) = \frac{1}{2\pi r_0^2(0,\Lambda)}(3M - r_0(0,\Lambda)), \tag{V.8}$$

$$\varrho_{BH}(0,\Lambda) = 0. \tag{V.9}$$

We check that :

$$r_0(0,\Lambda) = \frac{3M}{y}\frac{1+y^2}{\sqrt{3}+y},$$

with

$$y = \frac{\cos\left(\frac{1}{3}\arccos(3M\Lambda^{\frac{1}{2}})\right)}{\sqrt{1-\cos^2\left(\frac{1}{3}\arccos(3M\Lambda^{\frac{1}{2}})\right)}} \in]\sqrt{3},\infty[.$$

We deduce that the radius of the black-hole (resp. the temperature) is an increasing (resp. decreasing) function of the cosmological constant $\Lambda \in]0, \frac{1}{9M^2}[$, and we have :

$$2M < r_0(0,\Lambda) < 3M. \tag{V.10}$$

V.3 Weakly charged black-hole in an expanding universe

It is the case where $\Lambda > 0$ and $\mid Q \mid \to 0$.

$$r_0(Q,\Lambda) = r_0(0,\Lambda) - \frac{1}{2(3M - r_0(0,\Lambda))}Q^2 + O(Q^4), \tag{V.11}$$

$$\kappa_0(Q,\Lambda) = \kappa_0(0,\Lambda) + \frac{3}{2r_0^3(0,\Lambda)}\frac{r_0(0,\Lambda)-2M}{3M-r_0(0,\Lambda)}Q^2 + O(Q^4), \tag{V.12}$$

$$T_{BH}(Q,\Lambda) = T_{BH}(0,\Lambda) + \frac{3}{4\pi r_0^3(0,\Lambda)}\frac{r_0(0,\Lambda)-2M}{3M-r_0(0,\Lambda)}Q^2 + O(Q^4), \tag{V.13}$$

$$\varrho_{BH}(Q,\Lambda) = \frac{q^2Q}{\pi r_0(0,\Lambda)} + \frac{q^2Q^3}{2\pi r_0^2(0,\Lambda)(3M-r_0(0,\Lambda))} + O(Q^5). \tag{V.14}$$

By (V.10) we see that a small charge of the black-hole decreases its radius as in the Reissner-Nordstrøm case. But in opposite with (V.5), the temperature is an increasing function of the charge near zero.

V.4 Black-hole in a weakly expanding universe

It is the case where $0 \leq\mid Q \mid< M$ and $\Lambda \to 0^+$. We obtain :

$$r_0(Q,\Lambda) = r_0(Q,0) + \frac{\left(M+\sqrt{M^2-Q^2}\right)^4}{6\sqrt{M^2-Q^2}}\Lambda + O(\Lambda^2), \tag{V.15}$$

$$\kappa_0(Q,\Lambda) = \kappa_0(Q,0) + \frac{\Lambda}{6\sqrt{M^2-Q^2}}\left(-4M^2 - 4M\sqrt{M^2-Q^2} + 5Q^2\right) + O(\Lambda^2), \tag{V.16}$$

$$T_{BH}(Q,\Lambda) = T_{BH}(Q,0) + \frac{\Lambda}{12\pi\sqrt{M^2-Q^2}}\left(-4M^2 - 4M\sqrt{M^2-Q^2} + 5Q^2\right) + O(\Lambda^2), \tag{V.17}$$

$$\varrho_{BH}(Q,\Lambda) = \varrho_{BH}(Q,0)\left(1 - \frac{\left(M+\sqrt{M^2-Q^2}\right)^3}{6\sqrt{M^2-Q^2}}\Lambda\right) + O(\Lambda^2). \tag{V.18}$$

We constat that the radius of the black-hole is an increasing function of the cosmological constant near zero, and the absolute value of the charge density is decreasing. As regards the temperature, it appears that the ratio of the charge with respect to the mass plaies a rather subtle role : if the charge of the black-hole is not too large, more precisely

$$0 \leq \mid Q \mid < \sqrt{\frac{24}{25}}M, \tag{V.19}$$

the temperature is a decreasing function of Λ. But for a strongly charged black-hole, i.e.

$$\sqrt{\frac{24}{25}}M < \mid Q \mid < M, \tag{V.20}$$

the temperature is an *increasing* function of Λ. For the critical value

$$\mid Q \mid = \sqrt{\frac{24}{25}}M, \tag{V.21}$$

we evaluate that

$$\kappa_0\left(Q = \pm\sqrt{\frac{24}{25}}M, \Lambda\right) = \frac{5}{36M} - \frac{324}{125}M^3\Lambda^2 + O(\Lambda^3), \tag{V.22}$$

$$T_{BH}\left(Q = \pm\sqrt{\frac{24}{25}}M, \Lambda\right) = \frac{5}{72\pi M} - \frac{162}{125\pi}M^3\Lambda^2 + O(\Lambda^3), \tag{V.23}$$

hence the temperature is decreasing with respect to Λ again. At our knowledge, this phenomenon had not been previously noted.

VI Proof of the asymptotic estimates

This part is devoted to the proofs of the results of Parts III and IV. Taking advantage of the spherical invariance we reduce our 3D+1 problem to a 1D+1 problem. Therefore the strategy consists in using the expansion in spinoidal spherical harmonics.

It is well known (see e.g. [33] , [34]) that there exists two Hilbert bases $T^l_{\frac{1}{2},n}(\varphi,\theta)$, $T^l_{-\frac{1}{2},n}(\varphi,\theta)$ of $L^2(S^2)$,

$$l \in \mathbb{N}+\frac{1}{2},\;\; n\in\frac{1}{2}\mathbb{Z},\;\; l-\mid n\mid\in\mathbb{N}, \tag{VI.1}$$

such that :

$$\left(\frac{\partial}{\partial\theta}+\frac{1}{2\tan\theta}\right)T^{\ell}_{\pm\frac{1}{2},n}=\pm\frac{n}{\sin\theta}T^{\ell}_{\pm\frac{1}{2},n}-i\left(l+\frac{1}{2}\right)T^{\ell}_{\mp\frac{1}{2},n}, \tag{VI.2}$$

$$\frac{\partial}{\partial\varphi}T^{\ell}_{\pm\frac{1}{2},n}=-inT^{\ell}_{\pm\frac{1}{2},n}. \tag{VI.3}$$

Then we have

$$\left[L^2(S^2)\right]^4=\bigoplus_{\ell,n}L^2_{\ell,n},\quad L^2_{\ell,n}:=\mathbb{C}T^{\ell}_{-\frac{1}{2},n}\oplus\mathbb{C}T^{\ell}_{+\frac{1}{2},n}\oplus\mathbb{C}T^{\ell}_{-\frac{1}{2},n}\oplus\mathbb{C}T^{\ell}_{+\frac{1}{2},n}. \tag{VI.4}$$

If we denote $\Pi_{l,n}$ the projector from $\left[L^2(S^2)\right]^4$ onto $L^2_{\ell,n}$, a spin field $\Phi(t,x,\varphi,\theta)$ is solution of the Dirac equation (III.13) satisfying the boundary condition (III.14), (III.6), iff $u(t,x)$ given by

$$u(t,x):=e^{itqA(r_0)}e^{i\frac{\alpha_{\ell,n}}{2}\gamma^5}\Pi_{\ell,n}\left[\Phi(t,x,.)\right], \tag{VI.5}$$

is solution of

$$\frac{\partial u}{\partial t}+L\frac{\partial u}{\partial x}+iV_{\ell,n}u=0,\;\; t\in\mathbb{R},\;\; x>z(t), \tag{VI.6}$$

$$\forall t\in\mathbb{R},\;\; u_2(t,x=z(t))=\sqrt{\frac{1+\dot{z}(t)}{1-\dot{z}(t)}}u_4(t,x=z(t)),$$

$$u_3(t,x=z(t))=-\sqrt{\frac{1+\dot{z}(t)}{1-\dot{z}(t)}}u_1(t,x=z(t)), \tag{VI.7}$$

Here L is the matrix

$$L=\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}, \tag{VI.8}$$

and the potential is given by :

$$V_{\ell,n}(x) = q(A - A(r_0)) + mF^{\frac{1}{2}} \begin{pmatrix} 0 & 0 & ie^{i\alpha_{\ell,n}} & 0 \\ 0 & 0 & 0 & ie^{i\alpha_{\ell,n}} \\ -ie^{-i\alpha_{\ell,n}} & 0 & 0 & 0 \\ 0 & -ie^{-i\alpha_{\ell,n}} & 0 & 0 \end{pmatrix}$$

$$+ \left(l + \frac{1}{2}\right) \frac{F^{\frac{1}{2}}}{r} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{VI.9}$$

Therefore we have transformed our 3D+1 problem into a family of 1D+1 simple problems (VI.6), where the potential $V_{\ell,n}$ has nice properties of asymptotic behaviours as $x \to \pm\infty$. To get now the key estimate (III.34), it will be sufficient to prove a similar asymptotic result for these 1D+1 problems (Theorem VI.5).

VI.1 One dimensional problem

We consider the mixed hyperbolic problem with unknown $u = {}^t(u_1, u_2, u_3, u_4)$:

$$\frac{\partial u}{\partial t} + L\frac{\partial u}{\partial x} + iVu = 0, \quad t \in \mathbb{R}, \quad x > z(t), \tag{VI.10}$$

$$\forall t \in \mathbb{R}, u_2(t, x = z(t)) = \lambda(t) u_4(t, x = z(t)), u_3(t, x = z(t)) = -\lambda(t) u_1(t, x = z(t)), \tag{VI.11}$$

$$\forall x > z(s), \quad u(s, x) = f(x). \tag{VI.12}$$

Here V is a matrix valued map of x satisfying :

$$V \in C^1\left(\mathbb{R}_x; \mathbb{C}^{4\times 4}\right), \tag{VI.13}$$

$$\forall x \in \mathbb{R}, \quad V^*(x) = V(x). \tag{VI.14}$$

Moreover we assume the following asymptotic behaviours : there exists $\varepsilon > 0$, $C > 0$, $\mu \geq 0$, $\varrho \in \mathbb{R}$ and two hermitian matrices Γ, V_∞ such that :

$$\int_0^\infty \sup_{x<-t} \{| V(x) | + | V'(x) |\} dt < \infty, \tag{VI.15}$$

$$\lim_{x\to+\infty} V(x) = V_\infty, \quad \lim_{x\to+\infty} \frac{d}{dx} V(x) = 0, \tag{VI.16}$$

$$V_\infty = \mu\Gamma - \varrho Id, \quad \Gamma L + L\Gamma = 0, \quad \Gamma^2 = Id, \tag{VI.17}$$

and if $\mu > 0$, there exists $\eta \in]0, 1]$ such that :

$$\forall f \in \mathbb{C}^4, \; f_1 + f_3 = f_2 - f_4 = 0 \Rightarrow i < \Gamma L f, f >_{\mathbb{C}^4} \geq (\eta^2 - 1) \mid f \mid^2 . \tag{VI.18}$$

The boundary condition is given by functions $\lambda(t)$ and $z(t)$ satisfying :

$$\left\{\begin{array}{l} z\in C^2(\mathbb{R}),\\ \forall t\leq 0,\ z(t)=z(0)<0,\\ \forall t\in\mathbb{R},\ -1<\dot{z}(t)\leq 0,\\ z(t)=-t-Ae^{-2\kappa t}+\zeta(t),\ A>0,\ \ \kappa>0,\\ \mid\zeta(t)\mid+\mid\dot{\zeta}(t)\mid=O(e^{-4\kappa t}),\ t\longrightarrow+\infty, \end{array}\right. \tag{VI.19}$$

$$\lambda(t):=\sqrt{\frac{1+\dot{z}(t)}{1-\dot{z}(t)}}. \tag{VI.20}$$

We introduce the function spaces :

$$0\leq t,\ \ L^p_t:=\left[L^p(]z(t),\infty[_x,dx)\right]^4,\ \ 1\leq p\leq\infty, \tag{VI.21}$$

$$L^p_\infty:=\left[L^p(\mathbb{R}_x,dx)\right]^4. \tag{VI.22}$$

For $s<t$, L^p_s is naturally embedded in L^p_t; this amounts to extending the function by zero inside $[z(t),z(s)]$. For $0\leq t\leq\infty$ we denote $\|\ .\ \|$ the L^2_t norm and $|\ .\ |_\infty$ the norm in L^∞_t. We consider some spaces of more regular data :

$$0\leq t\leq\infty,\ \ k\in\mathbb{N}^*,\ \ H^k_t:=\left\{f\in L^2_t;\frac{d^k}{dx^k}f\in L^2_t\right\} \tag{VI.23}$$

and we denote $\|\ .\ \|_k$ the norm of H^k_t defined by :

$$\|\ f\ \|_k=\left(\int_{z(t)}^\infty\mid f(x)\mid^2+\mid\frac{d^k}{dx^k}f(x)\mid^2\,dx\right)^{\frac{1}{2}}. \tag{VI.24}$$

Thanks to the Sobolev embedding

$$H^1_t\subset\left[C^0([z(t),\infty[)\right]^4\cap L^\infty_t, \tag{VI.25}$$

we can introduce the family of densely defined operators on L^2_t :

$$\mathbb{H}_{V,t}:=iL\frac{d}{dx}-V, \tag{VI.26}$$

with domain

$$W^1_t:=\left\{f\in H^1_t;\ f_2(z(t))=\lambda(t)f_4(z(t)),\ \ f_3(z(t))=-\lambda(t)f_1(z(t))\right\},W^1_\infty:=H^1_\infty. \tag{VI.27}$$

Lemma VI.1. *The operator* $i\mathbb{H}_{V,t}$ *is maximal accretive for any* t*, and skew-adjoint for* $t \leq 0$ *and* $t = \infty$*. Moreover the point spectrum of* $\mathbb{H}_{V,\infty}$ *is empty.*

Proof of Lemma VI.1. Let f be in W_t^1. Since the Sobolev inequality assures that

$$f \in W_t^1 \Rightarrow \lim_{x\to\infty} \mid f(x) \mid = 0, \tag{VI.28}$$

an integration by part gives :

$$2\Re < i\mathbb{H}_{V,t}f, f >_{L_t^2} = \left(\lambda^2(t) - 1\right)\left(\mid f_1(z(t)) \mid^2 + \mid f_4(z(t)) \mid^2\right) = \dot{z}(t) \mid f(z(t) \mid^2 \leq 0, \tag{VI.29}$$

hence $i\mathbb{H}_{V,t}$ is accretive. Now we easily check that its adjoint $(i\mathbb{H}_{V,t})^*$ is defined in the sense of distributions by

$$(i\mathbb{H}_{V,t})^* f = -i\mathbb{H}_{V,t}f,$$

with domain :

$$D\left((i\mathbb{H}_{V,t})^*\right) = \left\{f \in H_t^1;\ f_2(z(t)) = \frac{1}{\lambda(t)}f_4(z(t)),\quad f_3(z(t)) = -\frac{1}{\lambda(t)}f_1(z(t))\right\}.$$

Hence we have again :

$$2\Re < (i\mathbb{H}_{V,t})^* f, f >_{L_t^2} = \dot{z}(t) \mid f(z(t) \mid^2 \leq 0.$$

Since both $i\mathbb{H}_{V,t}$ and its adjoint are accretive, we conclude that $i\mathbb{H}_{V,t}$ is maximal accretive and, if $\lambda(t) = 1$, skew-adjoint. In the same manner, $\mathbb{H}_{V,\infty}$ is selfadjoint on L_∞^2. If $u \in L_\infty^2$ is an eigenvector of $\mathbb{H}_{V,\infty}$ for the eigenvalue $\lambda \in \mathbb{R}$, then $v(x) = e^{-i\lambda Lx}u(x)$ is a solution in H_∞^1 to :

$$v'(x) + iLe^{i\lambda Lx}Ve^{-i\lambda Lx}v = 0.$$

Since $v(x) \to 0$ as $x \to -\infty$, and $\int_{-\infty}^0 \mid V(x) \mid dx < \infty$, we conclude by the Gronwall lemma that $v = 0$. □

The solution of (VI.10), (VI.11), (VI.12), is formally expressed via a propagator $U_V(t,s)$:

$$u(t) = U_V(t,s)f. \tag{VI.30}$$

More precisely the mixed problem is solved by the following :

Proposition VI.2. *For* $f \in W_s^1$*, there exists a unique solution* $u \in C^1(\mathbb{R}_t; L_\infty^2)$ *of (VI.10), (VI.11), (VI.12) satisfying for any real* t *:*

$$u(t) \in W_t^1. \tag{VI.31}$$

Moreover we have :

$$\| u(t) \| = \| f \|, \tag{VI.32}$$

and $U_V(t,s)$ *can be extended in an isometric strongly continuous propagator from* L^2_s *onto* L^2_t *satisfying*

$$\forall f \in W^1_s, \ \ (t \mapsto U_V(t,s)f) \in C^1\left(\mathbb{R}_t, L^2_\infty\right), \ \ \frac{d}{dt}U_V(t,s)f = i\mathbb{H}_{V,t}U_V(t,s)f, \tag{VI.33}$$

$$\forall f \in \left[C^\infty_0(]z(s_0),\infty[)\right]^4,$$

$$\exists h>0, (s \mapsto U_V(t,s)f) \in C^1\left(]s_0-h, s_0+h[, L^2_t\right), \frac{d}{ds}U_V(t,s)f = -iU_V(t,s)\mathbb{H}_{V,s}f, \tag{VI.34}$$

$$\big(x > R \Rightarrow f(x) = 0\big) \Rightarrow \big(x > R + \mid t-s \mid \Rightarrow [U_V(t,s)f](x) = 0\big). \tag{VI.35}$$

Proof of Proposition VI.2. The uniqueness follows from the conservation of the norm (VI.32) which is established by evaluating

$$\frac{d}{dt}\int_{z(t)}^\infty \mid u(t,x) \mid^2 dx = 2\Re < i\mathbb{H}_{V,t}u(t), u(t) >_{L^2_t} - \dot z(t) \mid u(t,z(t)) \mid^2 = 0.$$

To prove the existence we introduce the operators :

$$R(t) := \begin{pmatrix} \cos\theta(t) & 0 & \sin\theta(t) & 0 \\ 0 & \cos\theta(t) & 0 & \sin\theta(t) \\ -\sin\theta(t) & 0 & \cos\theta(t) & 0 \\ 0 & -\sin\theta(t) & 0 & \cos\theta(t) \end{pmatrix}, \ \ \theta(t) := \arctan\lambda(t),$$

$$\mathcal{T}(t) : f \in L^2_0 \mapsto \mathcal{T}(t)f \in L^2_t, \ \ [\mathcal{T}(t)f](x) = f(x - z(t) + z(0)).$$

We remark that

$$[R(t)]^{-1}\dot R(t) = -i\dot\theta(t)\boldsymbol{\gamma}^0,$$

$$\mathcal{T} \in C^1\left(\mathbb{R}_t, \mathcal{L}\left(\left[C^1_0(\mathbb{R}_x)\right]^4, \left[C^0_0(\mathbb{R}_x)\right]^4\right)\right), \ \ \dot{\mathcal{T}}(t) = -\dot z(t)\mathcal{T}(t)\partial_x.$$

Then u is a solution to the problem iff

$$w(t) = [R(t)]^{-1}[\mathcal{T}(t)]^{-1}u(t)$$

is solution to

$$\partial_t w + \mathbb{A}(t)w = 0, \ \ x > z(0),$$

$$w_2(t,z(0)) = w_3(t,z(0)) = 0,$$

where

$$\mathbb{A}(t) :=$$
$$\left([R(t)]^{-1}[\mathcal{T}(t)]^{-1}LR(t)\mathcal{T}(t) - \dot{z}(t)\right)\partial_x - i\dot{\theta}(t)\boldsymbol{\gamma}^0 + i[R(t)]^{-1}[\mathcal{T}(t)]^{-1}VR(t)\mathcal{T}(t).$$

We can easily show that the operator $\mathbb{A}(t)$ with dense domain (independent of t)

$$D(\mathbb{A}(t)) = \left\{f \in H_0^1;\ f_2(z(0)) = f_3(z(0)) = 0\right\}$$

is skew-adjoint on L_0^2. Moreover, since z is C^2 and V is uniformly continuous on $\mathbb{R}$, the map $t \mapsto \mathbb{A}(t)$ is norm continuous from $\mathbb{R}$ to $\mathcal{L}(D(\mathbb{A}(0)), L_0^2)$. Then the Theorems of T. Kato [28] assure that there exists a unique strongly continuous propagator $S(t,s)$ on L_0^2 such that :

$$w(t) = S(t,s)w(s),$$

and for $f \in D(\mathbb{A}(0))$, $S(t,s)f \in D(\mathbb{A}(0))$ is a strongly differentiable map from $\mathbb{R}_t \times \mathbb{R}_s$ to L_0^2 satisfying :

$$\frac{d}{dt}S(t,s)f = -\mathbb{A}(t)S(t,s)f, \quad \frac{d}{ds}S(t,s)f = S(t,s)\mathbb{A}(s)f.$$

Then the propagator defined by :

$$U_V(t,s) = R(t)\mathcal{T}(t)S(t,s)[R(s)]^{-1}[\mathcal{T}(s)]^{-1},$$

satisfies (VI.33) and (VI.34). To establish (VI.35) we check that :

$$\frac{d}{dt}\int_{R+|t-s|}^{\infty} \mid u(t,x) \mid^2 dx =$$
$$-(\mid u_2 \mid^2 + \mid u_3 \mid^2)(t, R+t-s)\mathbf{1}_{[0,\infty[}(t-s) + (\mid u_1 \mid^2 + \mid u_4 \mid^2)(t, R+s-t)\mathbf{1}_{[0,\infty[}(s-t).$$

□

It will be useful to have the explicit form of the free propagator $U_0(s,t)$:

Lemma VI.3. *For $t \leq s$, given $f \in L_s^2$, $u(t) = U_0(t,s)f$ is given by*

$$x > z(t) \Rightarrow u_2(t,x) = f_2(x-t+s), \quad u_3(t,x) = f_3(x-t+s), \tag{VI.36}$$

$$x > s + z(s) - t \Rightarrow u_1(t,x) = f_1(x+t-s), \quad u_4(t,x) = f_4(x+t-s), \tag{VI.37}$$

$$z(t) < x < s + z(s) - t \Rightarrow u_1(t,x) = -\sqrt{\frac{1-\dot{z}(\tau(x+t))}{1+\dot{z}(\tau(x+t))}}f_3(x+t+s-2(\tau(x+t))), \tag{VI.38}$$

$$z(t) < x < s + z(s) - t \Rightarrow u_4(t,x) = \sqrt{\frac{1-\dot{z}(\tau(x+t))}{1+\dot{z}(\tau(x+t))}} f_2(x+t+s-2(\tau(x+t))), \tag{VI.39}$$

where function τ is the unique solution of

$$z(0) \leq y < 0, \ \ z(\tau(y)) + \tau(y) = y, \tag{VI.40}$$

and satisfies

$$\tau(y) = -\frac{1}{2\kappa}\ln(-y) + \frac{1}{2\kappa}\ln(A) + O(y), \ \ y \to 0^-, \tag{VI.41}$$

$$1 + \dot{z}(\tau(y)) = -2\kappa y + O(y^2), \ \ y \to 0^-. \tag{VI.42}$$

Proof of Lemma VI.3. (VI.36) and (VI.37) are consequences of (VI.10), (VI.12) with $V = 0$. We have also by (VI.11) :

$$u_{1(4)}(t,x) = u_{1(4)}(\tau(x+t), z(\tau(x+t))) = -(+)\frac{1}{\lambda(\tau(x+t))} u_{3(2)}(\tau(x+t), z(\tau(x+t))).$$

Hence (VI.39) and (VI.38) follow from (VI.36). The properties of function τ are established in [2], Proposition I.2. □

We denote $U_0(t)$ the unitary group on L^2_∞ solving the Cauchy problem associated with the hyperbolic system on $\mathbb{R}^2$:

$$\frac{\partial u}{\partial t} + L\frac{\partial u}{\partial x} = 0, \ \ t \in \mathbb{R}, \ \ x \in \mathbb{R}, \tag{VI.43}$$

with infinitesimal generator

$$\mathbb{H}_{0,\infty} := iL\partial_x, \ \ D(\mathbb{H}_{0,\infty}) = H^1_\infty. \tag{VI.44}$$

We introduce the subspaces of the left and right propagating fields :

$$L^2_{in} = \left\{ f \in L^2_\infty; \ \ f_2 = f_3 = 0, \ \ x < 0 \Rightarrow f(x) = 0, \right\}, \tag{VI.45}$$

$$L^2_{out} = \left\{ f \in L^2_\infty; \ \ f_1 = f_4 = 0, \ \ x < 0 \Rightarrow f(x) = 0, \right\}. \tag{VI.46}$$

Proposition VI.4. *Assume $f \in L^2_{in}$. Then the strong limit :*

$$\Omega^{in}_V f = \lim_{T\to+\infty} U_V(0,T)U_0(T)f \ \ in \ \ L^2_0 \tag{VI.47}$$

exists and defines an isometry from L^2_{in} to L^2_0.

Proof of Proposition VI.4. Since $U_V(0,T)$ and $U_0(T)$ are isometric it is sufficient to consider

$$f \in L^2_{in} \cap [C_0^\infty(]0,R[)]^4 .$$

For such a data we have

$$U_0(T)f = U_0(T,0)f = f(T+.),$$

hence applying (VI.33), (VI.34) we obtain :

$$\frac{d}{dT}\left(U_V(0,T)U_0(T)f\right) = U_V(0,T)VU_0(T)f,$$

and by (VI.15) we conclude that

$$\int_0^\infty \parallel \frac{d}{dT}\Big(U_V(0,T)U_0(T)f\Big) \parallel dT \leq \int_{-R}^\infty \sup\{\mid V(x)\mid;\;\; x<-T\}dT < \infty.$$

The result follows from the Cook's method. □

We now state the main result of this part. Given $f \in L^2_\infty$ we denote

$$f^T(x) = f(x+T). \tag{VI.48}$$

Theorem VI.5. *For $f \in L^2_{out}$ we have :*

$$\lim_{T\to\infty} \parallel \mathbf{1}_{[\varrho,\infty[}(\mathbb{H}_{V,0})\left(U_V(0,T)f^T\right)\parallel^2 =< f, e^{\frac{2\pi}{\kappa}\mathbb{H}_{0,\infty}}\left(1+e^{\frac{2\pi}{\kappa}\mathbb{H}_{0,\infty}}\right)^{-1} f >_{L^2_\infty} . \tag{VI.49}$$

The proof is rather long and technical, so we begin by sketching the main steps of the method. The idea consists in comparing $U_V(0,T)f^T$ with f^*_T defined by :

$$f^*_T(x) = \begin{pmatrix} -\mid \kappa x\mid^{-\frac{1}{2}}\; f_3(2T+\frac{1}{\kappa}\ln(-x)-\frac{1}{\kappa}\ln(A)) \\ 0 \\ 0 \\ \mid \kappa x\mid^{-\frac{1}{2}}\; f_2(2T+\frac{1}{\kappa}\ln(-x)-\frac{1}{\kappa}\ln(A)) \end{pmatrix}, \tag{VI.50}$$

i.e. we rigorously justify the approximation of the geometrical optics.
We prove in Lemma VI.8 that $U_0(0,T)f^T \sim f^*_T$. We explicitly calculate $\parallel \mathbf{1}_{[0,\infty[}(\mathbb{H}_{0,0})\, f^*_T \parallel$ by Fourier transform in Lemma VI.6, and we show in Lemma VI.7 that $\parallel \mathbf{1}_{[\varrho,\infty[}(\mathbb{H}_{V_\infty,0})\, f^*_T \parallel$ tends to the same limit. To replace $\mathbf{1}_{[\varrho,\infty[}(\mathbb{H}_{V_\infty,0})$ by $\mathbf{1}_{[\varrho,\infty[}(\mathbb{H}_{V,0})$ we establish in Lemma VI.10 that the difference between these both operators is compact. At last, we make the link with $U_V(0,T)f^T$ and $U_0(0,T)f^T$ by using the fast decay of V as $x \to -\infty$.

Lemma VI.6. *For $f \in L^2_{out}$ we have :*

$$f^*_T \rightharpoonup 0 \ \ in \ \ L^2_0 - weak*, \ \ T \to \infty, \tag{VI.51}$$

moreover, for any $T > 0$ we have :

$$\| f^*_T \| = \| f \|, \tag{VI.52}$$

$$\| \mathbf{1}_{[0,\infty[}(\mathbb{H}_{0,0}) f^*_T \|^2 = < f, e^{\frac{2\pi}{\kappa}\mathbb{H}_{0,\infty}} \left(1 + e^{\frac{2\pi}{\kappa}\mathbb{H}_{0,\infty}}\right)^{-1} f >_{L^2_\infty} . \tag{VI.53}$$

Proof of Lemma VI.6. We may assume that f is smooth and compactly supported. By the change of variables $y = 2T + \frac{1}{\kappa}\ln(-x) - \frac{1}{\kappa}\ln(A)$ we directly obtain (VI.52) and

$$\int | f^*_T(x) | \, dx = e^{-\kappa T}\sqrt{A\kappa}\int e^{\frac{\kappa}{2}y} | f(y) | \, dy \to 0, \ \ T \to \infty,$$

which implies (VI.51).

To prove the key identity (VI.53) we introduce a map P from L^2_0 into L^2_∞ by putting for $g = Pf$:

$$x \leq z(0) \Rightarrow g(x) = f(x), \tag{VI.54}$$

$$x \leq z(0) \Rightarrow \begin{cases} g_1(x) = -f_3(2z(0) - x), \\ g_2(x) = f_4(2z(0) - x), \\ g_3(x) = -f_1(2z(0) - x), \\ g_4(x) = f_2(2z(0) - x). \end{cases} \tag{VI.55}$$

For $f \in W^1_0$ we have :

$$L\frac{d}{dx}Pf = PL\frac{d}{dx}f. \tag{VI.56}$$

Hence, using the Fourier transform $\mathcal{F}(\varphi) = \hat{\varphi}$, we see that $\mathbb{H}_{0,0}$ is unitarily equivalent to the operator :

$$-\frac{\xi}{\sqrt{4\pi}}L,$$

densely defined on the Hilbert space :

$$L^2_* := \left\{\hat{f} \in \left[L^2(\mathbb{R}_\xi)\right]^4 ; \hat{f}_1(\xi) + \hat{f}_3(-\xi) = \hat{f}_2(\xi) - \hat{f}_4(-\xi) = 0\right\}.$$

We have

$$\mathbf{1}_{[0,\infty[}(\mathbb{H}_{0,0}) = P^{-1}\mathcal{F}^{-1}\begin{pmatrix} \mathbf{1}_{[0,\infty[}(\xi) & 0 & 0 & 0 \\ 0 & \mathbf{1}_{]-\infty,0]}(\xi) & 0 & 0 \\ 0 & 0 & \mathbf{1}_{]-\infty,0]}(\xi) & 0 \\ 0 & 0 & 0 & \mathbf{1}_{[0,\infty[}(\xi) \end{pmatrix}\mathcal{F}P. \tag{VI.57}$$

Hence :

$$2\pi \parallel \mathbf{1}_{[0,\infty[}(\mathbb{H}_{0,0})\, f_T^* \parallel^2 = \int_0^\infty \mid \mathcal{F}(f_T^*)(\xi) \mid^2 d\xi$$

$$= \lim_{\varepsilon\to 0^+} A\kappa \int_0^\infty \mid \int_{-\infty}^\infty e^{i(A+i\varepsilon)\zeta e^{\kappa J y}} e^{\frac{\kappa}{2}y} f(y)dy \mid^2 d\zeta$$

$$= \lim_{\varepsilon\to 0^+} \frac{A\kappa}{2}\int_{\mathbb{R}\times\mathbb{R}} \frac{1}{\varepsilon\cosh\left[\frac{\kappa}{2}(y_1-y_2)\right] - iA\sinh\left[\frac{\kappa}{2}(y_1-y_2)\right]} f(y_1).\bar{f}(y_2)dy_1dy_2 \tag{VI.58}$$

$$= \lim_{\varepsilon\to 0^+} \frac{A\kappa}{4\pi}\int_{-\infty}^\infty \mid \hat{f}(\xi) \mid^2 \mathcal{F}\left(\frac{1}{\varepsilon\cosh(\frac{\kappa}{2}x) - iA\sinh(\frac{\kappa}{2}x)}\right)(-\xi)d\xi.$$

Now given $\varepsilon \neq 0$, $\xi < 0$ and $N > 0$, $M > 0$, we evaluate

$$\oint h(x)dx, \quad h(x) := \frac{e^{-ix\xi}}{\varepsilon\cosh(\frac{\kappa}{2}x) - iA\sinh(\frac{\kappa}{2}x)},$$

along the path

$$\{-N \le \Re x \le N,\ \Im x = 0, M\} \cup \{0 \le \Im x \le M,\ \Re x = \pm N\}.$$

First we have :

$$\left|\int_{\pm N}^{\pm N+iM} h(x)dx\right| \le Ce^{-\frac{\kappa}{2}N}\int_0^\infty e^{x\xi}dx \to 0, \quad N\to\infty,$$

$$\left|\int_{-N+iM}^{N+iM} h(x)dx\right| \le Ce^{M\xi}\int_{-\infty}^\infty e^{-\frac{\kappa}{2}|x|}dx \to 0, \quad M\to\infty.$$

We deduce that

$$\int_{-\infty}^\infty h(x)dx = 2i\pi\sum_{n=1}^\infty \rho_n(\varepsilon)$$

where $\rho_n(\varepsilon)$ are the residues of $h(x)$ at the poles $z_n(\varepsilon) \in \{z \in \mathbb{C}; \Im z > 0\}$. We easily check that :

$$z_n(\varepsilon) = \frac{2i}{\kappa}\left(n\pi - \arctan(\frac{\varepsilon}{A})\right),$$

$$\sup_{1\le n} \mid \rho_n(\varepsilon) - \frac{2i}{A\kappa}(-1)^n e^{\frac{2n\pi}{\kappa}\xi} \mid \le C\varepsilon,$$

hence we get that for $\xi < 0$ we have :

$$\left| \mathcal{F}\left(\frac{1}{\varepsilon\cosh(\frac{\kappa}{2}x) - iA\sinh(\frac{\kappa}{2}x)}\right)(\xi) - \frac{4\pi}{A\kappa}e^{\frac{2\pi}{\kappa}\xi}\left(1+e^{\frac{2\pi}{\kappa}\xi}\right)^{-1}\right| \leq C\varepsilon.$$

In the same manner, for $\xi > 0$, we choose $M < 0$ and considering the poles $z_n(\varepsilon) \in \{z \in \mathbb{C}; \Im z < 0\}$ we obtain :

$$\int_{-\infty}^{\infty} h(x)dx = 2i\pi \sum_{n=0}^{-\infty} \rho_n(\varepsilon),$$

$$\sup_{n\leq 0} \mid \rho_n(\varepsilon) - \frac{2i}{A\kappa}(-1)^n e^{-\frac{2n\pi}{\kappa}\xi} \mid \leq C\varepsilon,$$

$$\left| \mathcal{F}\left(\frac{1}{\varepsilon\cosh(\frac{\kappa}{2}x) - iA\sinh(\frac{\kappa}{2}x)}\right)(\xi) - \frac{4\pi}{A\kappa}\left(1+e^{-\frac{2\pi}{\kappa}\xi}\right)^{-1}\right| \leq C\varepsilon.$$

Finally we conclude that :

$$\mathcal{F}\left(\frac{1}{0 - iA\sinh(\frac{\kappa}{2}x)}\right)(\xi) = \frac{4\pi}{A\kappa}e^{\frac{2\pi}{\kappa}\xi}\left(1+e^{\frac{2\pi}{\kappa}\xi}\right)^{-1}, \tag{VI.59}$$

and

$\parallel \mathbf{1}_{[0,\infty[}(\mathbb{H}_{0,0})\, f_T^* \parallel^2 =$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{2\pi}{\kappa}\xi}\left(1+e^{-\frac{2\pi}{\kappa}\xi}\right)^{-1} \mid \hat{f}(\xi) \mid^2 d\xi = < f, e^{\frac{2\pi}{\kappa}\mathbb{H}_{0,\infty}}\left(1+e^{\frac{2\pi}{\kappa}\mathbb{H}_{0,\infty}}\right)^{-1} f >_{L^2_\infty}.$$

□

Lemma VI.7. *For $f \in L^2_{out}$, we have :*

$$\left| \parallel \mathbf{1}_{[\varrho,\infty[}(\mathbb{H}_{V_\infty,0})\, f_T^* \parallel - \parallel \mathbf{1}_{[0,\infty[}(\mathbb{H}_{0,0})\, f_T^* \parallel \right| \to 0, \;\; T \to \infty. \tag{VI.60}$$

Proof of Lemma VI.7. We have :

$$\mathbf{1}_{[\varrho,\infty[}(\mathbb{H}_{V_\infty,0}) = \mathbf{1}_{[0,\infty[}(\mathbb{H}_{\mu\Gamma,0}). \tag{VI.61}$$

We consider the case $\mu > 0$. We introduce the self-adjoint operators on L^2_∞ :

$$\mathbb{H}_{\mu\Gamma} := iL\partial_x - \mu\Gamma, \;\; D(\mathbb{H}_{\mu\Gamma}) = H^1_\infty, \tag{VI.62}$$

$$\mathbb{K}_{\mu\Gamma} := \mathbb{H}^-_{\mu\Gamma,0} \oplus \mathbb{H}_{\mu\Gamma,0}, \;\; D(\mathbb{K}_{\mu\Gamma,0}) = D(\mathbb{H}^-_{\mu\Gamma,0}) \oplus D(\mathbb{H}_{\mu\Gamma,0}), \tag{VI.63}$$

where $\mathbb{H}^-_{\mu\Gamma,0}$ is the self-adjoint operator on $\left[L^2(-\infty,z(0))\right]^4$ defined by :

$$\mathbb{H}^-_{\mu\Gamma,0} := iL\partial_x - \mu\Gamma, \tag{VI.64}$$

$$D(\mathbb{H}^-_{\mu\Gamma,0}) = \Big\{ f \in \left[L^2(-\infty,z(0))\right]^4 ; f' \in \left[L^2(-\infty,z(0))\right]^4 ,$$
$$f_1(z(0)) = f_3(z(0)), f_2(z(0)) = -f_4(z(0)) \Big\} . \tag{VI.65}$$

We have :

$$0 \oplus \mathbf{1}_{[0,\infty[}\left(\mathbb{H}_{\mu\Gamma,0}\right)\left(f^*_T\right) = \mathbf{1}_{[0,\infty[}\left(\mathbb{K}_{\mu\Gamma}\right)\left(0 \oplus f^*_T\right). \tag{VI.66}$$

For $f \in D(\mathbb{H}_{\mu\Gamma})$ we evaluate :

$$\| \mathbb{H}_{\mu\Gamma} f \|^2 = \| f' \|^2 + \mu^2 \| f \|^2,$$

and for $f \in D(\mathbb{H}_{\mu\Gamma,0})$ we get :

$$\| \mathbb{H}_{\mu\Gamma,0} f \|^2 = \| f' \|^2 + \mu^2 \| f \|^2 + i\mu < \Gamma L f(z(0)), f(z(0)) >_{\mathbb{C}^4} .$$

We deduce from (VI.18) that :

$$\forall f \in D(\mathbb{K}_{\mu\Gamma}), \ \ \| \mathbb{K}_{\mu\Gamma} f \| \geq \mu\eta \| f \| .$$

Hence choosing $\chi \in C^\infty(\mathbb{R})$, such that :

$$t \leq 0 \Rightarrow \chi(t) = 0, \ \ \mu\eta \leq t \Rightarrow \chi(t) = 1,$$

we have :

$$\mathbf{1}_{[0,\infty[}(\mathbb{H}_{\mu\Gamma}) = \chi(\mathbb{H}_{\mu\Gamma}), \ \ \mathbf{1}_{[0,\infty[}(\mathbb{K}_{\mu\Gamma}) = \chi(\mathbb{K}_{\mu\Gamma}).$$

We remark that

$$(\mathbb{H}_{\mu\Gamma} + i)^{-1} - (\mathbb{K}_{\mu\Gamma} + i)^{-1}$$

is of finite rank, thus compact on L^2_∞. Hence by the Weyl criterion (see e.g. [15], Theorem B.1.1) the operator

$$\mathbf{1}_{[0,\infty[}(\mathbb{H}_{\mu\Gamma}) - \mathbf{1}_{[0,\infty[}(\mathbb{K}_{\mu\Gamma}) = \chi(\mathbb{H}_{\mu\Gamma}) - \chi(\mathbb{K}_{\mu\Gamma}) \ \ \textit{is compact}. \tag{VI.67}$$

Then we deduce from (VI.51), (VI.61), (VI.66) that :

$$\| 0 \oplus \mathbf{1}_{[\varrho,\infty[}\left(\mathbb{H}_{V_\infty,0}\right)\left(f^*_T\right) - \mathbf{1}_{[0,\infty[}\left(\mathbb{H}_{\mu\Gamma}\right)\left(0 \oplus f^*_T\right) \| \rightarrow 0, \ T \rightarrow \infty. \tag{VI.68}$$

We calculate this last projector using the Fourier transform :

$$\mathbf{1}_{[0,\infty[}\left(\mathbb{H}_{\mu\Gamma}\right) = \mathcal{F}^{-1}\left(\frac{1}{2} - \frac{1}{2\sqrt{\xi^2+\mu^2}}(\xi L + \mu\Gamma)\right)\mathcal{F},$$

$$\mathcal{F}\left(f^*_T\right)(\xi) = \sqrt{\kappa A} e^{-\kappa T} \varphi\left(A e^{-2\kappa T}\xi\right), \ \ \varphi(\zeta) := \int_{-\infty}^{\infty} e^{i\zeta e^{\kappa y}} e^{\frac{\kappa}{2} y} f(y) dy.$$

We obtain :

$$\begin{aligned} &\| \left(\mathbf{1}_{[0,\infty[}(\mathbb{H}_{0,\infty}) - \mathbf{1}_{[0,\infty[}(\mathbb{H}_{\mu\Gamma})\right)(0 \oplus f_T^*) \|^2 \\ &\leq C \int \left(\left| \frac{\zeta}{\sqrt{\zeta^2 + A^2 e^{-4\kappa T}\mu^2}} - \frac{\zeta}{|\zeta|} \right|^2 + \frac{1}{e^{4\kappa T}\zeta^2 + a^2\mu^2} \right) |\varphi(\zeta)|^2 \, d\zeta \qquad \text{(VI.69)} \\ &\longrightarrow 0, \; T \to \infty. \end{aligned}$$

Since $(f_T^*)_2 = (f_T^*)_3 = 0$ we have :

$$\| \mathbf{1}_{[0,\infty[}(\mathbb{H}_{0,\infty})(0 \oplus f_T^*) \|^2 = \frac{1}{2\pi}\int_0^\infty |\mathcal{F}(f_T^*)(\xi)|^2 \, d\xi = \| \mathbf{1}_{[0,\infty[}(\mathbb{H}_{0,0}) f_T^* \|^2, \qquad \text{(VI.70)}$$

hence (VI.60) follows from (VI.68), (VI.69) and (VI.70). □

Lemma VI.8. *For $f \in L^2_{out}$ we have :*

$$\| U_0(0,T)f^T - f_T^* \| \to 0, \;\; T \to \infty, \qquad \text{(VI.71)}$$

$$U_0(0,T)f^T \rightharpoonup 0 \;\; in \;\; L^2_0 - weak*, \;\; T \to \infty. \qquad \text{(VI.72)}$$

Proof of Lemma VI.8. Given $\varepsilon > 0$ we choose $g \in L^2_{out}$, continuous, compactly supported in $[0,R]$ such that

$$\| f - g \| \leq \varepsilon.$$

Then for any $T > 0$ we have

$$\| f_T^* - g_T^* \| \leq \varepsilon,$$

hence we need to prove that :

$$\| U_0(0,T)g^T - g_T^* \| \to 0, \;\; T \to \infty.$$

Thanks to Lemma VI.3 we have for $T > (R - z(0))/2$:

$$\left[U_0(0,T)g^T\right]_2(x) = \left[U_0(0,T)g^T\right]_3(x) = 0,$$

$$\begin{aligned} &\left[U_0(0,T)g^T\right]_{1(4)}(x) = \\ &-(+)\sqrt{\frac{2 + 2\kappa x + O(x^2)}{-2\kappa x + O(x^2)}}\, g_{3(2)}\left(x + 2T + \frac{1}{\kappa}\ln(-x) - \frac{1}{\kappa}\ln(A) + O(x)\right). \end{aligned}$$

We deduce that :

$$\| U_0(0,T)g^T - g_T^* \|^2 = \kappa \int_\infty^\infty \left| g\left(y + O\left(e^{\kappa y - 2\kappa T}\right)\right) - g(y) \right|^2 dy \to 0, \;\; T \to \infty,$$

and (VI.72) follows from (VI.51) and (VI.71). □

We prove now a result of H^1-regularity of the solution. We essentially show that the polarized wave front set propagates according to the Hamilton flow and the usual law of reflection of singularities on the moving boundary. If the data of our problem, $V(x)$ and $z(t)$ were C^∞, we could invoke the deep general theorems of Ivriĭ [25]. Since V and z are less regular, we prefer to give an elementary proof for the solution of our simple system. We introduce the unitary group on L^2_∞ associated with $\mathbb{H}_{V,\infty}$:

$$U_V(t) := e^{it\mathbb{H}_{V,\infty}}. \tag{VI.73}$$

Lemma VI.9. *For $f \in L^2_\infty$ satisfying*

$$f_2 = f_3 = 0,$$

$$x < a \Rightarrow f(x) = 0,$$

we have for any $T > 0$:

$$\| \mathbf{1}_{\{x<a+T\}}(x)\partial_x \left[U_V(-T)f\right](x) \| \leq C_T \| f \| . \tag{VI.74}$$

Moreover, if $z(T) < a$, we have :

$$\| \mathbf{1}_{\{x<a+T\}}(x)\partial_x \left[U_V(0,T)f\right](x) \| \leq C_T \| f \| . \tag{VI.75}$$

Proof of Lemma VI.9. We denote $u(t,x) :=[U_V(t)f](x)$, and for $g = {}^t(g_1,g_2,g_3,g_4)$, we put :

$$[g]_{14} = {}^t(g_1,0,0,g_4), \quad [g]_{23} = {}^t(0,g_2,g_3,0).$$

We have :

$$(\partial_t - \partial_x)[u]_{14} + i[Vu]_{14} = 0, \quad x < a \Rightarrow [u]_{14}(0,x) = 0 \tag{VI.76}$$

$$(\partial_t + \partial_x)[u]_{23} + i[Vu]_{23} = 0, \quad [u]_{23}(0,.) = 0. \tag{VI.77}$$

We get for all x :

$$[\partial_x u]_{23}(-T,x) =$$

$$i\int_{-T}^0 [V'u]_{23}(s,x+s+T) + [V[\partial_x u]_{14}]_{23}(s,x+s+T) + [V[\partial_x u]_{23}]_{23}(s,x+s+T)ds.$$

We deduce from (VI.76) that :

$$[\partial_x u]_{14}(s,x+s+T) = \frac{1}{2}\frac{d}{ds}[u(s,x+s+T)]_{14} + \frac{i}{2}[Vu]_{14}(s,x+s+T),$$

hence integration by parts gives :

$$\begin{aligned}
&[\partial_x u]_{23}(-T,x)\\
&= \frac{i}{2}[V(x+T)[f(x+T)]_{14}]_{23} - \frac{i}{2}[V(x)[u(-T,x)]_{14}]_{23}\\
&+ i\int_{-T}^{0}\left([V'u]_{23} + \frac{i}{2}[V[Vu]_{14}]_{23} - \frac{1}{2}[V'[u]_{14}]_{23} + [V[\partial_x u]_{23}]_{23}\right)(s,x+s+T)ds.
\end{aligned}$$

Since $U_V(t)$ is unitary and V and V' are bounded, we get :

$$\| [\partial_x u]_{23}(-T) \| \leq C(1+T) \| f \| + C\int_{-T}^{0} \| [\partial_x u]_{23}(s) \| ds,$$

and we conclude by the Gronwall Lemma that :

$$\| [\partial_x u]_{23}(-T) \| \leq C_T \| f \| . \tag{VI.78}$$

On the other hand we deduce from (VI.76) that for $x < a+T$ we have :

$$\begin{aligned}
&[\partial_x u]_{14}(-T,x)\\
&= i\int_{-T}^{0}\left([V'u]_{14} + [V[\partial_x u]_{23}]_{14} + [V[\partial_x u]_{14}]_{14}\right)(s,s-s-T)ds.
\end{aligned}$$

Then we deduce from (VI.78) that

$$\| \mathbf{1}_{\{x<a+T\}}[\partial_x u]_{14}(-T) \| \leq C(T) \| f \| + C\int_{-T}^{0} \| \mathbf{1}_{\{x<a-s\}}[\partial_x u]_{14}(s) \| ds,$$

hence using one more the Gronwall Lemma we obtain :

$$\| \mathbf{1}_{\{x<a+T\}}[\partial_x u]_{14}(-T) \| \leq C_T \| f \| . \tag{VI.79}$$

and (VI.74) is established. To prove (VI.75) we choose $\alpha, \beta \in \mathbb{R}$, $\chi \in C^\infty(\mathbb{R})$ satisfying :

$$z(T)+T < \alpha < \beta < a+T,\;\; x \leq \alpha \Rightarrow \chi(x) = 0,\;\; x \geq \beta \Rightarrow \chi(x) = 1,$$

and we define :

$$v(t,x) = \left[U_V(t,T)f\right](x) - \chi(x+t)\left[U_V(t-T)f\right](x).$$

v is a solution to

$$v(T,x) = 0,\;\; x > z(t) \Rightarrow \partial_t v(t,x) = i\mathbb{H}_{V,t}v(t,x) + h(t),$$

$$v_2(t,z(t)) = \lambda(t)v_4(t,z(t)),\;\; v_3(t,z(t)) = -\lambda(t)v_1(t,z(t)),$$

with

$$h(t,x) := \chi'(x+t)L\left[U_V(t-T)f\right](x).$$

(VI.74) implies that :

$$h(t) \in D(\mathbb{H}_{V,t}), \quad \sup_{0\le t\le T} \parallel h(t) \parallel_{H^1_\infty} \le C_T \parallel f \parallel .$$

Whence

$$v(t) = \int_T^t U_V(t,s)h(s)ds \in H^1_t, \quad \sup_{0\le t\le T} \parallel v(t) \parallel_{H^1_t} \le C'_T \parallel f \parallel,$$

and we get (VI.75) using (VI.74) again. □

Lemma VI.10. *The operator*

$$\mathbf{1}_{[\varrho,\infty[}(\mathbb{H}_{V,0}) - \mathbf{1}_{[\varrho,\infty[}(\mathbb{H}_{V_\infty,0}) \tag{VI.80}$$

is compact on L^2_0.

Proof of Lemma VI.10. We have :

$$\mathbf{1}_{[\varrho,\infty[}(\mathbb{H}_{V,0}) - \mathbf{1}_{[\varrho,\infty[}(\mathbb{H}_{V_\infty,0}) = \mathbf{1}_{[0,\infty[}(\mathbb{H}_1) - \mathbf{1}_{[0,\infty[}(\mathbb{H}_2),$$

$$\mathbb{H}_1 := \mathbb{H}_{V+\varrho Id,0}, \quad \mathbb{H}_2 := \mathbb{H}_{\mu\Gamma,0}.$$

The identities :

$$a \in \mathbb{R}^*, \quad \frac{a}{\mid a \mid} = \int_0^\infty \frac{a}{\sqrt{\pi t}} e^{-a^2 t} dt,$$

$$0 \le t, \quad e^{-a^2 t} = -\frac{1}{2i\pi} \oint_\gamma \frac{e^{-z}}{a^2 t - z} dz,$$

where γ is the path in the complex plane given by :

$$\gamma = \left\{ z = e^{i\theta};\ \frac{\pi}{2} < \theta < \frac{3\pi}{2} \right\} \cup \left\{ z = \alpha \pm i;\ \ 0 \le \alpha \right\},$$

yield :

$$\mathbf{1}_{[0,\infty[}(\mathbb{H}_1) - \mathbf{1}_{[0,\infty[}(\mathbb{H}_2) =$$

$$\frac{i}{4\pi^{\frac{3}{2}}} \int_0^\infty \left(\oint_\gamma e^{-z} \left[\mathbb{H}_1 \left(t\mathbb{H}_1^2 - z \right)^{-1} - \mathbb{H}_2 \left(t\mathbb{H}_2^2 - z \right)^{-1} \right] dz \right) \frac{dt}{\sqrt{t}}. \tag{VI.81}$$

We have :

$$\mathbb{H}_2 = \mathbb{H}_1 + V_1, \quad V_1 := V + \varrho Id - \mu\Gamma,$$

$$\mathbb{H}_2^2 = \mathbb{H}_1^2 + \mathbb{W}, \quad \mathbb{W} := \mu^2 Id - (V+\varrho)^2 + iLV' + i[L(V+\varrho) + (V+\varrho)L]\partial_x,$$

and by (VI.16), (VI.17) :

$$\mid V_1(x) \mid + \mid \mu^2 Id - (V(x)+\varrho)^2 + iLV'(x) \mid$$

$$+ \mid L(V(x)+\varrho) + (V(x)+\varrho)L \mid \to 0,\ x \to \infty. \tag{VI.82}$$

For $t>0$, $z \in \gamma$ we consider :

$$\mathbb{A}(t,z) := \mathbb{H}_1 \left(t\mathbb{H}_1^2 - z\right)^{-1} - \mathbb{H}_2 \left(t\mathbb{H}_2^2 - z\right)^{-1}$$

$$= \left(t\mathbb{H}_1 \left(t\mathbb{H}_1^2 - z\right)^{-1}\right)\left(\mathbb{W}\left(t\mathbb{H}_2^2 - z\right)^{-1}\right) - V_1\left(t\mathbb{H}_2^2 - z\right)^{-1}.$$

Since $\left(t\mathbb{H}_2^2 - z\right)^{-1}$ is bounded from L^2_0 to H^2_0, the Sobolev embedding and the decay estimates (VI.82) imply that $\mathbb{W}$ is compact from H^2_0 to L^2_0 and $\left(\mathbb{W}\left(t\mathbb{H}_2^2 - z\right)^{-1}\right)$ and $V_1\left(t\mathbb{H}_2^2 - z\right)^{-1}$ are compact on L^2_0. Since $\left(t\mathbb{H}_1\left(t\mathbb{H}_1^2 - z\right)^{-1}\right)$ is bounded on L^2_0 we conclude that $\mathbb{A}(t,z)$ is compact on L^2_0. Now to prove the Lemma, it suffices to establish that

$$\frac{\mid e^{-z} \mid}{\sqrt{t}} \parallel \mathbb{A}(t,z) \parallel_{\mathcal{L}(L^2_0)} \in L^1(\mathbb{R}^+_t \times \gamma_z). \tag{VI.83}$$

On the one hand :

$$\parallel \mathbb{H}_j \left(t\mathbb{H}_j^2 - z\right)^{-1} \parallel_{\mathcal{L}(L^2_0)} \leq \sup_{\lambda\in\mathbb{R}} \mid \lambda(t\lambda^2 - z)^{-1} \mid = \frac{1}{\sqrt{2t(\mid z \mid - \Re z)}} \leq \frac{1}{2\sqrt{t}}, \tag{VI.84}$$

hence

$$0<t,\ \ z\in\gamma,\ \ \parallel \mathbb{A}(t,z) \parallel_{\mathcal{L}(L^2_0)} \leq \frac{1}{t}. \tag{VI.85}$$

On the other hand we write :

$$\mathbb{A}(t,z) = t\left(\mathbb{H}_1\left(t\mathbb{H}_1^2 - z\right)^{-1}\right)\mathbb{W}\left(t\mathbb{H}_2^2 - z\right)^{-1} - V_1\left(t\mathbb{H}_2^2 - z\right)^{-1}.$$

We have :

$$0<t,\ \ z\in\gamma,\ \ \parallel \left(t\mathbb{H}_2^2 - z\right)^{-1} \parallel_{\mathcal{L}(L^2_0)} \leq 1,$$

and we deduce from (VI.84) that :

$$\parallel \left(t\mathbb{H}_2^2 - z\right)^{-1} \parallel_{\mathcal{L}(L^2_0, H^1_0)} \leq 1 + \frac{1}{\sqrt{t}}.$$

Since $\mathbb{W} \in \mathcal{L}(H^1_0, L^2_0)$ we get :

$$0<t,\ \ z\in\gamma,\ \ \parallel \mathbb{A}(t,z) \parallel_{\mathcal{L}(L^2_0)} \leq C(1+\sqrt{t}), \tag{VI.86}$$

where $C > 0$ does not depend on $t > 0$, $z \in \gamma$. Therefore (VI.85), (VI.86) yield

$$\sup_{z\in\gamma} \| \mathbb{A}(t,z) \|_{\mathcal{L}(L^2_0)} \leq \frac{2C}{1+t},$$

and (VI.83) is established □

Proof of Theorem VI.5. For $0 < t_\varepsilon < T$ we write

$$U_V(0,T)f^T = O_T(\varepsilon) + U_V(0,t_\varepsilon)g_{\varepsilon,T}, \tag{VI.87}$$

with

$$O_T(\varepsilon) := U_V(0,t_\varepsilon)\left(U_V(t_\varepsilon,T) - U_0(t_\varepsilon,T)\right)f^T, \;\; g_{\varepsilon,T} := U_0(t_\varepsilon,T)f^T.$$

If f is supported in $[0,R]$, then $U_V(s,T)f^T$ is supported in $[z(s), -s+R]$, hence

$$\| O_T(\varepsilon) \| \leq \int_{t_\varepsilon}^T \| VU_V(s,T)f^T \| \, ds \leq \| f \| \int_{t_\varepsilon}^\infty \sup\{| V(x) |; x \leq -s+R\} ds.$$

Then, given $\varepsilon > 0$, we choose t_ε such that :

$$\sup_{T\leq t_\varepsilon} \| O_T(\varepsilon) \| \leq \varepsilon. \tag{VI.88}$$

Now for $r \geq 0$ we define $\theta_{r,T}$ as the unique solution to

$$z(\theta_{r,T}) - \theta_{r,T} = r - 2T.$$

By Lemma VI.3 we have :

$$\theta_{r,T} = T - \frac{r}{2} + O(e^{-2\kappa T}), \tag{VI.89}$$

and for $0 \leq t_\varepsilon \leq \theta_{R,T}$, $g_{\varepsilon,T}$ is supported in $[2(\theta_{0,T}-T)-t_\varepsilon, 2(\theta_{R,T}-T)+R-t_\varepsilon] \subset [-t_\varepsilon - O(e^{-2\kappa T}), -t_\varepsilon]$ and $[g_{\varepsilon,T}]_{23} = 0$. We introduce the function Y_ε given by

$$x < z(t_\varepsilon) + t_\varepsilon \Rightarrow Y_\varepsilon(x) = 0, \;\; z(t_\varepsilon) + t_\varepsilon \leq x \Rightarrow Y_\varepsilon(x) = 1.$$

The compact embedding $H^1 \subset L^2$ with (VI.72) and Lemma VI.9, assure that :

$$\| (1-Y_\varepsilon)U_V(0,t_\varepsilon)g_{\varepsilon,T} \| \to 0, \; T \to \infty, \tag{VI.90}$$

$$\| (1-Y_\varepsilon)U_V(-t_\varepsilon)g_{\varepsilon,T} \| \to 0, \; T \to \infty. \tag{VI.91}$$

We remark that :

$$Y_\varepsilon U_V(0,t_\varepsilon)g_{\varepsilon,T} = Y_\varepsilon U_V(-t_\varepsilon)g_{\varepsilon,T}.$$

Therefore, thanks to (VI.88), (VI.90), it is sufficient to prove that :

$$\lim_{T\to\infty} \parallel \mathbf{1}_{[\varrho,\infty[}(\mathbb{H}_{V,0})\left(Y_\varepsilon U_V(-t_\varepsilon)g_{\varepsilon,T}\right) \parallel^2 = < f, e^{\frac{2\pi}{\kappa}\mathbb{H}_{0,\infty}}\left(1+e^{\frac{2\pi}{\kappa}\mathbb{H}_{0,\infty}}\right)^{-1} f >_{L^2_\infty} . \tag{VI.92}$$

In the sequel, we write $u\sim v$ for $u,v\in C^0(\mathbb{R}_T, L^2_\infty)$ iff $\parallel u(T)-v(T)\parallel\to 0$ when $T\to\infty$. Since (VI.72) implies that $Y_\varepsilon U_V(-t_\varepsilon)g_{\varepsilon,T}$ tends weakly to 0 in L^2_0, we deduce from Lemma VI.10 and (VI.61) that :

$$\mathbf{1}_{[\varrho,\infty[}(\mathbb{H}_{V,0})\left(Y_\varepsilon U_V(-t_\varepsilon)g_{\varepsilon,T}\right) \sim \mathbf{1}_{[0,\infty[}(\mathbb{H}_{\mu\Gamma,0})\left(Y_\varepsilon U_V(-t_\varepsilon)g_{\varepsilon,T}\right). \tag{VI.93}$$

We have also :

$$0\oplus\mathbf{1}_{[0,\infty[}(\mathbb{H}_{\mu\Gamma,0})\left(Y_\varepsilon U_V(-t_\varepsilon)g_{\varepsilon,T}\right) = \mathbf{1}_{[0,\infty[}(\mathbb{K}_{\mu\Gamma})\left(Y_\varepsilon U_V(-t_\varepsilon)g_{\varepsilon,T}\right). \tag{VI.94}$$

We deduce from (VI.67) that :

$$\mathbf{1}_{[0,\infty[}(\mathbb{K}_{\mu\Gamma})\left(Y_\varepsilon U_V(-t_\varepsilon)g_{\varepsilon,T}\right) \sim \mathbf{1}_{[0,\infty[}(\mathbb{H}_{\mu\Gamma})\left(Y_\varepsilon U_V(-t_\varepsilon)g_{\varepsilon,T}\right), \tag{VI.95}$$

and with (VI.91) :

$$\mathbf{1}_{[0,\infty[}(\mathbb{H}_{\mu\Gamma})\left(Y_\varepsilon U_V(-t_\varepsilon)g_{\varepsilon,T}\right) \sim \mathbf{1}_{[0,\infty[}(\mathbb{H}_{\mu\Gamma})\left(U_V(-t_\varepsilon)g_{\varepsilon,T}\right). \tag{VI.96}$$

We introduce a potential V_ε on $\mathbb{R}$ given by :

$$x\geq -2t_\varepsilon \Rightarrow V_\varepsilon(x)=V(x),\ \ x\leq -2t_\varepsilon\Rightarrow V_\varepsilon(x)=V(2t_\varepsilon - x).$$

Then the finite speed of propagation implies that :

$$U_{V_\varepsilon}(-t_\varepsilon)g_{\varepsilon,T} = U_V(-t_\varepsilon)g_{\varepsilon,T}.$$

Now putting $\mathbb{H}_1 := \mathbb{H}_{V_\varepsilon+\varrho Id}$, $\mathbb{H}_2 := \mathbb{H}_{\mu\Gamma}$ in the proof of Lemma VI.10, we get that $\mathbf{1}_{[0,\infty[}(\mathbb{H}_{\mu\Gamma}) - \mathbf{1}_{[\varrho,\infty[}(\mathbb{H}_{V_\varepsilon})$ is compact on L^2_∞. Then we obtain :

$$\mathbf{1}_{[0,\infty[}(\mathbb{H}_{\mu\Gamma})\left(U_V(-t_\varepsilon)g_{\varepsilon,T}\right) \sim \mathbf{1}_{[\varrho,\infty[}(\mathbb{H}_{V_\varepsilon})U_{V_\varepsilon}(-t_\varepsilon)g_{\varepsilon,T}. \tag{VI.97}$$

We remark that :

$$\parallel \mathbf{1}_{[\varrho,\infty[}(\mathbb{H}_{V_\varepsilon})U_{V_\varepsilon}(-t_\varepsilon)g_{\varepsilon,T}\parallel = \parallel \mathbf{1}_{[\varrho,\infty[}(\mathbb{H}_{V_\varepsilon})g_{\varepsilon,T}\parallel, \tag{VI.98}$$

and by the previous argument, we have again :

$$\mathbf{1}_{[\varrho,\infty[}(\mathbb{H}_{V_\varepsilon})g_{\varepsilon,T} \sim \mathbf{1}_{[0,\infty[}(\mathbb{H}_{\mu\Gamma})g_{\varepsilon,T}. \tag{VI.99}$$

Now (VI.71) implies :

$$g_{\varepsilon,T}\sim U_0(t_\varepsilon)f_T^*$$

Since $U_0(t_\varepsilon)f_T^*(x) = f_T^*(x+t_\varepsilon)$ and $\mathbb{H}_{\mu\Gamma}$ commutes with ∂_x, we have

$$\| \mathbf{1}_{[0,\infty[}(\mathbb{H}_{\mu\Gamma})U_0(t_\varepsilon)f_T^* \| = \| \mathbf{1}_{[0,\infty[}(\mathbb{H}_{\mu\Gamma})f_T^* \|$$

and we get with (VI.69), (VI.70) and Lemma VI.53 that :

$$\| \mathbf{1}_{[0,\infty[}(\mathbb{H}_{\mu\Gamma})g_{\varepsilon,T} \| \to < f, e^{\frac{2\pi}{\kappa}\mathbb{H}_{0,\infty}} \left(1+e^{\frac{2\pi}{\kappa}\mathbb{H}_{0,\infty}}\right)^{-1} f >_{L^2_\infty}, \ \ T\to\infty. \quad \text{(VI.100)}$$

We conclude that (VI.92) is a consequence of (VI.93) to (VI.100). □

Corollary VI.11. *Given $f_{out}\in L^2_{out}$, $f_{in}\in L^2_{in}$, we have for $J=[\varrho,\infty[$ or $J=]\varrho,\infty[$:*

$$\lim_{T\to\infty} \| \mathbf{1}_J(\mathbb{H}_{V,0})U_V(0,T)\left(f^T_{out}+f^T_{in}\right)\|^2$$
$$= < f_{out}, e^{\frac{2\pi}{\kappa}\mathbb{H}_{0,\infty}} \left(1+e^{\frac{2\pi}{\kappa}\mathbb{H}_{0,\infty}}\right)^{-1} f_{out} >_{L^2_\infty}$$
$$+ \| \mathbf{1}_J(\mathbb{H}_{V,0})\left(\Omega_V^{in}f_{in}\right)\|^2 . \quad \text{(VI.101)}$$

Proof of Corollary VI.11. Since (VI.72) implies that $g_{\varepsilon,T}$ tends weakly to 0 in $L^2_{t_\varepsilon}$ as $T\to\infty$, (VI.87) and (VI.88) assure that

$$U_V(0,T)f^T_{out} \rightharpoonup 0 \ \ in \ \ L^2_0 - weak*, \ \ T\to\infty, \quad \text{(VI.102)}$$

and because $\mathbf{1}_{\{0\}}(\mathbb{H}_{V,0})$ is finite rank, we have :

$$\lim_{T\to\infty} \| \mathbf{1}_{\{0\}}(\mathbb{H}_{V,0})U_V(0,T)f^T_{out}\| = 0.$$

Then, using :

$$U_V(0,T)f^T_{in} = U_V(0,T)U_0(T)f_{in},$$

the result follows from (VI.102), Proposition VI.4 and Theorem VI.5. □

VI.2 Proofs of the asymptotic estimates

At present we are able to investigate the 3D+1 problem, so we return to the proofs of the results of Part III and IV.

Proof of Lemma III.1.

For $0\le t\le\infty$, we expand $\Phi\in\mathcal{L}^2_t$ in the following way :

$$\Phi = \sum_{\ell,n} e^{-i\frac{\alpha_{\ell,n}}{2}\gamma^5} \begin{pmatrix} u^\ell_{1,n}(x)T^\ell_{-\frac12,n}(\varphi,\theta) \\ u^\ell_{2,n}(x)T^\ell_{+\frac12,n}(\varphi,\theta) \\ u^\ell_{3,n}(x)T^\ell_{-\frac12,n}(\varphi,\theta) \\ u^\ell_{4,n}(x)T^\ell_{+\frac12,n}(\varphi,\theta) \end{pmatrix}, \quad \text{(VI.103)}$$

and we introduce :

$$\mathcal{I}_{l,n} : \Phi \in \mathcal{L}_t^2 \longmapsto \begin{pmatrix} u^\ell_{1,n}(x) \\ u^\ell_{2,n}(x) \\ u^\ell_{3,n}(x) \\ u^\ell_{4,n}(x) \end{pmatrix} \in L_t^2, \tag{VI.104}$$

$$\mathcal{R}_{\ell,n} : \begin{pmatrix} u^\ell_{1,n} \\ u^\ell_{2,n} \\ u^\ell_{3,n} \\ u^\ell_{4,n} \end{pmatrix} \in L_t^2 \longmapsto e^{-i\frac{\alpha_{\ell,n}}{2}\gamma^5} \begin{pmatrix} u^\ell_{1,n}(x)T^\ell_{-\frac{1}{2},n}(\varphi,\theta) \\ u^\ell_{2,n}(x)T^\ell_{+\frac{1}{2},n}(\varphi,\theta) \\ u^\ell_{3,n}(x)T^\ell_{-\frac{1}{2},n}(\varphi,\theta) \\ u^\ell_{4,n}(x)T^\ell_{+\frac{1}{2},n}(\varphi,\theta) \end{pmatrix} \in \mathcal{L}_t^2.$$

We make the link with the study of the one dimensional problem in the previous section by putting :

$$\mathcal{L}_t^2 = \bigoplus_{\ell,n} \mathcal{R}_{\ell,n} L_t^2, \;\; \mathbb{H}_t = \bigoplus_{\ell,n} \mathcal{R}_{\ell,n} \big(\mathbb{H}_{V_{\ell,n},t} - qA(r_0)Id\big)\mathcal{I}_{\ell,n}, \tag{VI.105}$$

where $\mathbb{H}_{V_{\ell,n},t}$ is given by (VI.26) and (VI.9). We note that the hypotheses (VI.13) to (VI.18) are satisfied for the choices :

$$\kappa = \kappa_0, \;\; \mu = m\sqrt{F(r_+)}, \;\; \varrho = qA(r_0), \;\; \eta = \inf(1, \sqrt{1+\cos\alpha_{\ell,n}}), \tag{VI.106}$$

$$\Gamma = \begin{pmatrix} 0 & 0 & ie^{i\alpha_{\ell,n}} & 0 \\ 0 & 0 & 0 & ie^{i\alpha_{\ell,n}} \\ -ie^{-i\alpha_{\ell,n}} & 0 & 0 & 0 \\ 0 & -ie^{-i\alpha_{\ell,n}} & 0 & 0 \end{pmatrix}. \tag{VI.107}$$

Hence the result follows from Lemma VI.1, in particular we have

$$2\Re \langle i\mathbb{H}_t\Phi, \Phi\rangle = \dot{z}(t) \parallel \Phi(z(t), .) \parallel^2_{L^2(S^2)} . \tag{VI.108}$$

□

Proof of Proposition III.2. The conservation law (III.23) is a consequence of (VI.108), and the existence of the solution is obtained by taking :

$$U(t,s) = e^{i(s-t)qA(r_0)} \bigoplus_{\ell,n} \mathcal{R}_{\ell,n} U_{V_{\ell,n}}(t,s)\mathcal{I}_{\ell,n}. \tag{VI.109}$$

Then the result follows from Proposition VI.2. □

Proof of Proposition III.3. We remark that :

$$\mathcal{L}^2_{BH} = \bigoplus_{\ell,n} \mathcal{R}_{\ell,n} L^2_{in}, \quad \mathcal{L}^2_{out} = \bigoplus_{\ell,n} \mathcal{R}_{\ell,n} L^2_{out}, \tag{VI.110}$$

$$\mathbb{H}_{BH}\Phi = \bigoplus_{\ell,n} \mathcal{R}_{\ell,n}\big(\mathbb{H}_{0,\infty} - qA(r_0)Id\big)\mathcal{I}_{\ell,n}, \quad U_{BH}(t) = e^{-iqA(r_0)t}\bigoplus_{\ell,n}\mathcal{R}_{\ell,n}U_0(t)\mathcal{I}_{\ell,n}. \tag{VI.111}$$

Therefore, since

$$U(0,T)U_{BH}(T) = \bigoplus_{\ell,n}\mathcal{R}_{\ell,n}U_{V_{l,n}}(0,T)U_0(T)\mathcal{I}_{\ell,n},$$

the existence of the wave operator follows from Proposition VI.4 by putting :

$$\Omega_{BH} = \bigoplus_{\ell,n}\mathcal{R}_{\ell,n}\Omega^{in}_{V_{\ell,n}}\mathcal{I}_{\ell,n}. \tag{VI.112}$$

□

Proof of Theorem III.4. We apply Corollary VI.11 and the dominated convergence theorem to get :

$$\begin{aligned}
&\parallel \mathbf{1}_J(\mathbb{H}_0)U(0,T)\left(\Phi^T_{out}+\Phi^T_{BH}\right)\parallel^2 \\
&\quad = \sum_{\ell,n}\parallel \mathbf{1}_J(\mathbb{H}_{V_{\ell,0}})U_{V_{\ell,n}}(0,T)\left(\mathcal{I}_{\ell,n}\Phi^T_{out}+\mathcal{I}_{\ell,n}\Phi^T_{BH}\right)\parallel^2 \\
&\quad \underset{T\to\infty}{\longrightarrow} \sum_{\ell,n} < \mathcal{I}_{\ell,n}\Phi_{out}, e^{\frac{2\pi}{\kappa_0}\mathbb{H}_{0,\infty}}\left(1+e^{\frac{2\pi}{\kappa_0}\mathbb{H}_{0,\infty}}\right)^{-1}\mathcal{I}_{\ell,n}\Phi_{out} >_{L^2_\infty} \\
&\quad + \parallel \mathbf{1}_J(\mathbb{H}_{V_{\ell,n},0})\left(\Omega^{in}_{V_{\ell,n}}\mathcal{I}_{\ell,n}\Phi_{BH}\right)\parallel^2 \\
&\quad =< \Phi_{out}, \zeta e^{\frac{2\pi}{\kappa_0}\mathbb{H}_{BH}}\left(1+\zeta e^{\frac{2\pi}{\kappa_0}\mathbb{H}_{BH}}\right)^{-1}\Phi_{out} >_{\mathcal{L}^2_\infty} \\
&\quad + \parallel \mathbf{1}_J(\mathbb{H}_0)\left(\Omega_{BH}\Phi_{BH}\right)\parallel^2 .
\end{aligned}$$

□

Proof of Theorem IV.1. By the identity of polarization it is sufficient to consider $\Phi^1_0 = \Phi^2_0 = \Phi_0$ and we assume that :

$$supp\,\Phi_0 \subset \left\{(t,x,\omega)\in[-R,+R]\times[0,R]\times S^2;\;\; 0<x+t\right\}.$$

For $T>0$ we introduce the map

$$\mathcal{T}_T : F\in\mathcal{L}^2_\infty \longmapsto (\mathcal{T}_T F)(x,\omega) = F(x+T,\omega).$$

Lemma VI.12.

$$\begin{aligned}
&\parallel \int_{-R}^{R} U(T+R,T+t)\mathcal{T}_T\Phi_0(t)dt \\
&\quad -\mathcal{T}_{T+R}\Big(e^{+iqA(r_0)R}U_{BH}(2R)S_{BH}P^{out}\Phi_0 \\
&\qquad\quad +e^{-iqA(r_0)R}S_{BH}P^{in}\Phi_0\Big)\parallel_{\mathcal{L}^2_{R+T}} \longrightarrow 0, \;\; T\to\infty.
\end{aligned} \tag{VI.113}$$

Proof of Lemma VI.12. We write by using (VI.109) :

$$\int_{-R}^{R} U(T+R,T+t)\mathcal{T}_T\Phi_0(t)dt = \sum_{\ell,n}\mathcal{R}_{\ell,n}\int_{-R}^{R} e^{iqA(r_0)(t-R)}U_{V_{\ell,n}}(T+R,T+t)\mathcal{I}_{\ell,n}\mathcal{T}_T\Phi_0(t)dt,$$

and we get by (VI.111) and (IV.11) :

$$\mathcal{T}_{T+R}\left(e^{+iqA(r_0)R}U_{BH}(2R)S_{BH}P^{out}\Phi_0 + e^{-iqA(r_0)R}S_{BH}P^{in}\Phi_0\right) = \sum_{\ell,n}\mathcal{R}_{\ell,n}\int_{-R}^{R} e^{iqA(r_0)(t-R)}U_0(R-t)\mathcal{I}_{\ell,n}\mathcal{T}_T\Phi_0(t)dt.$$

The hypothesis on the support of Φ implies that :

$$U_{V_{\ell,n}}(T+R,T+t)\mathcal{I}_{\ell,n}\mathcal{T}_T\Phi_0(t) = U_{V_{\ell,n}}(R-t)\mathcal{I}_{\ell,n}\mathcal{T}_T\Phi_0(t),$$

and the Duhamel formula with (II.11) assures that :

$$\| U_{V_{\ell,n}}(R-t)\mathcal{I}_{\ell,n}\mathcal{T}_T\Phi_0(t) - U_0(R-t)\mathcal{I}_{\ell,n}\mathcal{T}_T\Phi_0(t) \|_{L^2_\infty} \leq C(R,\Phi)e^{-\kappa_0 T}.$$

Hence we easily deduce (VI.113). □

To prove the Theorem, we get by (IV.7), Theorem III.4, Lemma VI.12, (IV.9) and (IV.4) :

$$\begin{aligned}
&\omega_{\mathcal{M}}\left(\mathbf{\Psi}_T^*(\Phi_T)\mathbf{\Psi}_T(\Phi_T)\right) \\
&\quad = \| \mathbf{1}_{]0,\infty[}(\mathbb{H}_0)\int_{-\infty}^{\infty} U(0,t)\Phi_T(t)dt \|^2 \\
&\quad = \| \mathbf{1}_{]0,\infty[}(\mathbb{H}_0)U(0,T+R)\int_{-\infty}^{\infty} U(T+R,t+T)\mathcal{T}_T\Phi_0(t)dt \|^2 \\
&\quad \underset{T\to\infty}{\longrightarrow} < S_{BH}P^{out}\Phi_0, e^{\frac{2\pi}{\kappa_0}qA(r_0)}e^{\frac{2\pi}{\kappa_0}\mathbb{H}_{BH}} \\
&\qquad\qquad \left(\mathbf{1} + e^{\frac{2\pi}{\kappa_0}qA(r_0)}e^{\frac{2\pi}{\kappa_0}\mathbb{H}_{BH}}\right)^{-1} S_{BH}P^{out}\Phi_0 >_{\mathcal{L}^2_\infty} \\
&\quad + \| \mathbf{1}_{]0,\infty[}(\mathbb{H}_0)\left(\Omega_{BH}P^{in}S_{BH}\Phi_0\right) \|^2 \\
&\quad = \omega_{BH}^{\frac{2\pi}{\kappa_0},qA(r_0)}\left(\mathbf{\Psi}_{BH}^*(P^{out}\Phi_0)\mathbf{\Psi}_{BH}(P^{out}\Phi_0)\right) \\
&\quad + \omega_{\Sigma_0}^0\left(\mathbf{\Psi}_0^*(\Omega_{BH}P^{in}S_{BH}\Phi_0)\mathbf{\Psi}_0(\Omega_{BH}P^{in}S_{BH}\Phi_0)\right).
\end{aligned}$$

□

VII Conclusion

We have considered a charged Dirac field outside a spherical charged star, stationary in the past and collapsing to a black hole in the future. The interaction between the field and the matter of the star is subsumed in a boundary condition belonging to a large class. We have rigorously established the famous result on the thermalization of the vacuum by the collapse : if the ground quantum state in the past is the Boulware vacuum, then, this state becomes of Unruh type near the future black-hole horizon. Moreover the temperature and the chemical potential are independent of the history of the collapse and of the boundary condition (in the class that we introduced). A static observer at infinity interprets this state as a stream of particules and antiparticles outgoing from the black hole to infinity. Furthermore, the black-hole preferentially emits fermions whose charge is of same sign as its own charge, rather than fermions of opposite charge. We have investigated the rather subtle role of the cosmological constant in the case of the DeSitter-Reissner-Nordstrøm Black-Hole : in the case of a weakly charged black hole in an expanding universe, the temperature is an increasing function of the charge, unlike the asymptotically flat case; in the case of a strongly charge, $24M^2 < 25Q^2 < 25M^2$, the temperature is an increasing function of the cosmological constant. We have only studied the two-point function which carries the information on the vacuum fluctuations. A subsequent work will be devoted to the investigation of the stress energy momentum tensor. Another interesting problem consists in treating the matter of the star (fluid or dust) instead of considering a boundary condition.

It goes without saying that we leave open the huge problem of the back reaction of this vacuum polarization on the metric, nevertheless we make some comments on the subject. The previous remarks suggest that the black-hole loses mass and charge [8]. Since the propagator of the Dirac system is unitary, there is no supperadiance of fermion fields [12], despite the existence of the generalized ergosphere for classical particles [14]. Therefore we may expect that the rate of the spontaneous loss of charge of the black-hole through charged fermion fields, is weak in the semiclassical regime, unlike the scalar case for wich supperradiant modes appear [21]. All these conjectures require the solution of monstrously non linear problems.

A Second Quantization of the Dirac Fields

To be able to construct the Boulware vacuum in the past, and the thermal state at the horizon, we describe here the essential features of the quantization of the Dirac field, convenient for the stationary space-times (for more details in the case of the flat space, see e.g. [6],[7],[9],[23],[38],[39]).

We first consider the case of one kind of non interacting fermions. Let $\mathfrak{h}$ be a complex Hilbert space (the one-fermion space), and we denote $<,>$ its scalar product, linear with respect to the first argument. We define the space of n-fermions as the antisymmetric n-tensor product of $\mathfrak{h}$:

$$\mathfrak{F}^{(0)}(\mathfrak{h}) := \mathbb{C},\ \ 1 \leq n \Rightarrow \mathfrak{F}^{(n)}(\mathfrak{h}) := \bigwedge_{\nu=1}^{n} \mathfrak{h}, \tag{A.1}$$

and the $Fermi - Fock$ space :

$$\mathfrak{F}(\mathfrak{h}) := \bigoplus_{n=0}^{\infty} \mathfrak{F}^{(n)}(\mathfrak{h}). \tag{A.2}$$

For $f \in \mathfrak{h}$ we construct the *fermion annihilation operator* $a_{\mathfrak{h}}(f)$, and the *fermion creation operator* $a^*_{\mathfrak{h}}(f)$ by putting :

$$a_{\mathfrak{h}}(f) : \mathfrak{F}^{(0)}(\mathfrak{h}) \mapsto \{0\},\ \ 1 \leq n,\ \ a_{\mathfrak{h}}(f) : \mathfrak{F}^{(n)}(\mathfrak{h}) \mapsto \mathfrak{F}^{(n-1)}(\mathfrak{h}), \tag{A.3}$$

$$a_{\mathfrak{h}}(f)\,(f_1 \wedge ... \wedge f_n) = \frac{\sqrt{n}}{n!} \sum_{\sigma} \varepsilon(\sigma) < f_{\sigma(1)}, f > f_{\sigma(2)} \wedge ... \wedge f_{\sigma(n)}, \tag{A.4}$$

$$0 \leq n,\ \ a^*_{\mathfrak{h}}(f) : \mathfrak{F}^{(n)}(\mathfrak{h}) \mapsto \mathfrak{F}^{(n+1)}(\mathfrak{h}), \tag{A.5}$$

$$a^*_{\mathfrak{h}}(f)\,(f_1 \wedge ... \wedge f_n) = \frac{\sqrt{n+1}}{n!} \sum_{\sigma} \varepsilon(\sigma) f \wedge f_{\sigma(1)} \wedge ... \wedge f_{\sigma(n)}, \tag{A.6}$$

where the sum is taken over all the permutations σ of $\{1, 2, ...n\}$ and $\varepsilon(\sigma)$ is one if σ is even and minus one if σ is odd. We have for $f^{(n)} \in \mathfrak{F}^{(n)}(\mathfrak{h})$:

$$\| a_{\mathfrak{h}}(f)(f^{(n)}) \|^2 + \| a^*_{\mathfrak{h}}(f)(f^{(n)}) \|^2 = \| f \|^2 \| f^{(n)} \|^2, \tag{A.7}$$

hence $a_{\mathfrak{h}}(f)$ and $a^*_{\mathfrak{h}}(f)$ have bounded extensions on $\mathfrak{F}(\mathfrak{h})$. Moreover these operators satisfy :

$$\| a_{\mathfrak{h}}(f) \| = \| a^*_{\mathfrak{h}}(f) \| = \| f \|, \tag{A.8}$$

$$a^*_{\mathfrak{h}}(f) = (a_{\mathfrak{h}}(f))^*, \tag{A.9}$$

and the canonical anti-commutation relations (CAR's) :

$$a_{\mathfrak{h}}(f)a_{\mathfrak{h}}(g) + a_{\mathfrak{h}}(g)a_{\mathfrak{h}}(f) = 0, \tag{A.10}$$

$$a^*_{\mathfrak{h}}(f)a^*_{\mathfrak{h}}(g) + a^*_{\mathfrak{h}}(g)a^*_{\mathfrak{h}}(f) = 0, \tag{A.11}$$

$$a^*_{\mathfrak{h}}(f)a_{\mathfrak{h}}(g) + a_{\mathfrak{h}}(g)a^*_{\mathfrak{h}}(f) = < f, g > \mathbf{1}. \tag{A.12}$$

The *CAR Algebra* on $\mathfrak{h}$ is the $\mathbb{C}^*$-algebra $\mathfrak{A}(\mathfrak{h})$ generated by the identity $\mathbf{1}$ and the $a_{\mathfrak{h}}(f)$, $f \in \mathfrak{h}$. There exist interesting operators that do not belong to $\mathfrak{A}(\mathfrak{h})$. For instance, given a closed separable subspace F of $\mathfrak{h}$, the *number operator* N_F defined on $\cup_{n=0}^{\infty}\mathfrak{F}^{(n)}$ by

$$N_F := \sum_{j=0}^{\infty} a_{\mathfrak{h}}^*(f_j)a_{\mathfrak{h}}(f_j) \tag{A.13}$$

where $(f_j)_{j\in\mathbb{N}}$ is an orthonormal basis of F (obviously N_F does not depend of the choice of the basis).

When the classical fields obey the Schrödinger type equation

$$\frac{d\psi}{dt} = i\mathbb{H}\psi,$$

where $\mathbb{H}$ is a selfadjoint operator on $\mathfrak{h}$, a gauge-invariant quasi-free state ω on $\mathfrak{A}(\mathfrak{h})$ satisfies the (β,μ)-KMS condition, $0 < \beta$, $\mu \in \mathbb{R}$, if it is characterized by the two-point function

$$\omega(a_{\mathfrak{h}}^*(f)a_{\mathfrak{h}}(g)) = \left\langle ze^{\beta\mathbb{H}}\left(\mathbf{1} + ze^{\beta\mathbb{H}}\right)^{-1} f, g\right\rangle \tag{A.14}$$

where z is the *activity* given by :

$$z = e^{\beta\mu}. \tag{A.15}$$

(Note that we have written the Schrödinger/Dirac equation as $\partial_t\psi = i\mathbb{H}\psi$, instead of the traditional form $i\partial_t\psi = \mathbb{H}\psi$ adopted in [9] or [39]. Hence we must change $\mathbb{H}$ into $-\mathbb{H}$ to find the conventions of these authors.) This state is a model for the *ideal Fermi gas* with *temperature* $0 < T = \beta^{-1}$ and *chemical potential* μ. In statistical mechanics, such a state is called *Gibbs grand canonical equilibrium state.*

In the case of charged spinor fields, the situation is more intricate since we have to consider both kinds of fermions, the particles and the antiparticles. We consider a complex Hilbert space $\mathfrak{H}$ (the space of the classical charged spin fields), and an antiunitary operator C on $\mathfrak{H}$ (the *charge conjugation*). We assume $\mathfrak{H}$ is split into two orthogonal spaces

$$\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-. \tag{A.16}$$

We define the one particle space

$$\mathfrak{h}_+ = \mathfrak{H}_+, \tag{A.17}$$

and the one antiparticle space

$$\mathfrak{h}_- = C\mathfrak{H}_-. \tag{A.18}$$

Then the space of n particles and m antiparticles is given by the tensor product of the previous spaces :

$$\mathfrak{F}^{(n,m)} := \mathfrak{F}^{(n)}(\mathfrak{h}_+) \otimes \mathfrak{F}^{(m)}(\mathfrak{h}_-), \tag{A.19}$$

and to be able to treat arbitrary numbers of particles and antiparticles simultaneously, we introduce the $Dirac - Fermi - Fock$ space :

$$\mathfrak{F}(\mathfrak{H}) := \bigoplus_{n,m=0}^{\infty} \mathfrak{F}^{(n,m)}. \tag{A.20}$$

If we denote the elements ψ of $\mathfrak{F}$ by a sequence :

$$\psi = (\psi^{(n,m)})_{n,m\in\mathbb{N}}, \ \ \psi^{(n,m)} \in \mathfrak{F}^{(n,m)},$$

the *vacuum vector* is the vector Ω_{vac} defined by :

$$\Omega_{vac}^{(0,0)} = 1, \ \ (n,m) \neq (0,0) \Rightarrow \Omega_{vac}^{(n,m)} = 0. \tag{A.21}$$

Now for $\varphi_{+/-} \in \mathfrak{h}_{+/-}$ we define the *particle annihilation operator*, $a(\varphi_+)$, the *particle creation operator*, $a^*(\varphi_+)$, the *antiparticle annihilation operator*, $b(\varphi_-)$, the *antiparticle creation operator*, $b^*(\varphi_-)$, by putting for $\psi_+^{(n)} \otimes \psi_-^{(m)} \in \mathfrak{F}^{(n,m)}$:

$$a(\varphi_+)\left(\psi_+^{(n)} \otimes \psi_-^{(m)}\right) = \left(a_{\mathfrak{h}_+}(\varphi_+)\left(\psi_+^{(n)}\right)\right) \otimes \psi_-^{(m)} \in \mathfrak{F}^{n-1,m}, \tag{A.22}$$

$$a^*(\varphi_+)\left(\psi_+^{(n)} \otimes \psi_-^{(m)}\right) = \left(a^*_{\mathfrak{h}_+}(\varphi_+)\left(\psi_+^{(n)}\right)\right) \otimes \psi_-^{(m)} \in \mathfrak{F}^{n+1,m}, \tag{A.23}$$

$$b(\varphi_-)\left(\psi_+^{(n)} \otimes \psi_-^{(m)}\right) = \psi_+^{(n)} \otimes \left(b_{\mathfrak{h}_-}(\varphi_-)\left(\psi_-^{(m)}\right)\right) \in \mathfrak{F}^{n,m-1}, \tag{A.24}$$

$$b^*(\varphi_-)\left(\psi_+^{(n)} \otimes \psi_-^{(m)}\right) = \psi_+^{(n)} \otimes \left(b^*_{\mathfrak{h}_-}(\varphi_-)\left(\psi_-^{(m)}\right)\right) \in \mathfrak{F}^{n,m+1}. \tag{A.25}$$

All these operators have bounded extensions on $\mathfrak{F}(\mathfrak{H})$ and satisfy the CAR's. The main object of the theory is the *quantized Dirac field operator* $\mathbf{\Psi}$:

$$f \in \mathfrak{H} \longmapsto \mathbf{\Psi}(f) := a(P_+f) + b^*(CP_-f) \in \mathcal{L}(\mathfrak{F}(\mathfrak{H})), \tag{A.26}$$

where we have denoted by $P_{+/-}$ the orthogonal projector from $\mathfrak{H}$ onto $\mathfrak{H}_{+/-}$. The mapping $f \in \mathfrak{H} \longmapsto \mathbf{\Psi}(f)$ is antilinear and bounded :

$$\parallel \mathbf{\Psi}(f) \parallel = \parallel f \parallel . \tag{A.27}$$

Its adjoint denoted by $\mathbf{\Psi}^*(f)$ is given by

$$\mathbf{\Psi}^*(f) = a^*(P_+f) + b(CP_-f), \tag{A.28}$$

and the CAR's are satisfied :

$$\mathbf{\Psi}(f)\mathbf{\Psi}(g) + \mathbf{\Psi}(g)\mathbf{\Psi}(f) = 0, \tag{A.29}$$

$$\mathbf{\Psi}^*(f)\mathbf{\Psi}^*(g) + \mathbf{\Psi}^*(g)\mathbf{\Psi}^*(f) = 0, \tag{A.30}$$

$$\mathbf{\Psi}^*(f)\mathbf{\Psi}(g) + \mathbf{\Psi}(g)\mathbf{\Psi}^*(f) = < f, g > \mathbf{1}. \tag{A.31}$$

The *Field Algebra* is the $\mathbb{C}^*$-algebra $\mathfrak{A}(\mathfrak{H})$ generated by $\mathbf{1}$ and the $\mathbf{\Psi}(f)$, $f \in \mathfrak{H}$. If we take f only in $\mathfrak{H}_+(-)$ we get a subalgebra isometric to $\mathfrak{A}(\mathfrak{h}_{+(-)})$. The *vacuum state* ω_{vac} on $\mathfrak{A}(\mathfrak{H})$ is defined by

$$A \in \mathfrak{A}(\mathfrak{H}), \;\; \omega_{vac}(A) := \langle A\Omega_{vac}, \Omega_{vac}\rangle \,, \tag{A.32}$$

or by the two point function :

$$\omega_{vac}(\mathbf{\Psi}^*(f)\mathbf{\Psi}(g)) =< P_- f, P_- g > . \tag{A.33}$$

Now we assume the classical fields to satisfy a Dirac type equation

$$\frac{d\Psi}{dt} = i\mathbb{H}\Psi \tag{A.34}$$

where $\mathbb{H}$ is selfadjoint on $\mathfrak{H}$ and leaves $\mathfrak{H}_+$ and $\mathfrak{H}_-$ invariant. Then

$$\mathbb{H}^+ := \mathbb{H}|_{\mathfrak{h}_+}, \quad \mathbb{H}^- := -C\mathbb{H}|_{\mathfrak{H}_-} C^{-1}, \tag{A.35}$$

are respectively selfadjoint on $\mathfrak{h}_+$ and $\mathfrak{h}_-$, and the classical fields of one particle, φ_+, and of one antiparticle, φ_-, are solutions to a Schrödinger type equation on $\mathfrak{h}_{+(-)}$:

$$\frac{d\varphi_{+(-)}}{dt} = i\mathbb{H}^{+(-)}\varphi_{+(-)}. \tag{A.36}$$

A usual splitting of $\mathfrak{H}$ (with the remark following A.15) is the choice

$$\mathfrak{H}_+ = \mathbf{1}_{]-\infty,0[}(\mathbb{H}) \quad \mathfrak{H}_- = \mathbf{1}_{]0,\infty[}(\mathbb{H}). \tag{A.37}$$

We say that a state $\omega_{\beta,\mu}$ on $\mathfrak{A}(\mathfrak{H})$ satisfies the (β,μ)-KMS condition, $0 < \beta$, $\mu \in \mathbb{R}$, if it is characterized by the two-point function

$$\omega_{\beta,\mu}(\mathbf{\Psi}^*(f)\mathbf{\Psi}(g)) = \left\langle ze^{\beta\mathbb{H}}\left(\mathbf{1} + ze^{\beta\mathbb{H}}\right)^{-1} f, g\right\rangle, \quad z = e^{\beta\mu}. \tag{A.38}$$

The link with the Gibbs equilibrium states for particles and antiparticles is given explicitly in the following :

Lemma A.1. *Given* $\varphi^j_{+(-)} \in \mathfrak{h}_{+(-)}$, *we have :*

$$\omega_{\beta,\mu}(a^*(\varphi^1_+)a(\varphi^2_+)) = \left\langle ze^{\beta\mathbb{H}^+}\left(\mathbf{1} + ze^{\beta\mathbb{H}^+}\right)^{-1}\varphi^1_+, \varphi^2_+\right\rangle, \tag{A.39}$$

$$\omega_{\beta,\mu}(b^*(\varphi^1_-)b(\varphi^2_-)) = \left\langle z^{-1}e^{\beta\mathbb{H}^-}\left(\mathbf{1} + z^{-1}e^{\beta\mathbb{H}^-}\right)^{-1}\varphi^1_-, \varphi^2_-\right\rangle. \tag{A.40}$$

Therefore the restrictions of $\omega_{\beta,\mu}$ to $\mathfrak{A}(\mathfrak{h}_+)$ and to $\mathfrak{A}(\mathfrak{h}_-)$, describe a double Gibbs equilibrium state : on the one hand, an ideal Fermi particle gas with temperature $0 < T = \beta^{-1}$ and chemical potential μ, and on the other hand an ideal Fermi antiparticle gas with the same temperature T but an opposite chemical potential $-\mu$.

Proof of Lemma A.1. Taking $f = \varphi_+^1$, $g = \varphi_+^2$ in (A.38), we obtain (A.39). Choosing $f = C^{-1}\varphi_-^2$, $g = C^{-1}\varphi_-^1$, we get :

$$\begin{aligned}\omega_{\beta,\mu}(b(\varphi_-^2)b^*(\varphi_-^1))&\\ &= \left\langle ze^{\beta\mathbb{H}}\left(\mathbf{1}+ze^{\beta\mathbb{H}}\right)^{-1}C^{-1}\varphi_-^2, C^{-1}\varphi_-^1\right\rangle\\ &= \left\langle \varphi_-^1, zCe^{\beta\mathbb{H}}\left(\mathbf{1}+ze^{\beta\mathbb{H}}\right)^{-1}C^{-1}\varphi_-^2\right\rangle\\ &= \left\langle \varphi_-^1, \left(\mathbf{1}+z^{-1}e^{\beta\mathbb{H}^-}\right)^{-1}\varphi_-^2\right\rangle.\end{aligned}$$

Then we deduce (A.40) using the normality of the state, $\omega_{\beta,\mu}(\mathbf{1}) = 1$, and the CAR :

$$b^*(\varphi_-^2)b(\varphi_-^1) + b(\varphi_-^1)b^*(\varphi_-^2) = <\varphi_-^2, \varphi_-^1> \mathbf{1}.$$

□

We apply these procedures to define the Boulware state in the past, and the thermal state at the horizon. First the quantization at time $t = 0$ is defined by choosing

$$\mathfrak{H} = \mathcal{L}_0^2, \quad \mathbb{H} = \mathbb{H}_0, \tag{A.41}$$

$$C\Phi = {}^t(\overline{\Phi}_4, \overline{\Phi}_3, -\overline{\Phi}_2, -\overline{\Phi}_1). \tag{A.42}$$

If we stress the charge of the spin field q by denoting the Hamiltonian (III.17) by $\mathbb{H}_0 = \mathbb{H}_0(q)$, we remark that $\mathbb{H}^- = \mathbb{H}_0(-q)$, and $C\Phi$ satisfies the boundary condition (III.14) for $\Phi \in D(\mathbb{H}_0)$. Hence C is actually a charge conjugation. As regards the definition of particles and antiparticles, appears a slight ambiguity due to the fact that 0 is a possible eigenvalue of $\mathbb{H}_0$, unlike the case of the whole Reissner-Nordstrøm manifold for which $\mathbb{H}_\infty$ has no eigenvalue (Lemma III.1). We leave open the problem of the point spectrum of $\mathbb{H}_0$ and we choose :

$$(P_+, P_-) = \left(\mathbf{1}_{]-\infty,0]}(\mathbb{H}_0), \mathbf{1}_{]0,\infty[}(\mathbb{H}_0)\right) \ \textit{or} \ \left(\mathbf{1}_{]-\infty,0[}(\mathbb{H}_0), \mathbf{1}_{[0,\infty[}(\mathbb{H}_0)\right), \tag{A.43}$$

and we denote $\mathbf{\Psi}_0$ the quantum field at time $t = 0$ constructed in the previous way. We define the Boulware quantum state ω_0 on the field algebra $\mathfrak{A}(\mathcal{L}_0^2)$ as the vacuum state :

$$\Phi_0^j \in \mathcal{L}_0^2, \quad \omega_0\left(\mathbf{\Psi}_0^*(\Phi_0^1)\mathbf{\Psi}_0(\Phi_0^2)\right) = \left\langle P_-\Phi_0^1, P_-\Phi_0^2\right\rangle. \tag{A.44}$$

To quantize at the black-hole horizon, we choose

$$\mathfrak{H}_{BH} := \mathcal{L}^2_\infty,\ \ \mathbb{H} = \mathbb{H}_{BH}, \tag{A.45}$$

$$P_+ = \mathbf{1}_{]-\infty,0[}(\mathbb{H}_{BH}),\ \ P_- = \mathbf{1}_{]0,\infty[}(\mathbb{H}_{BH}), \tag{A.46}$$

C is given by (A.42) again, and $\mathbf{\Psi}_{BH}(\Phi)$ denotes the quantum field defined by (A.28). We can easily express these operators using the partial Fourier transform with respect to x, $\hat{f}(\xi,\omega)$ of $f(x,\omega) \in L^2(\mathbb{R}_x \times S^2_\omega,\ dxd\omega)$:

$$\begin{aligned} P_+\mathfrak{H}_{BH} = \Big\{ \Phi \in \mathcal{L}^2_\infty;\ \ \xi \geq qA(r_0) \Rightarrow \hat{\Phi}_1(\xi,\omega) = \hat{\Phi}_4(\xi,\omega) = 0, \\ \xi \leq -qA(r_0) \Rightarrow \hat{\Phi}_2(\xi,\omega) = \hat{\Phi}_3(\xi,\omega) = 0 \Big\}, \end{aligned} \tag{A.47}$$

$$\begin{aligned} P_-\mathfrak{H}_{BH} = \Big\{ \Phi \in \mathcal{L}^2_\infty;\ \ \xi \leq qA(r_0) \Rightarrow \hat{\Phi}_1(\xi,\omega) = \hat{\Phi}_4(\xi,\omega) = 0, \\ \xi \geq -qA(r_0) \Rightarrow \hat{\Phi}_2(\xi,\omega) = \hat{\Phi}_3(\xi,\omega) = 0 \Big\}. \end{aligned} \tag{A.48}$$

In fact it will be useful to split the fields into a part outgoing to infinity, and a part falling into the black-hole, as $t \to +\infty$, by putting :

$$\mathfrak{H}^{out} := \left\{ \Phi \in \mathcal{L}^2_\infty;\ \ \Phi_1 = \Phi_4 = 0 \right\}, \tag{A.49}$$

$$\mathfrak{H}^{in} := \left\{ \Phi \in \mathcal{L}^2_\infty;\ \ \Phi_2 = \Phi_3 = 0 \right\}. \tag{A.50}$$

We denote P^{out} and P^{in} the orthogonal projectors from $\mathfrak{H}_{BH}$ onto $\mathfrak{H}^{out}$ and $\mathfrak{H}^{in}$. We are mainly concerned with the outgoing (anti)particles. Let ω^{out} be a state on $\mathfrak{A}(\mathfrak{H}^{out})$. Given a Lebesgue measurable subset Λ of $\mathbb{R} \times S^2$, of measure $\mid \Lambda \mid \in]0,\infty[$, we introduce

$$\mathfrak{H}^{out}_\Lambda := \left\{ \Phi \in \mathfrak{H}^{out};\ \ (x,\omega) \notin \Lambda \Rightarrow \Phi(x,\omega) = 0 \right\}. \tag{A.51}$$

We choose an orthonormal basis $(\Phi^j)_{j\in\mathbb{N}}$ of $\mathfrak{H}^{out}_\Lambda$, we define the following numbers :

$$N^+_\Lambda(\omega^{out}) :=\mid \Lambda \mid^{-1} \sum_j \omega^{out}(a^*(P_+\Phi^j)a(P_+\Phi^j)), \tag{A.52}$$

$$N^-_\Lambda(\omega^{out}) :=\mid \Lambda \mid^{-1} \sum_j \omega^{out}(b^*(CP_-\Phi^j)b(CP_-\Phi^j)), \tag{A.53}$$

and if these numbers are finite

$$\varrho_\Lambda(\omega^{out}) := q\left(N^+_\Lambda(\omega^{out}) - N^+_\Lambda(\omega^{out}) \right). \tag{A.54}$$

We could understand these quantities as, respectively, the density of particles, the density of antiparticles, the charge density (in Λ). But this interpretation is somewhat misleading since according to the Paley-Wiener theorem $P_+\mathfrak{H}_\Lambda^{out}\cap\mathfrak{H}_\Lambda^{out}=P_-\mathfrak{H}_\Lambda^{out}\cap\mathfrak{H}_\Lambda^{out}=\{0\}$, hence no particle and no antiparticle is localized in Λ. Nevertheless, in the case of the Gibbs states, these quantities do not depend on Λ, hence these concepts are meaningful:

Lemma A.2. *$N_\Lambda^\pm(\omega^{out})$ is independent of the choice of the basis $(\Phi^j)_{j\in\mathbb{N}}$. Moreover, given a $(\beta,\mu)-KMS$ state $\omega^{out}_{\beta,\mu}$ on $\mathfrak{A}(\mathfrak{H}^{out})$, we have*

$$N_\Lambda^+(\omega^{out}_{\beta,\mu})=\frac{1}{\pi\beta}\ln\left(1+e^{\beta\mu}\right),\quad N_\Lambda^-(\omega^{out}_{\beta,\mu})=\frac{1}{\pi\beta}\ln\left(1+e^{-\beta\mu}\right),\quad \varrho_\Lambda(\omega^{out}_{\beta,\mu})=\frac{1}{\pi}q\mu. \tag{A.55}$$

Proof of Lemma A.2. If $(\Psi^n)_{n\in\mathbb{N}}$ is another orthonormal basis of $\mathfrak{H}_\Lambda^{out}$ we have :

$$\begin{aligned}N_\Lambda^+(\omega^{out})&=\mid\Lambda\mid^{-1}\sum_{n,m}\left(\sum_j<\Phi^j,\Psi^n>\overline{<\Phi^j,\Psi^m>}\right)\omega^{out}(a^*(P_+\Psi^n)a(P_+\Psi^m))\\&=\mid\Lambda\mid^{-1}\sum_n\omega^{out}(a^*(P_+\Psi^n)a(P_+\Psi^n)).\end{aligned}$$

By using the Fourier transform and (A.39), (A.47) and (A.49), we calculate

$$\begin{aligned}&N_\Lambda^+(\omega^{out}_{\beta,\mu})=\\&\frac{1}{2\pi\mid\Lambda\mid}\int_{-qA(r_0)}^\infty\frac{e^{\beta(\mu-qA(r_0))}e^{-\beta\xi}}{\left(1+e^{\beta(\mu-qA(r_0))}e^{-\beta\xi}\right)}\sum_j\left(\mid\hat{\Phi}_2^j(\xi)\mid^2+\mid\hat{\Phi}_3^j(\xi)\mid^2\right)d\xi.\end{aligned}\tag{A.56}$$

Now given $f_h\in L^2(\Lambda)$ we have :

$$\parallel f_2\parallel^2+\parallel f_3\parallel^2=\sum_j|<f_2,\Phi_2^j>+<f_3,\Phi_3^j>|^2,$$

hence by choosing $f_k(x)=e^{-ix\xi}\mathbf{1}_\Lambda(x)$, $f_l=0$ we deduce that :

$$\sum_j\left(\mid\hat{\Phi}_2^j(\xi)\mid^2+\mid\hat{\Phi}_3^j(\xi)\mid^2\right)=2\mid\Lambda\mid$$

and we easily get the value of $N_\Lambda^+(\omega^{out}_{\beta,\mu})$. Finally we obtain $N_\Lambda^-(\omega^{out}_{\beta,\mu})$ thanks to Lemma A.1 by changing μ into $-\mu$. □

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